Feynman path integrals as analysis on path space

by time slicing approximation

* Introduction and Known results *

Naoto Kumano-go (Kogakuin University, Japan)

Day 1. Introduction and Sketch of known results.

Day 2. Our results and Process of proofs.

Day 3. Proofs and Calculation techniques.

Day 4. Phase space Feynman path integrals.

The 1st lecture is the introduction.

Feynman path integral does not have the unique definition.

Furthermore, the known results are famous but are not my results.

It is difficult for me to explain their philosophies.

Therefore I hope to sketch many short topics.

If you get one image of Feynman path integral somewhere, I am happy.

In the 2nd lecture, we will state the main results of our theory.

In the 3rd lecture, we will explain the calculation techniques in our proof.

In the 4th lecture, we will apply our approach to the phase space path integral.

1. Introduction to Feynman path integral beyond mathematics

Let $T > 0$ and $x \in \mathrm{R}^d$. We consider the initial value problem of the Schrödinger equation

$$\left(i\hbar\partial_T + \frac{\hbar^2}{2}\triangle - V(x)\right)u(T,x) = 0\,, \quad u(0,x) = v(x)$$

with the Planck parameter $0 < \hbar < 1$. We denote the kernel function of the fundamental solution by $K(T,0,x,x_0)$, i.e.,

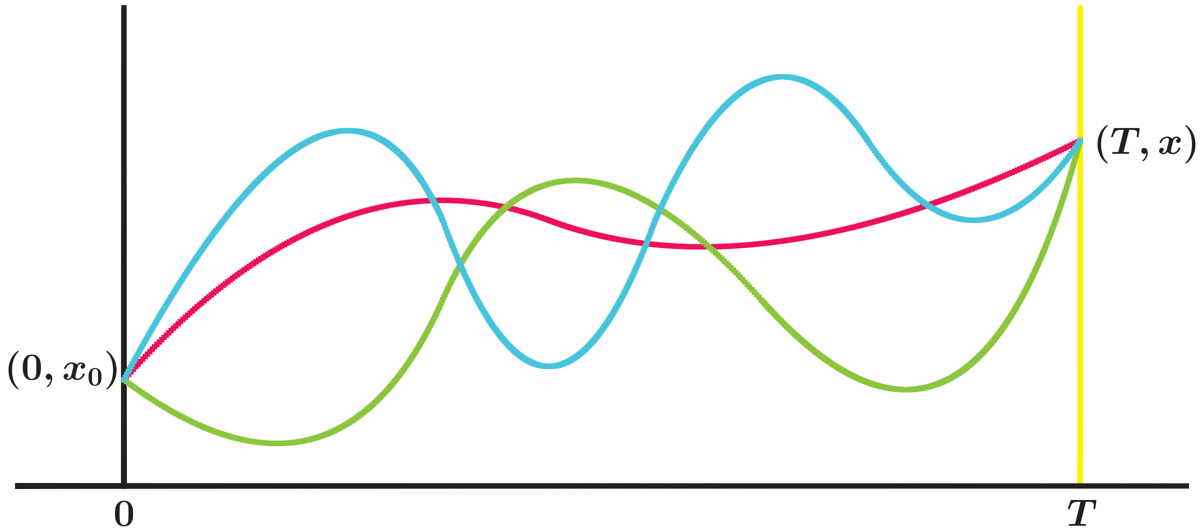
$$u(T,x) = \int_{\mathrm{R}^d} K(T,0,x,x_0)v(x_0)dx_0\,.$$

R. P. Feynman, Rev. Mod. Phys. 20 (1948), 367-387.

In 1948, R. P. Feynman expressed the kernel function $K(T, 0, x, x_0)$ using a path integral as follows:

$$K(T, 0, x, x_0) = \int e^{\frac{i}{\hbar}S[\gamma]}\mathcal{D}[\gamma]\,.$$

Here $\gamma : [0, T] \to \mathrm{R}^d$ is a path with $\gamma(0) = x_0$ and $\gamma(T) = x$, $S[\gamma] = \int_0^T \frac{1}{2}\left|\frac{d\gamma}{dt}\right|^2 - V(\gamma)dt$ is the action, the path integral $\int \sim \mathcal{D}[\gamma]$ is a new sum over all the paths γ. Feynman defined his new integral as follows.



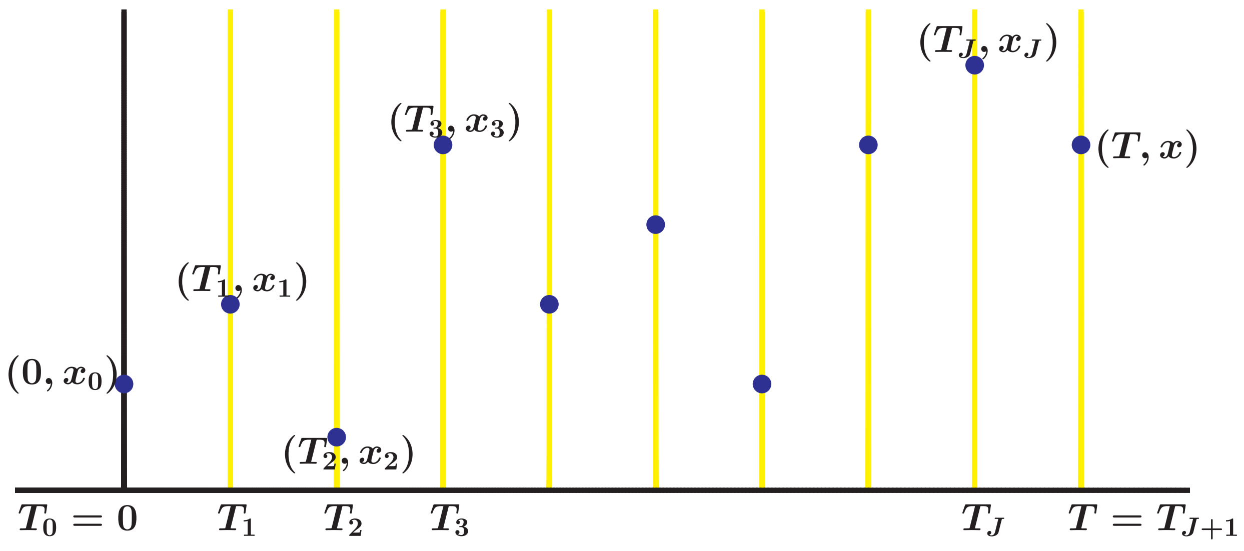
Let $\Delta_{T,0} : T = T_{J+1} > T_J > \cdots > T_1 > T_0 = 0$ be any division of $[0, T]$.

