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# Stabilization of Compressible Navier-Stokes System in one dimension

Mythily Ramaswamy
TIFR Centre for Applicable Mathematics, Bangalore, India

## Collaborators and Financial Support

Collaborators:

Jean-Pierre Raymond (IMT, Toulouse, France),

Michael Renardy (Virginia Tech, USA),

Shirshendu Chowdhury, Debayan Maity, Debanjana Mitra (TIFR-CAM, Bangalore)

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# Compressible Navier-Stokes system

- A model for flow of compressible fluid in $\Omega \subset \mathbb{R}$:
  Density $\rho(x,t)$, velocity $u(x,t)$ of the fluid in $\Omega \times \ (0,T)$ :

$$\partial_t \rho + \partial_x(\rho u) \ = \ 0$$

$$(\rho u)_t \ + \ (\rho u^2)_x \ + \ (p(\rho))_x \ - \nu u_{xx} \ = \ 0.$$

- Pressure $p$ is

$$p(\rho) \ = \ (a\ \rho^\gamma) \ \text{ for } \gamma \geq 1, a > 0.$$

- Can we control the fluid?
- Can we stabilize the nonlinear system?

# Compressible Navier-Stokes system

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# Scope of our work

- Linearize the system around constant steady states
- With suitable boundary conditions get the spectrum and a Fourier basis
- Study controllability of the linearized system : interior and boundary null controllability and approximate controllability
- Study feedback stabilization of the linearized system
- Using this study, analyse local stabilization of the nonlinear system

# Initial boundary value problem for the linearized system

- Domain $\Omega = (0,\pi)$
- $(\rho_s,\ u_s)$ : a constant steady state solution with $\rho_s > 0, u_s \geq 0$
- Linearized system around this solution :

$$\begin{aligned} \partial_t \rho \quad &+ \quad u_s \rho_x + \rho_s\, u_x \quad = 0 \\ \partial_t u \quad &- \quad \frac{\nu}{\rho_s} u_{xx} + u_s\, u_x + a\gamma\, \rho_s^{\gamma-2} \rho_x \ = \ f \chi_O \end{aligned}$$

with $O \subset \Omega$

- Initial, boundary conditions :

$$\begin{aligned} \rho(x,0) &= \rho_0(x)\ ; \quad u(x,0) \ = \ u_0(x),\ x \in \Omega \\ u(0,t) &= q_0(t)\ ;\ u(\pi,t) = q_1(t) \quad \forall\, t > 0 \end{aligned}$$

- Additional boundary conditions for $\rho$ at $x = 0$ when $u_s > 0$
- Distributed control $f$ ; Boundary controls $q_0,\ q_1$

# Function space framework

- Function space for the case $u_s = 0$ : $\mathbf{Z} = L^2(\Omega) \ \times \ L^2(\Omega)$
- Equip with inner product, denoting $b = a\gamma\rho_s^{\gamma-2}$

$$\left\langle \begin{pmatrix} \rho \\ u \end{pmatrix}, \begin{pmatrix} \sigma \\ v \end{pmatrix} \right\rangle_{\mathbf{Z}} = b\int_0^\pi \rho(x)\sigma(x)dx + \rho_s \int_0^\pi u(x)v(x)dx$$

- Call $\nu_0 = \frac{\nu}{\rho_s}$. Define the subspace :

$$\mathcal{D}(A) = \{ \begin{pmatrix} \rho(x) \\ u(x) \end{pmatrix} \in \mathbf{Z} : u(x) \in H_0^1(\Omega), (-b\rho(x) + \nu_0 u'(x)) \in H^1(\Omega) \}$$

- Define $A : \mathcal{D}(A) \to \mathbf{Z}$ :

$$A = \begin{bmatrix} 0 & -\rho_s \frac{d}{dx} \\ -a\gamma\rho_s^{\gamma-2}\frac{d}{dx} & \frac{\nu}{\rho_s}\frac{d^2}{dx^2} \end{bmatrix}$$

- $\mathcal{D}(A)$ is dense in $\mathbf{Z}$ ; $A$ is maximal dissipative
- $(A, \mathcal{D}(A))$ is the infinitesimal generator of $C^0$ semigroup $S(t)$ on $\mathbf{Z}$.

