CLT for Lipschitz-Killing curvatures of excursion sets of Gaussian random fields[1]

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Objective

f: a 'smooth', isotropic, real valued Gaussian random field defined on the the paramater space M_t, which for simplicity, we shall take as $\mathbb{S}^d(t)$ (d-dimensional sphere with radius t).
Then, define

$$A_{u,t} = A_u(f, \mathbb{S}^d(t)) = \{x \in \mathbb{S}^d(t) : \ f(x) \geq u\}$$

Main goal: CLT for the (appropriately normalized) Lipschitz-Killing Curvatures (LKCs) of $A_{u,r}$ as $r \to \infty$

State of the art

- Pham (2014) proved central limit theorem for the volume of the excursion sets, under two different settings: growing parameter space with a fixed threshold; growing parameter space with an increasing threshold.
- Estrade and León (2015) proved CLT for the Euler-Poincaré characteristic of the excursion sets (fixed threshold; growing parameter space).

Proofs of Estrade-Leon, at some level, are related of Kratz and León (2001), where the authors proved a central limit theorem for functionals of random field, where the functionals depend on the field through the field, its first and second order derivatives.

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But why does one care?

Lipschitz-Killing curvatures. What are they?

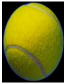
Notion of Euler-Poincaré characteristic (from Keith Worsley's page)

Euler Characteristic in 3D:
EC = #blobs - #tunnels or handles + #hollows

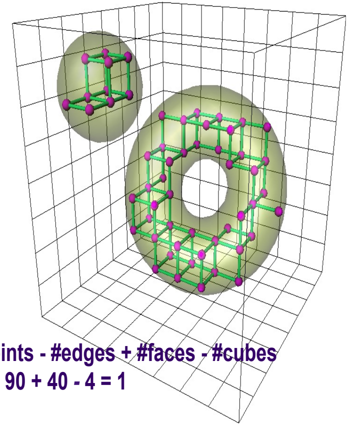
EC() = 1 – 0 + 0 = 1

EC() = 1 – 1 + 0 = 0

EC() = 1 - 3 + 0 = -2

EC() = 1 - 0 +1 = 2

Euler Characteristic in 3D:
EC = #blobs - #tunnels or handles + #hollows
= 2 - 1 + 0 = 1



EC = #points - #edges + #faces - #cubes
= 55 - 90 + 40 - 4 = 1

Lipschitz-Killing curvatures via Crofton's/Hadwiger's formula (1860s)

- Graff(n, k): affine Grassmannian, of all k-dimensional subspaces of $\mathbb{R}^n$.
- Equip Graff(n, k) with a measure $\lambda_{n,k}$, which is invariant under the set of rigid motions $E(n)$.
- This measure can be factored as ν_k^n on the Grassmannian Gr(n, k) and Lebesgue measure on $\mathbb{R}^n$, and can be normalized so that $\nu_k^n(\mathrm{Gr}(n, k)) = \left[\begin{array}{c} n \\ k \end{array} \right] = \binom{n}{k} \frac{\omega_n}{\omega_k \omega_{n-k}}$.
- Let $M \subset \mathbb{R}^n$, *nice* and compact, then we have

$$\int_{\mathrm{Graff}(n,n-k)} \mathcal{L}_0(M \cap V)\, d\lambda_{n-k}^n(V) = \mathcal{L}_k(M).$$

where $\mathcal{L}_0$ is the Euler-Poincaré characteristic, and $\mathcal{L}_k$ is the k-th Lipschitz-Killing curvature (LKC).

LKCs: properties

- For an m-dimensional subset $A \subset \mathbb{R}^n$, $\mathcal{L}_0(A)$ is its Euler–Poincaré characteristic, and $\mathcal{L}_m(A)$ is its m-dimensional volume.
- $\mathcal{L}_i$, of say a set A, is an intrinsic, integral geometric characteristics of the set.
- Writing R for the Riemannaian curvature tensor, LKCs for a smooth Riemannian manifold M can be defined as

$$\mathcal{L}_k(M) = c(n,k) \int_M \mathrm{Tr}\left(R^{\frac{n-k}{2}}\right) Vol_g$$

whenever $\frac{n-k}{2}$ is an integer, and it is zero otherwise.

- Scaling: $\mathcal{L}_k(\lambda A) = \lambda^k \mathcal{L}_k(A)$.

Lipschitz–Killing curvatures (LKCs): examples

- A box B with dimensions (a, b, c): $\mathcal{L}_0(B) = 1$, $\mathcal{L}_1(B) = (a + b + c)$, $\mathcal{L}_2(B) = (ab + bc + ac)$, $\mathcal{L}_3(B) = abc$.
- A ball $B_n(r)$ of radius r in $\mathbb{R}^n$:

$$\mathcal{L}_j(B_n(r)) = r^j \begin{pmatrix} n \\ j \end{pmatrix} \frac{\omega_n}{\omega_{n-j}}$$

- A sphere $S^{n-1}(r)$ of radius r in $\mathbb{R}^n$:

$$\mathcal{L}_j(S^{n-1}(r)) = 2r^j \begin{pmatrix} n \\ j \end{pmatrix} \frac{\omega_n}{\omega_{n-j}},$$

  for even values of $(n - j - 1)$, and 0 otherwise.
- For a unit codimensional manifold, every alternate $\mathcal{L}_i$ vanishes.

