CONSERVATION LAWS DRIVEN BY LÉVY WHITE NOISE.

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July 28, 2014

This work was supported by EADS chair funds and performed within the EADS chair for "**mathematics of complex systems**".

**joint work with K. H. Karlsen and A. Majee.

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Balance laws with noise

A nonlinear balance law is of the form

$$\frac{\partial u(t,x)}{\partial t} + \mathrm{div}_x F(u(t,x)) = q(t,x,u(t,x)); \;\; t > 0, \;\; x \in \mathbb{R}^d. \quad (1)$$

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Example: $q(t,x,u) = \sigma(t,x,u)\frac{dB_t}{dt}$.Then (1) is better understood as the SPDE

$$du(t,x) + \mathrm{div}_x F(u(t,x))\,dt = \sigma(x,u(t,x))\,dB_t; \;\; t > 0, \;\; x \in \mathbb{R}^d. \quad (2)$$

Balance laws with Poisson noise

Our specific interest is the initial value problem

$$du(t,x) + \mathrm{div}_x F(u(t,x))\, dt = \int_{|z|>0} \eta(x,u;z)\, \tilde{N}(dz,dt) \qquad (3)$$

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We want to find a $L^p(\mathbb{R}^d)$-valued process $u(t,\cdot)$ which satisfies (3) /solves (3). What do we exactly mean by solution here?

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Usual deterministic calculus needs to be replaced by Itô-Lévy calculus.

The entropy inequalities will have non-localities.

Viscous problem

Consider the viscous perturbation

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$$\begin{aligned}
d\big[\beta(u_\epsilon(t,x))\psi\big] &= \partial_t\psi\beta(u_\epsilon)\,dt - \psi\mathrm{div}_x\zeta(u_\epsilon)\,dt \\
&+ \int_{|z|>0} \psi\big(\beta(u_\epsilon+\eta(x,u_\epsilon;z)) - \beta(u_\epsilon)\big)\,\tilde{N}(dz,dt) \\
&+ \int_{|z|>0} \psi\big(\beta(u_\epsilon+\eta(x,u_\epsilon;z)) - \beta(u_\epsilon) - \eta(x,u_\epsilon;z)\beta'(u_\epsilon)\big)\,m(dz)\,dt \\
&+ \psi(t,x)\Big(\epsilon\Delta_{xx}\beta(u_\epsilon) - \epsilon\beta''(u_\epsilon)|\nabla_x u_\epsilon|^2\Big)\,dt.
\end{aligned}$$

Entropy inequalities

Definition (entropy solution)

A $L^2(\mathbb{R}^d)$*-valued* $\{\mathcal{F}_t : t \geq 0\}$*-predictable process* $u(t,x)$ *is an entropy solution of* (3) *if*

(1) For each $T > 0$, $p = 2, 3, 4, \ldots$, $\sup_{0 \leq t \leq T} E\left[||u(t)||_p^p\right] < \infty$.

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(2) For $0 \leq \psi \in C_c^{1,2}([0,\infty) \times \mathbb{R}^d)$ *and convex entropy pair* (β, ζ)

$$\langle \psi(0), \beta(u(0)) \rangle + \int_{t=0}^{T} \langle \partial_t \psi(t), \beta(u(t)) \rangle \, dt + \int_{r=0}^{T} \langle \zeta(u(r)), \nabla_x \psi(r) \rangle \, dr$$

$$+ \int_{r=0}^{T} \int_{|z|>0} \langle \beta(u(r) + \eta(., u(r); z)) - \beta(u(r, .)), \psi(r, \cdot) \rangle \tilde{N}(dz, dr)$$

$$+ \int_{r=0}^{T} \int_{|z|>0} \langle \beta(u(r) + \eta(\cdots)) - \beta(u(r)) - \eta(\cdots)\, \beta'(u(r)), \psi(r) \rangle \, m(dz)$$

$$\geq 0 \quad P - a.s$$

Assumptions

- $F(s) \in C^2(\mathbb{R} : \mathbb{R}^d)$ with polynomially growing derivatives.

- There exist $K > 0$ and $\lambda^* \in [0,1)$ s.t for all $x, y \in \mathbb{R}^d$; $u, v \in \mathbb{R}$; $z \in \mathbb{R}$,

$$|\eta(x,u;z) - \eta(y,v;z)| \leq (\lambda^*|u-v| + K|x-y|)(|z| \wedge 1)$$

- The Lévy measure $m(dz)$ satisfies $\int_{\mathbb{R}_z} (|z|^2 \wedge 1)\, m(dz) < \infty$.

- There is $g \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ s.t. $|\eta(x,u;z)| \leq g(x)(1+|u|)(|z| \wedge 1)$.

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The viscous perturbation

$$du_\epsilon + \operatorname{div}_x F_\epsilon(u_\epsilon)\,dt = \int_{|z|>0} \eta_\epsilon(x, u_\epsilon; z)\,\tilde{N}(dz,\, dt) + \epsilon \Delta_{xx} u_\epsilon\, dt, \quad (4)$$

with $\sup_{\epsilon>0} \sup_{0\le t\le T} E\Big[||u_\epsilon(0,\cdot)||_p^p\Big] < \infty$ for $p = 2, 4, \ldots$.. Then

Compactness

We establish compactness as Young measures with an additional parameter.

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Theorem (B, Karlsen & Majee; 2014)

If $\bigcap_{p=1,2,..} L^p(\mathbb{R}^d)$*-valued* $\mathcal{F}_0$*-measurable random variable* u_0 *satisfies*

$$E\left[||u_0||_p^p + ||u_0||_2^p\right] < \infty, \qquad \text{for } p = 1, 2, \ldots \; . \tag{5}$$

Then there exists a entropy process solution of (3)*.*

Uniqueness and existence

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The $\bigcap_{p=1,2,..} L^p(\mathbb{R}^d)$-valued $\mathcal{F}_0$-measurable random variable u_0 satisfies (5). Then the entropy process solution of (3) is unique. Moreover, it is the unique stochastic entropy solution.

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