Note that

$$
\begin{aligned}
u(T,x) &= \int_{\mathrm{R}^d} K(T,0,x,x_0)v(x_0)dx_0 = \int_{\mathrm{R}^d} K(T,T_1,x,x_1)u(T_1,x_1)dx_1 \\
&= \int_{\mathrm{R}^d}\int_{\mathrm{R}^d} K(T,T_1,x,x_1)K(T_1,0,x_1,x_0)v(x_0)dx_0dx_1\,.
\end{aligned}
$$

Therefore, we can write

$$
\begin{aligned}
\int e^{\frac{i}{\hbar}S[\gamma]}\mathcal{D}[\gamma] = K(T,0,x,x_0) &= \int_{\mathrm{R}^d} K(T,T_1,x,x_1)K(T_1,0,x_1,x_0)dx_1 \\
&= \int_{\mathrm{R}^{dJ}} \prod_{j=1}^{J+1} K(T_j,T_{j-1},x_j,x_{j-1}) \prod_{j=1}^{J} dx_j\,.
\end{aligned}
$$



Classical path

Feynman considered $K(T_j, T_{j-1}, x_j, x_{j-1})$ when $|T_j - T_{j-1}|$ is small.

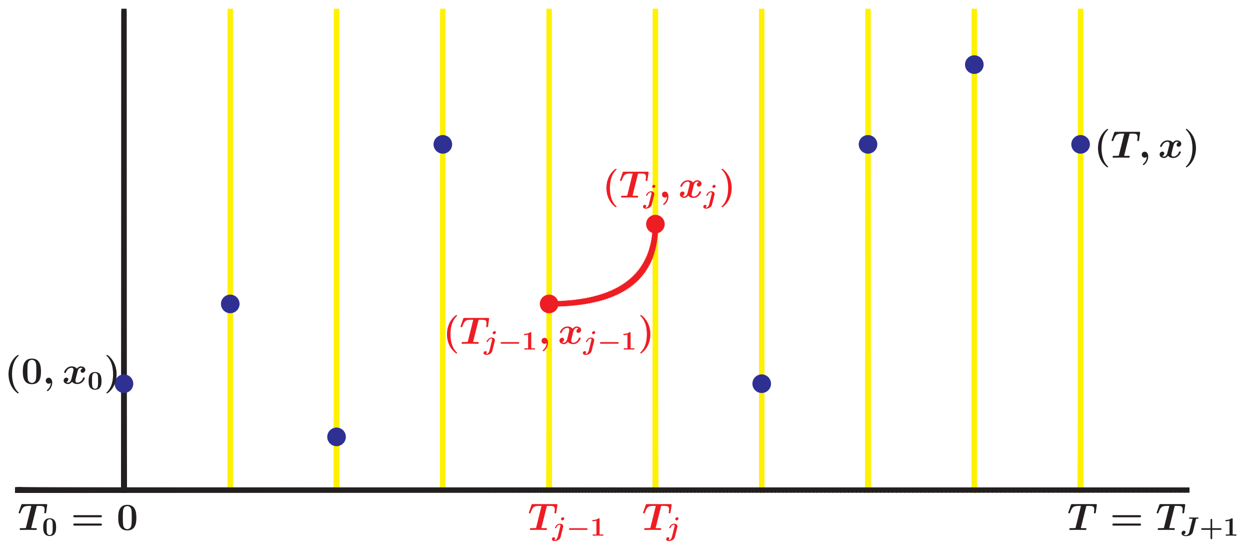
Let $\gamma_{T_j,T_{j-1}}$ be the classical path defined by the Euler equation

$$\ddot{\gamma}_{T_j,T_{j-1}}(t) + (\partial_x V)(\gamma_{T_j,T_{j-1}}) = 0\,, \quad T_{j-1} \leq t \leq T_j$$

with $\gamma_{T_j,T_{j-1}}(T_{j-1}) = x_{j-1}$ and $\gamma_{T_j,T_{j-1}}(T_j) = x_j$.

Let $S_{T_j,T_{j-1}}(x_j, x_{j-1})$ be the action along the classical path $\gamma_{T_j,T_{j-1}}$ by

$$S_{T_j,T_{j-1}}(x_j, x_{j-1}) = \int_{T_{j-1}}^{T_j} \frac{1}{2}|\dot{\gamma}_{T_j,T_{j-1}}|^2 - V(\gamma_{T_j,T_{j-1}})dt\,.$$



Time slicing approximation via piecewise classical paths

P. A. M. Dirac, Physikalische Zeitschrift der Sowietunion 3 (1933) 64-72.

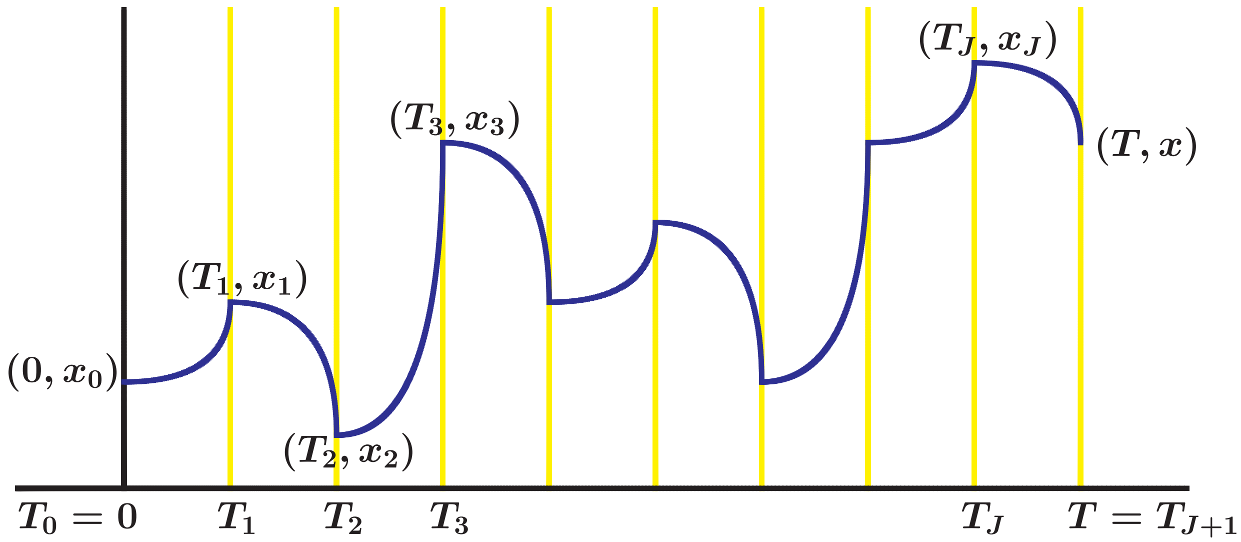
Set $t_j = T_j - T_{j-1}$ and $|\Delta_{T,0}| = \max_{1 \le j \le J+1} t_j$. Using Dirac's approximation

$$K(T_j, T_{j-1}, x_j, x_{j-1}) \approx \left(\frac{1}{2\pi i\hbar t_j} \right)^{d/2} e^{\frac{i}{\hbar} S_{T_j, T_{j-1}}(x_j, x_{j-1})}$$

as $|\Delta_{T,0}| \to 0$, Feynman defined the path integral by

$$\int e^{\frac{i}{\hbar} S[\gamma]} \mathcal{D}[\gamma] \equiv \lim_{|\Delta_{T,0}| \to 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i\hbar t_j} \right)^{d/2} \int_{\mathrm{R}^{dJ}} e^{\frac{i}{\hbar} \sum_{j=1}^{J+1} S_{T_j, T_{j-1}}(x_j, x_{j-1})} \prod_{j=1}^{J} dx_j \, .$$

This method is now called the time slicing approximation.



