Algorithms for Mean-Field Type Control Problems

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July 2015

Production of Exhaustible Resource

An oil field is exploited by N producers. Each producer^j optimize his profit so as to fulfill his goal R_0 at T; so his known resource is $x_t^j \in (0, R_0)$, $x_0^j = R_0$.

$$\max_{v^j \geq 0}\{\int_0^T \mathbf{E}\left[p_t v_t^j - C(v_t^j)\right]\mathbf{e}^{-rt}dt \ : \ dx_t^j = -v_t^j dt + \sigma x_t^j dw_t, \ x_0^j = R_0, \ x_T^j = 0.\}$$

- w_t is a Brownian, $C(v_t^j) = \alpha v_t^j + \beta {v_t^j}^2$ is production cost,
- price p_t is such that demand balances production[1] $v_t = \frac{1}{N}\sum_{j=1}^N v_t^j$ by

$$p_t = a\mathbf{e}^{-bt}(\mathbf{E}(v_t))^{-c} \ \Rightarrow \ J^j = \mathbf{E}\int_0^\infty (a\mathbf{e}^{-bt}(\mathbf{E}[v_t])^{-c} v_t^j - \alpha v_t^j - \beta {v_t^j}^2)\mathbf{e}^{-rt}dt$$

- Let $x_t = \frac{1}{N}\sum_{j=1}^N x_t^j$, $J = \frac{1}{N}\sum_{j=1}^N J^j$. By linearity $dx_t = -v_t dt + \sigma x_t dw_t$, so

$$\max_{v \geq 0}\{J := J(0) \ : \ dx_t = -v_t dt + \sigma x_t dw_t\} \qquad \text{Penalty} \downarrow$$

$$J(\tau) = \int_\tau^T \left(\mathbf{E}[a\mathbf{e}^{-bt}(\mathbf{E}[v_t])^{-c} v_t - \alpha v_t - \beta v_t^2]\right)\mathbf{e}^{-rt}dt - \gamma\mathbf{E}[(x_T)^d]$$

meanfield game theory $\uparrow$

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Calculus of Variations on the Deterministic Control Problem

Let $\rho(x,t)v$ the PDF of x_t; let $Q = \mathbf{R}\times(0,T)$. Change of sign gives

$$\min_{v\ge 0}\{J = \int_Q \left(\alpha v + \beta v^2 - a\mathbf{e}^{-bt}\chi^{-c}v\right)\mathbf{e}^{-rt}\rho \mathrm{d}x\mathrm{d}t + \gamma\int_{\mathbf{R}} x^d\rho_T \mathrm{d}x,$$

$$\chi = \int_{\mathbf{R}} v\rho \mathrm{d}x \ : \ \partial_t\rho - \frac{\sigma^2}{2}\partial_{xx}(x^2\rho) - \partial_x(\rho v) = 0, \quad \rho(0) = \rho_0\}$$

Calculus of variations. Let $\alpha_t = \alpha - a\mathbf{e}^{-bt}\chi^{-c}$. Then

$$\delta J = \gamma\int_{\mathbf{R}} x^d\delta\rho_T \mathrm{d}x$$
$$+\int_Q \mathbf{e}^{-rt}\left((\alpha_t + 2\beta v)\rho\delta v + (\alpha_t+\beta v)v\delta\rho + ac\mathbf{e}^{-bt}\chi^{-c}(\rho\delta v + v\delta\rho)\right)\mathrm{d}x\mathrm{d}t$$

So introduce p with $p_T = \gamma x^d$ and

$$-\partial_t p - \frac{\sigma^2 x^2}{2}\partial_{xx}(p) + v\partial_x p = \mathbf{e}^{-rt}(\alpha + \beta v - a(1-c)\mathbf{e}^{-bt}\chi^{-c})v \ \Rightarrow$$

$$\delta J = \int_Q \left(\mathbf{e}^{-rt}(\alpha + 2\beta v - a(1-c)\mathbf{e}^{-bt}\chi^{-c}) - \partial_x p\right)\rho\delta v \mathrm{d}x\mathrm{d}t + o(\|\delta v\|)$$

Algorithm: Apply the gradient method on u

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Calculus of variations. Let $\alpha_t = \alpha - a\mathbf{e}^{-bt}\chi^{-c}$. Then

$$\begin{aligned}\delta J &= \gamma\int_{\mathbf{R}} x^d\delta\rho_T\mathrm{d}x \\ &+ \int_Q \mathbf{e}^{-rt}\Big((\alpha_t + 2\beta v)\rho\delta v + (\alpha_t+\beta v)v\delta\rho + ac\mathbf{e}^{-bt}\chi^{-c}(\rho\delta v + v\delta\rho)\Big)\mathrm{d}x\mathrm{d}t\end{aligned}$$

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HJB equations, the stochastic approach

Denote $u = -v_t$, $\chi = \mathbf{E}[v_t] = -\int_{\mathbf{R}} u\rho \mathrm{d}x$ where ρ is the PDF of x_t

$$\partial_t \rho - \frac{\sigma^2}{2}\partial_{xx}(x^2\rho) + \partial_x(\rho u) = 0 \text{ in } \mathbf{R}, \quad \rho_{|0} \text{ given}$$