# Operator Equation

- Call $\mathbf{U}(x,t) = \begin{pmatrix} \rho(x,t) \\ u(x,t) \end{pmatrix}$
- System without controls :

$$\begin{aligned} \frac{d\mathbf{U}(t)}{dt} &= A\mathbf{U}(t), \quad t > 0 \\ \mathbf{U}(0) &= \mathbf{U}_0 \in \mathbf{Z}. \end{aligned}$$

- For every $\mathbf{U}_0 \in \mathbf{Z}$, there is a unique solution $\mathbf{U}$ in $C([0,\infty),\mathbf{Z})$

# Spectrum of $A$ when $u_s = 0$

The point spectrum of $A$

- lies on the left half plane
- consists of a finite number of pairs of complex eigenvalues :

$$|\mathrm{Real}(\lambda_k)| \geq \frac{\nu}{2\rho_s} := \frac{\nu_0}{2}, \quad |\mathrm{Im}(\lambda_k)| \leq \frac{2a\gamma\rho_s}{\nu}.$$

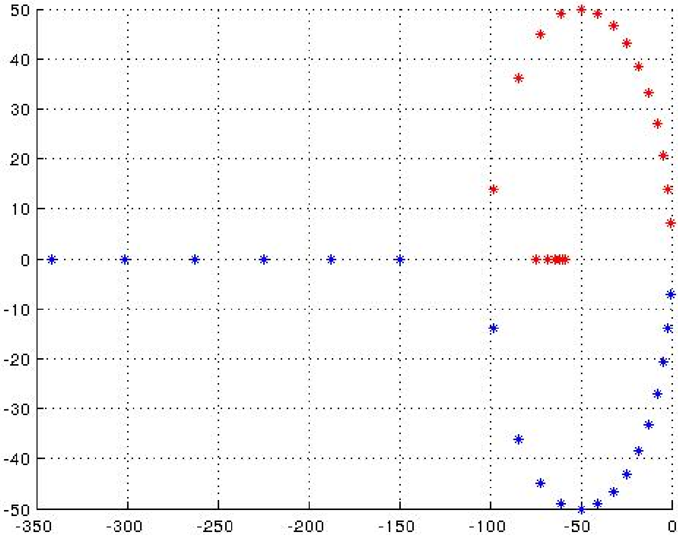
- and an infinite number of pairs of real eigenvalues :

$$\lim_{n\to\infty} \lambda_n = -\frac{a\gamma\rho_s}{\nu} := -\omega_0, \qquad \mu_n \to -\infty \text{ as } n \to \infty.$$

Eigenfunctions corresponding to $\lambda_n$ and $\mu_n$ :

$$\boldsymbol{\xi_n}(x) = \begin{pmatrix} cos(nx) \\ \frac{\lambda_n}{\rho_s n} sin(nx) \end{pmatrix}, \quad \boldsymbol{\zeta_n}(x) = \begin{pmatrix} cos(nx) \\ \frac{\mu_n}{\rho_s n} sin(nx) \end{pmatrix}.$$

# $a = \rho_s = \nu = 1, \gamma = 50, \nu_0/2 = .5, \omega_0 = 50.$



# Orthonormal basis

- Define a Fourier basis $\{\mathbf{\Phi}_n\}$ in $\mathbf{Z}$ :

$$\mathbf{\Phi}_0(x) = \frac{1}{\sqrt{b\pi}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ ;$$

$$\mathbf{\Phi}_{2n}(x) = \sqrt{\frac{2}{b\pi}} \begin{pmatrix} cos(nx) \\ 0 \end{pmatrix}, \quad \mathbf{\Phi}_{2n-1}(x) = \sqrt{\frac{2}{\rho_s\pi}} \begin{pmatrix} 0 \\ sin(nx) \end{pmatrix}$$

  for $n \geq 1$.