2. Re-introduction to path integral beyond mathematics

Let $U(T,0)$ be the fundamental solution for the Schrödinger equation

$$\left(i\hbar\partial_T + \frac{\hbar^2}{2}\triangle - V(x)\right)U(T,0) = 0\,, \quad U(0,0) = I\,.$$

By the Fourier transform and inverse Fourier transform, we can write

$$v(x) = \left(\frac{1}{2\pi\hbar}\right)^d \int_{\mathrm{R}^{2d}} e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0} v(x_0)\,dx_0\,d\xi_0\,,$$

$$\left(-\frac{\hbar^2}{2}\triangle + V(x)\right)v(x) = \left(\frac{1}{2\pi\hbar}\right)^d \int_{\mathrm{R}^{2d}} e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0}\left(\frac{\xi_0^2}{2} + V(x)\right)v(x_0)\,dx_0\,d\xi_0\,.$$

As an approximation of $U(T,0)v(x)$, one may use the operator

$$\begin{aligned}
I(T,0)v(x) &= \left(\frac{1}{2\pi\hbar}\right)^d \int_{\mathrm{R}^{2d}} e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0} e^{-\frac{i}{\hbar}T(\frac{\xi_0^2}{2}+V(x))} v(x_0)\,dx_0\,d\xi_0 \\
&= \left(\frac{1}{2\pi\hbar}\right)^d \int_{\mathrm{R}^d}\int_{\mathrm{R}^d} e^{-\frac{i}{\hbar}\frac{T}{2}(\xi_0-\frac{(x-x_0)}{T})^2+\frac{i}{\hbar}(\frac{(x-x_0)^2}{2T}-V(x)T)} d\xi_0\, v(x_0)\,dx_0 \\
&= \left(\frac{1}{2\pi i\hbar T}\right)^{d/2} \int_{\mathrm{R}^d} e^{\frac{i}{\hbar}(\frac{(x-x_0)^2}{2T}-V(x)T)} v(x_0)\,dx_0\,.
\end{aligned}$$

Here we used the following formula.

Formula For any real number a,

$$\int_{-\infty}^{\infty} e^{-iax^2} dx = \sqrt{\frac{\pi}{ia}}\,.$$

Let $\Delta_{T,0} : T = T_{J+1} > T_J > \cdots > T_1 > T_0 = 0$ be any division of $[0, T]$.

Set $t_j = T_j - T_{j-1}$ and $|\Delta_{T,0}| = \max_{1 \le j \le J+1} t_j$. Using the approximation

$$I(T_j, T_{j-1})v(x_j) = \left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2} \int_{\mathrm{R}^d} e^{\frac{i}{\hbar}\left(\frac{(x_j - x_{j-1})^2}{2t_j} - V(x_j)t_j\right)} v(x_{j-1}) dx_{j-1}\,,$$

we can probably write

$$U(T, 0)v(x) = \lim_{|\Delta_{T,0}| \to 0} I(T, T_J)I(T_J, T_{J-1}) \cdots I(T_2, T_1)I(T_1, 0)v(x)\,.$$

Set $x_{J+1} = x$. Then we can formally rewrite

$$\int_{\mathrm{R}^d} K(T, 0, x, x_0)v(x_0)dx_0 = U(T, 0)v(x)$$

$$= \lim_{|\Delta_{T,0}| \to 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2} \int_{\mathrm{R}^{d(J+1)}} e^{\frac{i}{\hbar}\sum_{j=1}^{J+1}\left(\frac{(x_j - x_{j-1})^2}{2t_j} - V(x_j)t_j\right)} v(x_0) \prod_{j=0}^{J} dx_j\,.$$

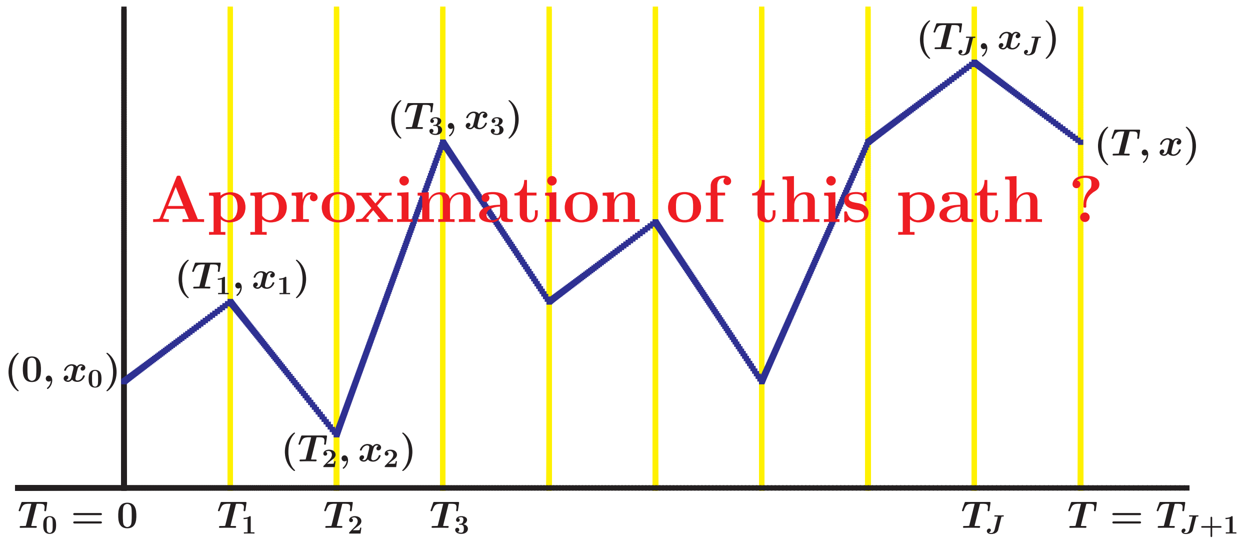
Hence we can probably interpret

$$\int e^{\frac{i}{\hbar}S[\gamma]}\mathcal{D}[\gamma] \equiv \lim_{|\Delta_{T,0}|\to 0} \prod_{j=1}^{J+1}\left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2} \int_{\mathrm{R}^{dJ}} e^{\frac{i}{\hbar}\sum_{j=1}^{J+1}(\frac{(x_j-x_{j-1})^2}{2t_j}-V(x_j)t_j)} \prod_{j=1}^{J} dx_j \,.$$

Here we used the approximation

$$S[\gamma] = \int_0^T \frac{1}{2}\left|\frac{d\gamma}{dt}\right|^2 - V(\gamma)dt \approx \sum_{j=1}^{J+1}(\frac{(x_j - x_{j-1})^2}{2t_j} - V(x_j)t_j) \,.$$

This method is also called the time slicing approximation.