Let $V(\rho_t(\cdot)) = \min_u\{J(t) : \rho(t) = \rho_t\}$. Let V' be defined by $V'_\rho \cdot \mu = \int_{\mathbf{R}} V' \mathrm{d}\mu$

By [1] $V'_{|T} = \gamma x^d$, $u = \frac{1}{2\beta}[\alpha - e^{rt}\partial_x V' - a(1-c)e^{-bt}\chi^{-c}]$

$$\partial_t V' + \frac{\sigma^2 x^2}{2}\partial_{xx} V' + u\partial_x V' = -(a(1-c)\mathbf{e}^{-bt}\chi^{-c}u - \alpha u + \beta u^2)\mathbf{e}^{-rt}, \quad \tau = T - t$$

$$V^* = \frac{e^{-rt}}{\beta}[V' - (\alpha - a(1-c)\mathbf{e}^{-bt}\chi^{-c})x], \quad f = \int_0^\tau (\alpha - e^{-b(T-s)}a(1-c)\chi^{-c})e^{r(\tau-s)}$$

$$\partial_\tau V^* + (\partial_x V^*)^2 - \frac{\sigma^2 x^2}{2}\partial_{xx} V^* + rV^* = f(\tau)x^2, \quad V^*|_0 = \frac{\gamma x^d}{\beta} - \frac{x^2}{\beta}(\alpha - a(1-c)R_0^{-c}$$

Algorithm: initialize u. compute ρ by Fokker Planck and χ, then V' by above, then u by above and loop.

[1] M. Lauriere and O. Pironneau: Dynamic programming for mean-field type control. C.R.A.S Serie I, 1-7, Oct. 2014

Case d=2: Ricatti Equation

For the time being forget the constraint $u \geq 0$. Then

$$\partial_t V' + \frac{\sigma^2 x^2}{2}\partial_{xx} V' = \beta \mathbf{e}^{-rt} v^2 = \frac{\mathbf{e}^{-rt}}{4\beta}(\alpha - a(1-c)\mathbf{e}^{-bt}\chi^{-c} - \mathbf{e}^{rt}\partial_x V')^2$$

Assume that $V' = P(t)x^2 + z(t)x + s(t)$, then by identification,

$$\dot{P} + \sigma^2 P = \frac{\mathbf{e}^{rt}}{4\beta}P^2, \quad P(T) = \gamma$$

Let $Q_t = \mathbf{e}^{rt}P$. Then with $\mu = \sigma^2 - r$,

$$\dot{Q}_t + \mu Q_t - \frac{Q_t^2}{\beta} = 0,\ Q_T = \gamma \mathbf{e}^{rT} \ \Rightarrow\ \frac{\dot{Q}_t}{\mu}\left(\frac{1}{Q_t} + \frac{1}{\beta\mu - Q_t}\right) = -1$$

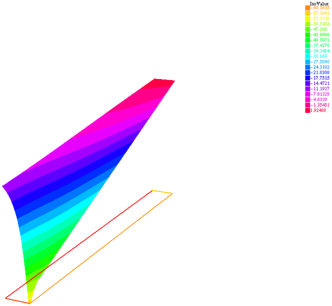
So long as $Q_t > 0$, i.e. if μT is small enough,

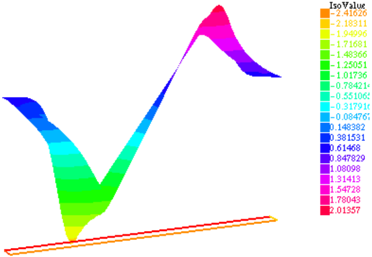
$$\frac{Q_t}{Q_t - \beta\mu} = \frac{\gamma \mathbf{e}^{rT}}{\gamma \mathbf{e}^{rT} - \beta\mu}\mathbf{e}^{(T-t)\mu} := A\mathbf{e}^{(T-t)\mu} \ \Rightarrow\ P = \mathbf{e}^{-rt}\frac{\beta\mu A\mathbf{e}^{(T-t)\mu}}{A\mathbf{e}^{(T-t)\mu} - 1}$$

Let us compute the optimal u by another method and check that $\partial_x u = -2\mathbf{e}^{rt}P$

Results

$$T = 1, c = \frac{3}{2}, \alpha = 1, \beta = 1, \gamma = 0.5, a = 1, b = 1, R_0 = 1, r = 0.02, \sigma^2 = 0.09$$





Left: $\partial_x v$ Ricatti Solution with proper treatment of boundary conditions.
Right: v vs time (toward us) and space (towards left) in $\mathbf{R} \times (0, T)$ computed by the gradient-control method. The feedback zone is very clear.

Conclusion

- Calcul of Variations applies and provides necessary conditions.
- Dynamic Programming applies to mean-field type control and provides sufficient conditions.
- Extention to problems with global system noise: see A. Bensoussan, C. Fresch & C.P. Yam.
- Full math proofs hard but doable: see Y. Achdou and M. Laurière (to appear)
- Opens the path to numerical solutions
- Need to find new algorithms
- Apply to interest rate of refundable loans

Thanks for your Attention