- Define the subspaces :

$$\mathbf{V}_0 = span\ \{\mathbf{\Phi_0}\}; \quad \mathbf{V}_n = span\ \{\mathbf{\Phi}_{2n}, \mathbf{\Phi}_{2n-1}\}, \quad n \geq 1$$

- $\mathbf{Z}$ is the orthogonal sums of the subspaces $\{\mathbf{V}_n\}_{n\geq 0}$.
- $\mathbf{Z_0}$, is the orthogonal sum of $\{\mathbf{V}_n\}_{n\geq 1}$ :

$$\mathbf{Z}_0 := \{\begin{pmatrix} \rho \\ u \end{pmatrix} \ \in \ \mathbf{Z} \ : \ \int_0^\pi \rho(x)dx = 0\}.$$

# Null Controllability of the Linearized system at $(\rho_s, 0)$:

Null Controllable if and only if the initial density is in $H^1$ and the control acts on the whole domain!

**Theorem**

[SC,MR,JPR]
For every $T > 0$, the system is null controllable in time $T$, using interior control $f \in L^2((0,\infty), L^2(\Omega))$ for velocity, if and only if
$\mathbf{U_0} = \begin{pmatrix} \rho_0 \\ u_0 \end{pmatrix} \in H^1_m(\Omega) \times L^2(\Omega)$,

$$H^1_m(\Omega) = \{\rho \in H^1(\Omega) \ : \ \int_0^\pi \rho(x)dx = 0\}$$

.

Hence the system is exponentially stabilizable using a control for velocity acting on the whole domain.

# Stabilization of the linearized system at $(\rho_s, 0)$

The linearized system is

- stable with decay rate $e^{-\omega t}$ for $0 < \omega < \min \{ \nu_0/2, \ \omega_0 \}$
  $\omega_0$, the accumulation point for the real eigenvalues of $A$.
- not stabilizable with decay rate $e^{-\omega t}$ for $\omega > \omega_0$ [2]

**Qn :** Is the linearized system stabilizable when $\nu_0/2 \ < \ \omega_0$ ?

**Difficulty :** Some eigenvalues will become unstable.

# Stabilization of the linearized system at $(\rho_s, 0)$

A system of controlled PDE in $Z$

$$z'(t) = Az(t) + Bu(t),\ t > 0, \quad z(0) = z_0 \ \in \ Z$$

**stabilizable by feedback** when there exists an operator $K \in \mathcal{L}(Z, U)$ such that $A + BK$ is exponentially stable in $Z$.

**Main Result :** Linearized system is stabilizable by a feedback control even when $\nu_0/2 < \omega_0$, with decay rate $e^{-\omega t}$ for $0 < \omega < \omega_0$.

**Idea of the proof :**

- Define $A_\omega \ = \ A + \omega I$ for $0 < \omega < \omega_0$.
- Finitely many eigenvalues of $A_\omega$ are unstable in this case.
- Project the system onto the unstable subspace, $Z_u$ and stable subspace $Z_s$.

# Construction of Feedback Control

- Use Hautus test to show $(A_\omega, B)$ is stabilizable in $Z_0$.
- Finite dimensional projected system $(\Pi_u A_\omega, \Pi_u B)$ is also stabilizable.
- Hence there exists a feedback $K_u$ such that $(\Pi_u A_\omega + \Pi_u B K_u)$ is stable.
- $K_u$ can be obtained by solving a finite dimensional Riccati equation.
- Define $K_m = K_u \Pi_u$.
- For all $z_0 \in Z_0$, solution of

  $$z'(t) = Az(t) + BK_m z(t),\ t > 0, \quad z(0) = z_0$$

  decays exponentially :

  $$\|z(t)\| \leq Ce^{-\gamma_1 t}\|z_0\|.$$

# Stabilization for nonlinear system

**Qn :** Is the nonlinear system stabilizable near constant steady states? At what rate of decay?

Usual Strategy

- Compute the feedback control for the linearized system
- Put it into the nonlinear system, treating the nonlinear terms as source term on the right hand side for an iterative process
- In each iteration get the solution of a linear system
- Show the convergence of the iterates in a suitable small neighbourhood of the steady state solution

# Initial Reductions

Let $(\rho_s, u_s)$ be a constant steady state.