3. Oscillatory integral

We used the following formula. This integral does not converge absolutely.

In our approach, we treat integrals of this type as an oscillatory integral.

Formula For any real number a,

$$\int_{-\infty}^{\infty} e^{-iax^2} dx = \sqrt{\frac{\pi}{ia}}\,.$$

Interpretation of Formula

$$\int_{-\infty}^{\infty} e^{-iax^2} dx \equiv \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} e^{-\epsilon^2 x^2} e^{-iax^2} dx = \lim_{\epsilon \to 0} \sqrt{\frac{\pi}{\epsilon^2 + ia}} = \sqrt{\frac{\pi}{ia}}\,. \quad \square$$

Definition (Oscillatory integral) Let $\chi(x)$ be a rapidly decreasing function on R^d with $\chi(0) = 1$, for example, $\chi(x) = e^{-x^2}$. The oscillatory integral $\int_{\mathrm{R}^d} a(x)dx$ is defined by

$$\int_{\mathrm{R}^d} a(x)dx \equiv \lim_{\epsilon \to 0} \int_{\mathrm{R}^d} \chi(\epsilon x) a(x) dx\,,$$

if the right-hand side exists independently of the choice of $\chi(x)$.

4. Example $d = 1$, $V(x) = 0$ via piecewise classical paths

By the Euler equation

$$\ddot{\gamma}_{T_j,T_{j-1}}(t) = 0\,,$$

with $\gamma_{T_j,T_{j-1}}(T_j) = x_j$ and $\gamma_{T_j,T_{j-1}}(T_{j-1}) = x_{j-1}$, we have the classical path

$$\gamma_{T_j,T_{j-1}}(t) = \frac{(t - T_{j-1})x_j + (T_j - t)x_{j-1}}{T_j - T_{j-1}}\,,$$

and the phase function

$$S_{T_j,T_{j-1}}(x_j, x_{j-1}) = \frac{1}{2}\int_{T_{j-1}}^{T_j} |\dot{\gamma}_{T_j,T_{j-1}}|^2 dt = \frac{(x_j - x_{j-1})^2}{2t_j}\,.$$

Set $x_1^\circ \equiv \dfrac{(T_2 - T_1)x_2 + (T_1 - T_0)x_0}{T_2 - T_0}$. Then we can write

$$S_{T_2,T_1}(x_2, x_1) + S_{T_1,0}(x_1, x_0) = \frac{(x_2 - x_1)^2}{2t_2} + \frac{(x_1 - x_0)^2}{2t_1}$$
$$= \frac{(x_2 - x_0)^2}{2T_2} + \frac{T_2}{2t_2t_1}(x_1 - x_1^\circ)^2 = S_{T_2,0}(x_2, x_0) + \frac{T_2}{2t_2t_1}(x_1 - x_1^\circ)^2\,.$$

Performing the oscillatory integration with respect to x_1, we have

$$\begin{aligned}
&\left(\frac{1}{2\pi i\hbar t_2}\right)^{1/2}\left(\frac{1}{2\pi i\hbar t_1}\right)^{1/2}\int_{\mathrm{R}} e^{\frac{i}{\hbar}S_{T_2,T_1}(x_2,x_1)+\frac{i}{\hbar}S_{T_1,0}(x_1,x_0)}dx_1 \\
&=\left(\frac{1}{2\pi i\hbar t_2}\right)^{1/2}\left(\frac{1}{2\pi i\hbar t_1}\right)^{1/2} e^{\frac{i}{\hbar}S_{T_2,0}(x_2,x_0)}\int_{\mathrm{R}} e^{\frac{i}{\hbar}\frac{T_2}{2t_2t_1}(x_1-x_1^{\circ})^2}dx_1 \\
&=\left(\frac{1}{2\pi i\hbar T_2}\right)^{1/2} e^{\frac{i}{\hbar}S_{T_2,0}(x_2,x_0)}.
\end{aligned}$$

Using this relation inductively, we obtain the kernel function $K(T,0,x,x_0)$ of the fundamental solution.

$$\begin{aligned}
K(T,0,x,x_0) &= \lim_{|\Delta_{T,0}|\to 0}\prod_{j=1}^{J+1}\left(\frac{1}{2\pi i\hbar t_j}\right)^{1/2}\int_{\mathrm{R}^J} e^{\frac{i}{\hbar}\sum_{j=1}^{J+1}S_{T_j,T_{j-1}}(x_j,x_{j-1})}\prod_{j=1}^{J}dx_j \\
&= \lim_{|\Delta_{T,0}|\to 0}\left(\frac{1}{2\pi i\hbar T}\right)^{1/2} e^{\frac{i}{\hbar}S_{T,0}(x,x_0)} \\
&= \left(\frac{1}{2\pi i\hbar T}\right)^{1/2}\exp\frac{i}{\hbar}\left(\frac{(x-x_0)^2}{2T}\right). \quad \square
\end{aligned}$$

5. Example $d = 1$, $V(x) = |x|^2/2$ via piecewise classical paths

Let $|T_j - T_{j-1}| < \pi$. By the Euler equation

$$\ddot{\gamma}_{T_j,T_{j-1}}(t) + \gamma_{T_j,T_{j-1}}(t) = 0$$

with $\gamma_{T_j,T_{j-1}}(T_j) = x_j$ and $\gamma_{T_j,T_{j-1}}(T_{j-1}) = x_{j-1}$, we have the classical path

$$\gamma_{T_j,T_{j-1}}(t) = \frac{x_j \sin(t - T_{j-1}) + x_{j-1} \sin(T_j - t)}{\sin(T_j - T_{j-1})},$$

and the phase function

$$\begin{aligned} S_{T_j,T_{j-1}}(x_j, x_{j-1}) &= \frac{1}{2} \int_{T_{j-1}}^{T_j} |\dot{\gamma}_{T_j,T_{j-1}}|^2 - |\gamma_{T_j,T_{j-1}}|^2 dt = \frac{1}{2} \Big[\dot{\gamma}_{T_j,T_{j-1}} \gamma_{T_j,T_{j-1}} \Big]_{T_{j-1}}^{T_j} \\ &= \frac{(x_j^2 + x_{j-1}^2) \cos(T_j - T_{j-1}) - 2x_j \cdot x_{j-1}}{2 \sin(T_j - T_{j-1})}. \end{aligned}$$

Let $x_1^\dagger = x_1^\dagger(x_2, x_0)$ be the critical point defined by

$$\partial_{x_1}(S_{T_2,T_1} + S_{T_1,0})(x_2, x_1^\dagger, x_0) = 0\,.$$

Then we can write

$$\begin{aligned} &S_{T_2,T_1}(x_2, x_1) + S_{T_1,0}(x_1, x_0) \\ &= S_{T_2,0}(x_2, x_0) + \frac{1}{2}\left(\frac{\cos(T_2 - T_1)}{\sin(T_2 - T_1)} + \frac{\cos(T_1 - T_0)}{\sin(T_1 - T_0)}\right)(x_1 - x_1^\dagger)^2 \\ &= S_{T_2,0}(x_2, x_0) + \frac{1}{2}\frac{\sin(T_2 - T_0)}{\sin(T_2 - T_1)\sin(T_1 - T_0)}(x_1 - x_1^\dagger)^2\,. \end{aligned}$$