Plan of action

- For $V \in \text{Graff}(d+1, k)$, let us define

$$\mathcal{L}_0^{\#}(u, M_t; V) = \frac{\mathcal{L}_0\left(A_u(f; M_t \cap V)\right) - \mathbb{E}\left(\mathcal{L}_0\left(A_u(f; M_t \cap V)\right)\right)}{\left|M_t \cap V\right|^{1/2}}.$$

  where $|M_t \cap V)|$ is $(k-1)$-dimensional volume.

- Plan: prove the convergence of fdd of $\mathcal{L}_0^{\#}(u, M_t; \cdot)$; invoke tightness; check that the variance of $\mathcal{L}_{d+1-k}^{\#}(u, M_t)$ has a uniform upper bounded, and also that the variance is not degenerate; get the required CLT by observing that $\mathcal{L}_{d+1-k}^{\#}(u, M_t)$ is a "nice" functional of the random field $\mathcal{L}_0^{\#}(u, M_t; \cdot)$

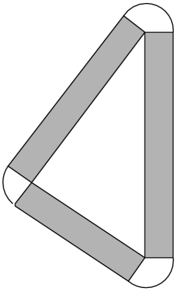
Another approach

Lipschitz-Killing curvatures via tube formula

Steiner's tube formula

- Let A be a *nice* subset of $\mathbb{R}^n$; λ_n be the n-dimensional Lebesgue measure.
- Let

$$\text{Tube}(A, \rho) = A + B_n(0, \rho) = \{x \in \mathbb{R}^n : \ \text{dist}(x, A)$$



Then Steiner's tube formula is given by:

$$\lambda_n(\text{Tube}(A, \rho)) = \sum_{i=0}^{\dim(A)} \omega_{n-i} \rho^{n-i} \, \mathcal{L}_i(A),$$

$(\mathcal{L}_i(A))_{i=1}^{\dim(A)}$ = Lipschitz–Killing curvatures, and ω_{n-i} is the volume of a unit ball in $\mathbb{R}^{n-i}$.

A tube formula for the excursion set on M_t

Note that

$$\mathrm{Tube}(A_u(f;M_t);\rho) = A_u(f;M_t) + B_{d+1}(0,\rho)$$

where $B_{d+1}(0,\rho) = \{x \in \mathbb{R}^{d+1} : \|x\| \le \rho\}$.
Then, recalling that M_t is just a d-dimensional sphere with radius t, the volume of the tube can be simplified as:

$$\begin{aligned} & \mathrm{Vol}\left(\mathrm{Tube}(A_u(f;M_t);\rho)\right) \\ = \; & \mathcal{H}_d(A_u(f;M_t)) \int_{-\rho}^{\rho} \det(I + u\,\nabla E_r)\,\mathcal{H}_1(du) \\ & + \int_0^{\rho}\int_0^{\pi}\int_{\partial A_u(f;M_t)} \det\left[I + u\nabla\left(\cos\theta E_r - \sin\theta\sum_{i=1}^{d}[E_i f]E_i\right)\right] \\ & \times \mathcal{H}_{d-1}(dx)\mathcal{H}_1(d\theta)\mathcal{H}_1(du) \end{aligned}$$

Recipe for CLT

- Then, $\mathcal{L}_k$ is identified (modulo constants) as the coefficient of $u^{(d+1-k)}$.
- Notice that since all the terms appearing in the expression above are solely dependent on the field f through f, ∇f, and $\nabla^2 f$, which falls in the realm of Kratz-Leon (though in the setting of a manifold).
- Hermite expansion: the terms coming from the determinant are clearly, polynomials involving the field, its derivative and second derivative.But the second integral in the tube formula is over $\partial A_u(f; M_t)$. But notice that

$$\int_{\partial A_u(f;M_t)} g(x)\, \mathcal{H}_{d-1}(dx) = \int_{M_t} g(x)\, \delta_{(f(x)=u)}\, \|\nabla f(x)\|\, \mathcal{H}_d(dx)$$

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- Need also a Hermite expansion for δ, which has also been done earlier by Kratz-Leon, Estrade-Leon, ...

Back to the recipe for CLT

- Once we have the Hermite expansion for the random variables inside the integral, we invoke the standard Nourdin-Peccati type of scheme.
- Projection onto each chaos converges to Gaussian.
- Check non-degeneracy of the variance (already done in the earlier setup).
- Conclude convergence of $\mathcal{L}^{\#}_{d+1-k}(A_u(f; M_t))$ to a Gaussian limit as $t \to \infty$.

THANK YOU FOR YOUR ATTENTION!

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