Rewrite the equation for the perturbation $(\sigma, v)$ ;

$$\sigma = \rho - \rho_s \quad ; \quad v = u - u_s$$

around this steady state

To get exponential decay $e^{-\omega t}$, rewrite the equation for

$$\hat{\sigma} = \sigma e^{\omega t}, \quad \hat{v} = v e^{\omega t},$$

Then it is enough to show that bounded solution $(\hat{\sigma}, \hat{v})$ exists for all $t$ for the last nonlinear system.

This is usually done by some iteration process.

# Stabilization of nonlinear system

Main Difficulty

What is a good space to set up the iteration process?

Equation for density $\rho$ is a transport equation

One of the nonlinear terms $\rho_x v$ ;

there is no gain in regularity for $\rho$ ;

The derivative is in a less regular space

This is an obstacle to set up an iteration process

# Change of coordinates

Use the transformation of coordinates :

$$y \to X_{\widehat{v}}(y,t)$$

for each $t > 0$ and for $\widehat{v}$ in a suitable space,

$$\frac{\partial Y_{\widehat{v}}}{\partial t}(x,t) = e^{-\omega t}\widehat{v}(Y_{\widehat{v}}(x,t),t), \qquad Y_{\widehat{v}}(x,0) = x, \qquad \text{for} \quad t > 0.$$

Then the transformed system does not have the difficult nonlinear term !

New Difficulties :

- The control domain is transformed to a time dependent domain
- The transformed density variable is no more of zero average.

# Outline of the Strategy

For initial velocity sufficiently small, find a fixed set $O$ lying in every transformed control interval for each $t > 0$.

Split the transformed density into two parts :

- one part with average zero;
- the other part, depending only on time, lying in a suitable weighted Lebesgue space.
- $\sigma(x,t) = \sigma_m(x,t) + \sigma_\Omega(t)$, with $\sigma_\Omega(t) = \dfrac{1}{\pi}\displaystyle\int_\Omega \sigma(x,t)\,dx.$

# Change of coordinates

Denote

$$\Omega_x := (0,\pi) \quad \text{and} \quad Q_x^\infty := \Omega_x \times (0,\infty),$$

Under some conditions on $\widehat{v}$, for each $t > 0$,
$Y_{\widehat{v}}(\cdot,t)$ maps $\Omega_x$ onto $\Omega_y$ smoothly.

For $t > 0$, $X_{\widehat{v}}(\cdot,t)$ is the inverse mapping of $Y_{\widehat{v}}(\cdot,t)$. Set

$$\widetilde{\sigma}(x,t) = \widehat{\sigma}(Y_{\widehat{v}}(x,t),t), \quad \widetilde{v}(x,t) = \widehat{v}(Y_{\widehat{v}}(x,t),t), \quad \widetilde{g}(x,t) = \widehat{g}(Y_{\widehat{v}}(x,t),t),$$

The control domain $(\ell_1,\ell_2)$ is also transformed :

$$\widetilde{\ell}_{1,\widetilde{v}}(t) = X_{\widehat{v}}(\ell_1,t) \quad \text{and} \quad \widetilde{\ell}_{2,\widetilde{v}}(t) = X_{\widehat{v}}(\ell_2,t),$$

# Transformed system

$(\widetilde{\sigma}, \widetilde{v}, \widetilde{g})$, together with $(X, Y) = (X_{\widehat{v}}, Y_{\widehat{v}})$, satisfy the following new nonlinear system

$$
\begin{aligned}
&\widetilde{\sigma}_t + \rho_s \widetilde{v}_x - \omega \widetilde{\sigma} \; = \; \mathcal{F}_1(\widetilde{\sigma}, \widetilde{v}, t), \quad \text{in } Q_x^\infty, \\
&\widetilde{v}_t \; + b\widetilde{\sigma}_x - \nu_0 \widetilde{v}_{xx} - \omega \widetilde{v} \; = \; \mathcal{F}_2(\widetilde{\sigma}, \widetilde{v}, t) + \chi_O \, \widetilde{g}, \quad \text{in } Q_x^\infty, \\
&\widetilde{\sigma}(0) = \sigma_0, \;\; \widetilde{v}(0) = v_0 \qquad \text{in } \Omega_x, \quad \int_{\Omega_x} \sigma_0(x) dx = 0, \\
&\widetilde{v}(0, t) = 0, \quad \widetilde{v}(\pi, t) = 0, \qquad \forall t > 0, \\
&Y(x, t) = x + \int_0^t e^{-\omega s} \widetilde{v}(x, s) \, ds, \quad t > 0, \; x \in \Omega_x, \\
&X(Y(x, t), t) = x, \; x \in \Omega_x, \qquad Y(X(y, t), t) = y, \; y \in \Omega_y, \; t > 0, \\
&\widetilde{\ell}_{j, \widetilde{v}}(t) = X(\ell_j, t), \; \text{for } j = 1, 2.
\end{aligned}
$$