Performing the oscillatory integration with respect to x_1, we have

$$\begin{aligned} &\left(\frac{1}{2\pi i\hbar t_2}\right)^{1/2}\left(\frac{1}{2\pi i\hbar t_1}\right)^{1/2}\int_{\mathrm{R}} e^{\frac{i}{\hbar}S_{T_2,T_1,0}(x_2,x_1)+\frac{i}{\hbar}S_{T_1,0}(x_1,x_0)}dx_1 \\ &= \left(\frac{1}{2\pi i\hbar T_2}\right)^{1/2} e^{\frac{i}{\hbar}S_{T_2,0}(x_2,x_0)}\left(\frac{T_2}{\prod_{j=1}^2 t_j} \times \frac{\prod_{j=1}^2 \sin t_j}{\sin T_2}\right)^{1/2}\,. \end{aligned}$$

Using this relation inductively and taking $|\Delta_{T,0}| = \max_{1\le j\le J+1} t_j \to 0$, we obtain the kernel function $K(T,0,x,x_0)$ of the fundamental solution.

$$K(T,0,x,x_0) = \lim_{|\Delta_{T,0}|\to 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i\hbar t_j}\right)^{1/2} \int_{\mathrm{R}^{dJ}} e^{\frac{i}{\hbar}\sum_{j=1}^{J+1} S_{T_j,T_{j-1}}(x_j,x_{j-1})} \prod_{j=1}^{J} dx_j$$

$$= \lim_{|\Delta_{T,0}|\to 0} \left(\frac{1}{2\pi i\hbar T}\right)^{1/2} e^{\frac{i}{\hbar}S_{T,0}(x,x_0)} \left(\frac{T}{\prod_{j=1}^{J+1} t_j} \times \frac{\prod_{j=1}^{J+1}\sin t_j}{\sin T}\right)^{1/2}$$

$$= \left(\frac{1}{2\pi i\hbar \sin T}\right)^{1/2} \exp\frac{i}{\hbar}\left(\frac{(x^2+x_0^2)\cos T - 2x\cdot x_0}{2\sin T}\right). \quad \square$$

Remark This calculation is not valid when T is large ($T=\pi$). However, if we regard this as composition of many operators in $L^2(\mathrm{R}^d)$, the composite operator converges to the fundamental solution $U(T,0)$ even when T is large (cf. Fujiwara 1979).

6. Significance of Feynman path integrals

R. P. Feynman and A. Hibbs, Quantum mechanics and Path integrals, McGraw-Hill, New York (1965).

Using the general functional $F[\gamma]$ as an integrand, Feynman suggested a new analysis with the functional integration

$$\int F[\gamma] e^{\frac{i}{\hbar} S[\gamma]} \mathcal{D}[\gamma]$$

and the functional differentiation $(DF)[\gamma][\eta]$. Especially, he gave a new view of quantum mechanics from classical Lagrangian. For example, if we take $\hbar \longrightarrow 0$, we can formally see the correspondence from the quantum mechanics to the classical mechanics, i.e., the semiclassical approximation.

L. S. Schulman, Technique and Applications of Path Integration, 2nd edition, Dover (2007).

Trouble in mathematics

R. H. Cameron, J. of Math. and Phys. Sci. 39(1960), 126-140.

However, in 1960, R. H. Cameron proved that the measure $e^{\frac{i}{\hbar}S[\gamma]}\mathcal{D}[\gamma]$ of the path integral does not exist in the sense of mathematics.

Remark **Using a measure, we can prove many properties of integrals:**

- **The interchange of the order of integrations.**
- **The interchange of integrals and limits.**
- **The existence of integrals.**

There are many mathematical approaches to Feynman path integral.

Mathematical approaches via operators

- **Nelson(1964)** $\cdots$

by the time slicing approximation via Lie-Trotter formula.

- **Nelson(1964)** • **Cameron-Storvick(1983)** $\cdots$

via analytic continuation of Wiener measure.

- **Johnson-Lapidus(1999)** • **Gill-Zachary(2002)** $\cdots$ **as operational calculus.**

G. W. Johnson and M. L. Lapidus, The Feynman Integral and Feynman's Operational Calculus, Oxford University Press, New York (2000).

Remark **The above approaches can construct the solution for the general potential. However the convergences are senses of operators. It is difficult to treat the semiclassical approximation $\hbar \longrightarrow 0$.**

Mathematical approaches via oscillatory integrals

• Itô(1967) • Albeverio-Høegh Krohn(1976) • Truman(1972)

• Rezende(1985) $\cdots$ as an infinite dimensional oscillatory integral via Fresnel transform.

S. A. Albeverio, R. J. Hoegh-Krohn and S. Mazzucchi, Mathematical Theory of Feynman Path Integrals, 2nd edition, Springer (2008).

• Fujiwara(1979,1991) • Kitada-H. Kumano-go(1981) • Yajima(1991)

• N. Kumano-go(1995) • Fujiwara-Tsuchida(1997) • W. Ichinose(1997)

by time slicing approximation via oscillatory integrals.

Remark **The above approaches need the smooth condition for the potential. However the convergences are sharper. We can treat the semiclassical approximation $\hbar \to 0$.**

7. Nelson's explanation by time slicing approximation

E. Nelson, J. Math. Phys. 5 (1964), 332-343.

Theorem(Nelson 1964) Let $V(x)$ is a real-valued function. Suppose that $H = -\frac{\hbar^2}{2}\triangle + V(x)$ is an essentially self-adjoint operator on $D = D(-\frac{\hbar^2}{2}\triangle) \cap D(V)$. Then, for any $v \in L^2(\mathrm{R}^d)$, we have

$$u(T,x) = (e^{-\frac{i}{\hbar}TH}v)(x)$$
$$= \text{s-}\lim_{J\to\infty} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i\hbar T/(J+1)}\right)^{d/2} \int_{\mathrm{R}^d} \cdots \int_{\mathrm{R}^d} e^{\frac{i}{\hbar}S_{J+1}(x_{J+1},x_J,\ldots,x_1,x_0)} v(x_0) \prod_{j=0}^{J} dx_j \,.$$

where

$$S_{J+1}(x_{J+1},x_J,\ldots,x_1,x_0) = \sum_{j=1}^{J+1}\left(\frac{(x_j - x_{j-1})^2}{2T/(J+1)} - V(x_{j-1})\frac{T}{J+1}\right).$$

and the integrals are taken in the sense of limit in mean, i.e.,

$\int_{\mathrm{R}^d} \sim dx = \lim_{R\to\infty} \int_{|x|\le R} \sim dx$ in $L^2(\mathrm{R}^d)$.

