Marco Cannone

Smooth and singular solutions for the Navier-Stokes equations

So far, only two ways for attacking the Cauchy problem for the Navier-Stokes equations are known: the first is due to J. Leray (1933), and the second is due to T. Kato (1984). None of them can be considered the "golden rule" for solving the Navier-Stokes equations because they both leave open the following celebrated question. In three dimensions, does the velocity field of a fluid flow that starts with smooth initial data (velocity and external force) remain smooth and unique for all time ? Based on a priori energy estimates, Leray's theory gives the existence of global weak, possibly irregular and possibly non-unique solutions to the Navier-Stokes equations, whereas Kato's approach, based on the fixed point scheme, imposes a priori a regularization effect on solutions we look for. In other words, Kato's solutions are considered as fluctuations around the solution of the heat equation with same initial data, and are as such a priori regular. There exist however two exceptions, more exactly two critical spaces where Kato's method applies without imposing any a priori regularizing condition : the Lorentz space $L^{3,\infty}$, considered from an analytical viewpoint by M. Yamazaki (1999) and Y. Meyer (1999), and the pseudomeasure space, introduced by Y. Le Jan and A.S. Sznitman (1997), and associated to a probabilistic representation of solutions of the Navier-Stokes equations. In this lecture, based on a joint work with G. Karch (J. Diff. Eq., to appear, 2003), we will show how Kato's approach gives existence and uniqueness of a small solution in a larger space which, in our case, contains genuinely singular solutions that are not smoothed out by the action of the nonlinear semigroup associated. More exactly, using the pseudomeasure space of Le Jan-Sznitman we can prove the following results. The existence of singular solutions associated to singular (e.g. the Dirac delta) external forces, thus allowing to describe the solutions considered by L.D. Landau (1944) and by G. Tian and Z. Xin (1998). The existence of regular solutions for more regular external forces. The asymptotic stability of small solutions including stationary ones. A pointwise loss of smoothness for solutions for large data. Applying the same techniques we will prove similar results for a model equation of gravitating particles, following the lines of a joint work with P. Biler, I. Guerra and G. Karch. Moreover, in the case of this particular model, we will show that the loss of smoothness for large data holds in the distributional sense as well.