# Stabilization of Transformed System

Let $\omega \in (0, \omega_0)$. There exists a bounded linear operator $K$ from $L^2(\Omega_x) \times L^2(\Omega_x)$ into $L^2(\Omega_x)$ of the form

$$K(\sigma, v)(x) = \int_0^\pi k_\sigma(x, \xi)\, \sigma(\xi)\, d\xi + \int_0^\pi k_v(x, \xi)\, v(\xi)\, d\xi,$$

with $k_\sigma \in L^2(\Omega_x \times \Omega_x)$ and $k_v \in L^2(\Omega_x \times \Omega_x)$, and there exist constants $\mu_0 > 0$ and $\widetilde{C}_1 > 0$, depending on $K$, such that, for all $0 < \overline{\mu} < \mu_0$ and all initial conditions $(\sigma_0, v_0)$ satisfying

$$\|(\sigma_0, v_0)\|_{H^1_m(\Omega_x) \times H^1_0(\Omega_x)} \leq \widetilde{C}_1\, \bar{\mu},$$

the closed loop nonlinear system after setting

$$\widetilde{g}(t) = K(\widetilde{\sigma}(t), \widetilde{v}(t))$$

admits a unique solution $(\widetilde{\sigma}, \widetilde{v}, X, Y)$ in the ball $D_\mu$.

# Back to Original system

Find a feedback control for the original system by making a reverse change of variables :

$$\widehat{\sigma}(\zeta,t) = \widetilde{\sigma}(X(\zeta,t),t)$$

$$\widehat{v}(\zeta,t) = \widetilde{v}(X(\zeta,t),t),$$

for all $\zeta \in \Omega_y, \ \forall t \in (0,\infty)$.

Then feedback control is transformed in the form

$$\widehat{K}(\widehat{\sigma}(t),\widehat{v}(t),X_{\widehat{v}}(t))(y) = K\left(\widetilde{\sigma}(\cdot,t),\widetilde{v}(\cdot,t)\right)\circ X(y,t), \quad \forall\,(y,t) \in Q_y^{\infty}.$$

# Closed Loop Nonlinear system

$$\rho_t + (\rho v)_y \;=\; 0, \quad \text{in } (0,\pi)\times(0,\infty),$$

$$\rho(v_t \;+\; v v_y) \;+\; (p(\rho))_y \;-\nu v_{yy} \;=$$

$$\rho\,\chi_{(\ell_1,\ell_2)}\,\widehat{K}\big(e^{\omega t}(\rho(t)-\bar{\rho}), e^{\omega t}v(t), Y(t), X(t)\big) \quad \text{in } (0,\pi)\times(0,\infty),$$

$$\rho(0) = \rho_0, \;\; v(0) = v_0, \quad \text{in } (0,\pi),$$

$$v(0,t) = 0, \quad v(\pi,t) = 0, \qquad \forall t > 0,$$

$$Y(x,t) = x + \int_0^t v(Y(x,s),s)\,ds, \quad t > 0,\; x \in \Omega_x,$$

$$X(Y(x,t),t) = x,\; x \in \Omega_x,$$

$$Y(X(y,t),t) = y,\; y \in \Omega_y,\; t > 0.$$

# Stabilization Theorem

Let $\omega$ belong to $(0, \omega_0)$. There exist

(i) a continuous nonlinear mapping $\widehat{K}$ of the variables $(\rho, v, X, Y)$ from $H^1_m(\Omega_y) \times H^1_0(\Omega_y) \times H^1(\Omega_x) \times H^1(\Omega_y)$ into $L^2(\Omega_x)$ and