Lemma (Lie-Trotter formula) Let A and B be self-adjoint operators on the Hilbert space $\mathcal{H}$. Suppose that $A+B$ is essentially self-adjoint operator with $D(A+B)=D(A)\cap D(B)$. Then, for any $T\in \mathrm{R}$ and $v\in\mathcal{H}$, we have

$$e^{-iT(A+B)}v = \text{s-}\lim_{J\to\infty}\left(e^{-i\frac{T}{J+1}A}e^{-i\frac{T}{J+1}B}\right)^{J+1}v\,.$$

Sketch of Proof Note that

$$(e^{-\frac{i}{\hbar}T(-\frac{\hbar^2}{2}\triangle)}v)(x) = \lim_{R\to\infty}\left(\frac{1}{2\pi i\hbar T}\right)^{d/2}\int_{|y|\le R}\exp\left(\frac{i}{\hbar}\frac{(x-y)^2}{2T}\right)v(y)dy\,.$$

$$(e^{-\frac{i}{\hbar}TV}v)(x) = e^{-\frac{i}{\hbar}TV(x)}v(x)\,.$$

By the Lie-Trotter formula, we have

$$u(T,x) = e^{-\frac{i}{\hbar}T(-\frac{\hbar^2}{2}\triangle+V)}v = \text{s-}\lim_{J\to\infty}\left(e^{-\frac{i}{\hbar}\frac{T}{J+1}(-\frac{\hbar^2}{2}\triangle)}e^{-\frac{i}{\hbar}\frac{T}{J+1}V}\right)^{J+1}v\,.$$

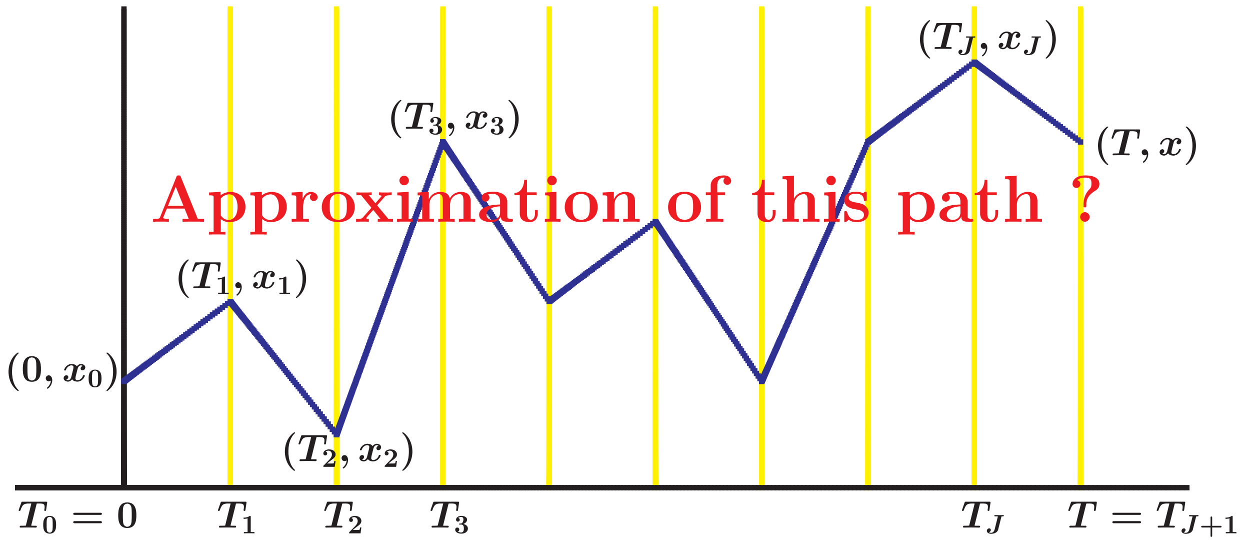
If we write down the right-hand side, we obtain the result. □

Remark This approach can treat a general potential $V(x)$.

Remark This is not a functional integration. From the view of the classical mechanics, we may hope the semiclassical approximation as $\hbar \to 0$. However it is difficult because the convergence is a sense of operators.

Remark $S_{J+1}(x_{J+1}, x_J, \ldots, x_1, x_0)$ is an approximation.

$$S[\gamma] = \int_0^T \frac{1}{2}\left|\frac{d\gamma}{dt}\right|^2 - V(\gamma)dt \approx \sum_{j=1}^{J+1}\left(\frac{(x_j - x_{j-1})^2}{2T/(J+1)} - V(x_{j-1})\frac{T}{J+1}\right).$$



8. Nelson's analytic continuation from Wiener measure

E. Nelson, J. Math. Phys. 5 (1964), 332-343.

Nelson considered the Schrödinger equation with the complex mass m.

$$\left(i\hbar\partial_T + \frac{\hbar^2}{2m}\triangle - V(x) \right) u(T,x) = 0\,, \quad u(0,x) = v(x)\,.$$

If m is a real number, this equation is a usual Schrödinger equation.

Set

$$(K_m^T v)(x) = \left(\frac{m}{2\pi i\hbar T} \right)^{d/2} \int_{\mathrm{R}^d} \exp\left(\frac{mi}{\hbar}\frac{(x-y)^2}{2T} \right) v(y)dy\,,$$

$$(M^T v)(x) = \exp\left(-\frac{iT}{\hbar} V(x) \right) v(x)\,.$$

For the sake of simplicity, assume that the real-valued function $V(x)$ is continuous.

The case with the purely imaginary mass $m = im'$ $(m' > 0)$

First we consider the equation with the purely imaginary mass $m = im'$ $(m' > 0)$, i.e., the heat equation with the diffusion constant $\hbar/(2m') > 0$.

$$\left(\partial_T - \frac{\hbar}{2m'}\triangle + \frac{i}{\hbar}V(x)\right)u(T,x) = 0\,, \quad u(0,x) = v(x)\,.$$

M. Kac had shown how to solve the equation.

M. Kac, In: Proc. 2nd Berk. Symp. on Math. Statist. and Prob. Univ. Calif. Press, Berk. (1950), 189-215.

Note that

$$(K_m^T v)(x) = \left(\frac{m'}{2\pi\hbar T}\right)^{d/2}\int_{\mathrm{R}^d}\exp\left(-\frac{m'}{\hbar}\frac{(x-y)^2}{2T}\right)v(y)dy\,,$$

$$(M^T v)(x) = \exp\left(-\frac{iT}{\hbar}V(x)\right)v(x)\,.$$

Theorem 1(Nelson 1964) Let $m = im'$ $(m' > 0)$. Then for any $v \in L^2(\mathrm{R}^d)$, there exists

$$U_m^T v \equiv \lim_{J\to\infty} (K_m^{T/(J+1)} M^{T/(J+1)})^{J+1} v \quad \text{in } L^2(\mathrm{R}^d) \,.$$

Furthermore $u(T, x) = (U_m^T v)(x)$ satisfies the Schrödinger equation with the purely imaginary mass $m = im'$ in the sense of distribution.