(ii) positive constants $\widehat{\mu}_0, \widehat{C}_1$

such that, for all $0 < \widehat{\mu} < \widehat{\mu}_0$, for all initial condition $(\rho_0, v_0) \in H^1(\Omega_y) \times H^1_0(\Omega_y)$ satisfying

$$\|(\rho_0 - \rho_s, v_0)\|_{H^1_m(\Omega_y)\times H^1_0(\Omega_y)} \leq \widehat{C}_1 \widehat{\mu},$$

the nonlinear closed loop system admits a unique solution $(\rho, v, X, Y)$ satisfying, for all $(y, t) \in Q_y^\infty$,

$$\|(\rho(\cdot, t) - \rho_s, v(\cdot, t))\|_{H^1_m(\Omega_y)\times H^1_0(\Omega_y)} \leq C\,\widehat{\mu}\,e^{-\omega t}, \quad \rho(y, t) \geq \frac{\rho_s}{2}.$$

# Linearized system at $(\rho_s, u_s)$

For the linearized system around $(\rho_s, u_s)$ with periodic boundary conditions for $\rho, u$ and $u_x$ in $(0, 2\pi)$

## The Point Spectrum of $A$

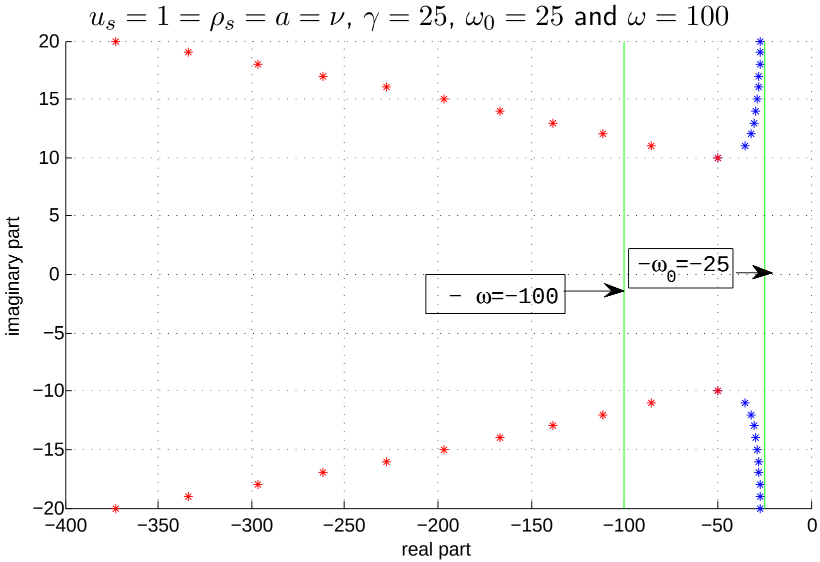
- Consists of eigenvalues $\{-\lambda_n\}$, in the left side of the complex plane
- One sequence is

$$\lambda_n^h = \omega_0 - \varepsilon_n^h - i\ n\ u_s$$

  with $\varepsilon_n^h \to 0$, as $|n| \to \infty$, for $n \in \mathbb{Z}$ ;
- The other sequence is

$$\lambda_n^p = \nu_0 n^2 - \omega_0 + \varepsilon_n^p - i\ n\ u_s$$

  with $\varepsilon_n^p \to 0$, as $|n| \to \infty$, for $n \in \mathbb{Z}$ ;
- No accumulation point in the spectrum
- Absolute value of the eigenvalues goes to infinity.