Moreover there exists a measure $d\mu(\omega)$ on the path space $\Omega = \{\omega \in C([0, T] \to \mathrm{R}^d) \; ; \; \omega(T) = x\}$ such that

$$(U_m^T v)(x) = \int_\Omega \exp\left(-\frac{i}{\hbar}\int_0^T V(\omega(s))ds\right) v(\omega(0))d\mu(\omega) \,.$$

Remark This approach can treat a general potential $V(x)$. Furthermore, this is a functional integral with the measure on the path space.

Sketch of Proof We construct the measure $d\mu(\omega)$ on the path space $\Omega = \{\omega \in C([0,T] \to \mathrm{R}^d) \ ; \ \omega(T) = x\}$ using

$$p(T,x,dy) = \left(\frac{m'}{2\pi\hbar T}\right)^{d/2} \exp\left(-\frac{m'}{\hbar}\frac{(x-y)^2}{2T}\right) dy$$

for the cylinder sets. Since $V(\omega(s))$ is continuous for a. e. ω,

$$\int_0^T V(\omega(s))ds = \lim_{J\to\infty} \sum_{j=1}^{J+1} V(\omega(j\frac{T}{J+1}))\frac{T}{J+1}$$

for a. e. ω. By the Lebesgue dominated convergence theorem and the definition of the Wiener integral, we have

$$\int_\Omega \exp\left(-\frac{i}{\hbar}\int_0^T V(\omega(s))ds\right) v(\omega(0))d\mu(\omega)$$

$$= \lim_{J\to\infty} \int_\Omega \exp\left(-\frac{i}{\hbar}\sum_{j=1}^{J+1} V(\omega(j\frac{T}{J+1}))\frac{T}{J+1}\right) v(\omega(0))d\mu(\omega)$$

$$= \lim_{J\to\infty} (K_m^{T/(J+1)} M^{T/(J+1)})^{J+1} v(x) \equiv U_m^T v(x) \, . \ \square$$

The case with the complex mass m with $\mathrm{Im}\, m > 0$

Theorem 2(Nelson 1964) Let $\mathrm{Im}\, m > 0$. Then for any $v \in L^2(\mathrm{R}^d)$, there exists

$$U_m^T v \equiv \lim_{J \to \infty} (K_m^{T/(J+1)} M^{T/(J+1)})^{J+1} v \quad \text{in } L^2(\mathrm{R}^d)\,.$$

Furthermore $u(T,x) = (U_m^T v)(x)$ satisfies the Schrödinger equation with the complex mass m ($\mathrm{Im}\, m > 0$) in the sense of distribution.

Remark If $\mathrm{Re}\, m \neq 0$, this is not a functional integral any more because the measure does not exist.

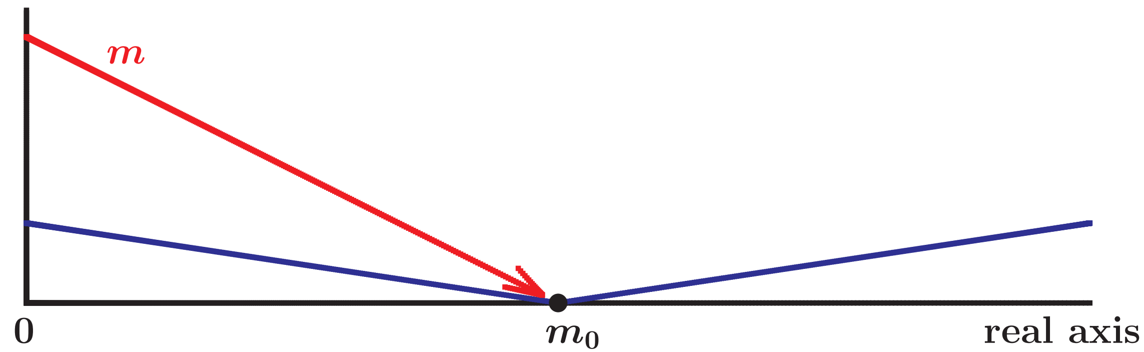
R. H. Cameron, J. of Math. and Phys. Sci. 39(1960), 126-140.

The case with the real mass $m_0 > 0$

Theorem 3(Nelson 1964) There is a set N of real numbers of Lebesgue measure 0 such that for all $m_0 \in \mathrm{R}^+ - N$, there exists

$$U^T_{m_0} v \equiv \lim_{m \searrow m_0} U^T_m v \quad \text{in } L^2(\mathrm{R}^d)$$

when m approaches m_0 non-tangentially from the upper half-plane($\mathrm{Im}\ m > 0$, $m \to m_0$ and $|\mathrm{Re}\ (m - m_0)|/\mathrm{Im}\ m$ is bounded). Furthermore $u(T, x) = (U^T_{m_0} v)(x)$ satisfies the Schrödinger equation with the real mass m_0 in the sense of distribution.



Remark This approach can treat a general potential $V(x)$.

9. Fujiwara's results via piecewise classical paths

D. Fujiwara, J. D'Analyse Math. 35 (1979), 41-96.

Fujiwara proved that the convergence of the time slicing approximation becomes sharper if we use the piecewise classical paths under a smooth condition.

Assumption of $S[\gamma] = \int_0^T \frac{1}{2}\left|\frac{d\gamma}{dt}\right|^2 - V(t,\gamma)dt.$

$V(t,x) : \mathrm{R}\times\mathrm{R}^d \to \mathrm{R}$, $\partial_x^\alpha V(t,x)$ is continuous and

$|\partial_x^\alpha V(t,x)| \le C_\alpha(1+|x|)^{\max(2-|\alpha|,0)}$.

Examples

$$V(t,x) = \sum_{j,k=1}^{d} a_{j,k}(t)x_j x_k + \sum_{j=1}^{d} b_j(t)x_j + c(t,x)\,.$$

Here $a_{j,k}(t)$, $b_j(t)$ and $\partial_x^\alpha c(t,x)$ are real-valued continuous bounded functions.

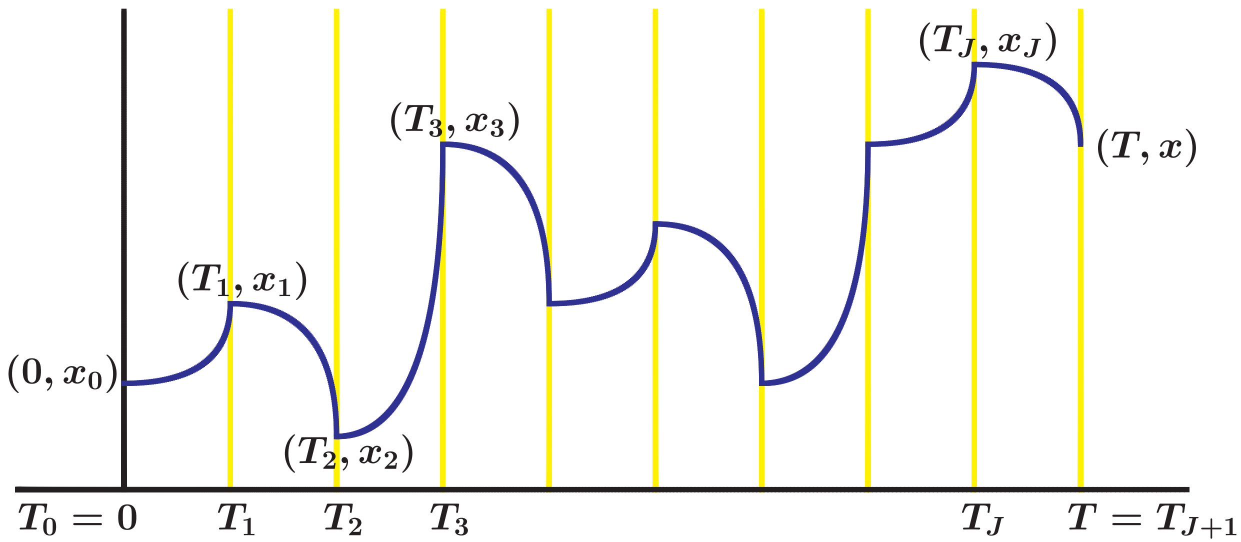
Piecewise classical paths

Let $|\Delta_{T,0}|$ be small. Let $\gamma_{T_j,T_{j-1}}$ be the classical path defined by the Euler equation

$$\ddot{\gamma}_{T_j,T_{j-1}}(t) + (\partial_x V)(t, \gamma_{T_j,T_{j-1}}) = 0\,, \quad T_{j-1} \le t \le T_j$$

with $\gamma_{T_j,T_{j-1}}(T_{j-1}) = x_{j-1}$ and $\gamma_{T_j,T_{j-1}}(T_j) = x_j$. Let $S_{T_j,T_{j-1}}(x_j, x_{j-1})$ be the action along the classical path $\gamma_{T_j,T_{j-1}}$ by

$$S_{T_j,T_{j-1}}(x_j, x_{j-1}) = \int_{T_{j-1}}^{T_j} \frac{1}{2}|\dot{\gamma}_{T_j,T_{j-1}}|^2 - V(t, \gamma_{T_j,T_{j-1}})dt\,.$$



The convergence in the uniform topology of L^2-operator

Theorem (Fujiwara 1979) For any $v \in L^2(\mathrm{R}^d)$, we have

$$u(T,x) = \lim_{|\Delta_{T,0}|\to 0} \prod_{j=1}^{J+1} \left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2} \int_{\mathrm{R}^d} \cdots \int_{\mathrm{R}^d} e^{\frac{i}{\hbar} S_{\Delta_{T,0}}(x_{J+1},x_J,\ldots,x_1,x_0)} v(x_0) \prod_{j=0}^{J} dx_j \,,$$

where

$$S_{\Delta_{T,0}}(x_{J+1},x_J,\ldots,x_1,x_0) = \sum_{j=1}^{J+1} S_{T_j,T_{j-1}}(x_j,x_{j-1})\,,$$

and the topology of the convergence as $|\Delta_{T,0}| \to 0$ is the uniform topology of L^2-operator.

Remark Note that Nelson's case as $J \to \infty$ is the strong topology.

The semiclassical approximation $\hbar \to 0$

D. Fujiwara, In: Functional analysis and related topics, Lecture Notes in Math. 1540, Springer (1991), 39-53.

Assume that T is small. Fujiwara considered the phase function, the main term and the remainder term of the time slicing approximation.

$$\prod_{j=1}^{J+1} \left(\frac{1}{2\pi i\hbar t_j}\right)^{d/2} \int_{\mathrm{R}^{dJ}} e^{\frac{i}{\hbar} S_{\Delta_{T,0}}(x_{J+1},x_J,\ldots,x_1,x_0)} \prod_{j=1}^{J} dx_j$$

$$= \left(\frac{1}{2\pi i\hbar T}\right)^{d/2} e^{\frac{i}{\hbar} S_{T,0}(x,x_0)} \left(D_{\Delta_{T,0}}(x,x_0)^{-1/2} + \hbar \Upsilon_{\Delta_{T,0}}(\hbar,x,x_0)\right) .$$

Lemma 1 (Fujiwara) The phase function $S_{T,0}(x,x_0)$ is the action $S[\gamma_{T,0}]$ defined by the classical path $\gamma_{T,0}$ with $\gamma_{T,0}(0) = x_0$ and $\gamma_{T,0}(T) = x$, i.e.,

$$S_{T,0}(x,x_0) = S[\gamma_{T,0}] .$$

Lemma 2 (Fujiwara) Let $(x_J^\dagger, \ldots, x_1^\dagger)$ be the critical point defined by

$$(\partial_{(x_J,\ldots,x_1)} S_{\Delta_{T,0}})(x_{J+1}, x_J^\dagger, \ldots, x_1^\dagger, x_0) = 0\,.$$

We define the main term $D_{\Delta_{T,0}}(x_{J+1}, x_0)$ by

$$D_{\Delta_{T,0}}(x_{J+1}, x_0) = \left(\frac{\prod_{j=1}^{J+1} t_j}{T_{J+1}} \right)^d \det(\partial^2_{(x_J,\ldots,x_1)} S_{\Delta_{T,0}})(x_{J+1}, x_J^\dagger, \ldots, x_1^\dagger, x_0)\,.$$

Then we have

$$|D_{\Delta_{T,0}}(x, x_0) - 1| \leq CT^2\,,$$

$$|D_{\Delta_{T,0}}(x, x_0) - D(T, x, x_0)| \leq C'|\Delta_{T,0}|T\,.$$

Lemma 3 (Fujiwara) The remainder term satisfies the following estimates:

$$|\Upsilon_{\Delta_{T,0}}(\hbar, x, x_0)| \leq CT^2\,,$$

$$|\Upsilon_{\Delta_{T,0}}(\hbar, x, x_0) - \Upsilon(T, \hbar, x, x_0)| \leq C'\frac{1}{\hbar}|\Delta_{T,0}|T\,.$$

Theorem (Fujiwara 1991) (The estimate for the remainder term)

Let T be sufficiently small. Then we have

$$\int e^{\frac{i}{\hbar}S[\gamma]}\mathcal{D}[\gamma] = \left(\frac{1}{2\pi i\hbar T}\right)^{d/2} e^{\frac{i}{\hbar}S[\gamma_{T,0}]}\left(D(T,x,x_0)^{-1/2} + \hbar\Upsilon(T,\hbar,x,x_0)\right).$$

Here $\gamma_{T,0}$ is the classical path with $\gamma_{T,0}(0) = x_0$ and $\gamma_{T,0}(T) = x$,

$D(T,x,x_0)$ is the Morette-Van Vleck determinant and

$$|\Upsilon(T,\hbar,x,x_0)| \leq C\,.$$