$u_s = 1 = \rho_s = a = \nu$, $\gamma = 25$, $\omega_0 = 25$ and $\omega = 100$
imaginary part
real part
− ω=−100
−ω0=−25
20
15
10
5
0
−5
−10
−15
−20
−400
−350
−300
−250
−200
−150
−100
−50
0

# Null Controllability

Can work with Fourier basis and Moment method to conclude null controllability in for regular initial conditions. ( Chowdhury - Mitra)

**Theorem**

[SC,DM,MR,MRE]
For any $T > \frac{2\pi}{u_s}$ and any initial condition $(\rho_0, u_0) \in \dot{H}^1_{per}(I_{2\pi}) \times L^2(I_{2\pi})$, the system with periodic boundary condition is null controllable at time $T$ by a localized interior control $f(\cdot) \in L^2(0, T; L^2(\mathcal{O}))$ acting only on the velocity equation, where $\mathcal{O}$ is any nonempty open subset of $I_{2\pi}$.

Hence the system is exponentially stabilizable using localized interior control for velocity.

# Stabilization of linearized system

**Qn :** The linearized system at $(\rho_s, u_s)$ is stabilizable at what rate of decay?

- Stabilizable in $\dot{H}^1_{per} \times L^2$ with any rate of decay
- this is the optimal space for stabilization with arbitrary decay
- Spectrum decouples into 2 distinct parts "hyperbolic part" and "parabolic part".

# Stabilization of linearized system

- Difficulty : Hyperbolic part contains infinitely many eigenvalues for $\omega > \omega_0$
- Use projection onto unstable eigensubspaces to compute feedback stabilization
- Take the infinite sum of these orthogonal components and show the convergence
- Orthogonal components of feedback control for hyperbolic eigenvalues are summable if density lies in $H^1$

# Coordinate transformation

For any smooth function $\widehat{v}$, $L$-periodic in the space variable and bounded in $L^2(0,\infty;H^2_{per}(\Omega_y))$, the $L$-periodic mapping $Y_{\widehat{v}}(\cdot,t)$ from $\Omega_x$ to $\Omega_y$ satisfies

$$\begin{aligned}
&\frac{\partial Y_{\widehat{v}}(x,t)}{\partial t} + u_s \frac{\partial Y_{\widehat{v}}(x,t)}{\partial x} = u_s + e^{-\omega t}\widehat{v}(Y_{\widehat{v}}(x,t),t), \quad \forall\,(x,t)\in Q_x^\infty,\\
&Y_{\widehat{v}}(x,0) = I(x), \quad \forall\, x\in\Omega_x,\\
&Y_{\widehat{v}}(x,\cdot) = Y_{\widehat{v}}(x+L,\cdot), \quad \forall\, x\in\Omega_x,
\end{aligned}$$

where $I(x)$ is the identity mapping in $\mathbb{R}/(L\mathbb{Z})$.

For every $t>0$, the mapping $x\to Y_{\widehat{v}}(x,t)$ is a smooth bijection from $\Omega_x$ to $\Omega_y$.

Denote by $X_{\widehat{v}}(\cdot,t)$ the $L$-periodic inverse of $Y_{\widehat{v}}(\cdot,t)$.

# Stabilization Theorem

Let $\omega$ be any positive number. There exist positive constants $\widehat{\mu}_0$ and $\widehat{\kappa}$, depending on $\omega$, $\rho_s$, $u_s$, $\ell_1$, $\ell_2$ and $L$, such that, for $0 < \widehat{\mu} \leq \widehat{\mu}_0$ and any initial condition $(\rho_0, u_0) \in H^1_{per}(\Omega_y) \times H^1_{per}(\Omega_y)$, where $(\rho_0, u_0)$ obeys

$$\|(\rho_0 - \rho_s, u_0 - u_s)\|_{\dot{H}^1_{per}(\Omega_y) \times H^1_{per}(\Omega_y)} \leq \widehat{\kappa}\, \widehat{\mu},$$

there exists a control $f \in L^2(0, \infty; L^2(\Omega_y))$ for which the nonlinear system admits a unique solution $(\rho, u)$ satisfying

$$\|(\rho(\cdot, t) - \rho_s, u(\cdot, t) - u_s)\|_{\dot{H}^1_{per}(\Omega_y) \times H^1_{per}(\Omega_y)} \leq C e^{-\omega t},$$

for some positive constant $C$ depending on $\widehat{\mu}$. Moreover

$$\rho(y, t) \geq \frac{\rho_s}{2}$$

for all $(y, t) \in Q_y^\infty$.

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