

Phase space Feynman path integrals via piecewise bicharacteristic paths  
and their semiclassical approximations

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- [1] N. Kumano-go and D. Fujiwara, Phase space Feynman path integrals via piecewise bicharacteristic paths and their semiclassical approximations. *Bulletin des Sciences Mathematique*, 132 (2008) 313-357.

# 1. Introduction to phase space Feynman path integral

Let  $T > 0$  and  $x \in \mathbb{R}^d$ . We consider the fundamental solution  $U(T, 0)$  for the Schrödinger equation

$$\left( i\hbar\partial_T - H(T, x, \frac{\hbar}{i}\partial_x) \right) U(T, 0) = 0, \quad U(0, 0) = I,$$

with the Planck parameter  $0 < \hbar < 1$ . By the Fourier transform and inverse Fourier transform with  $x_0, \xi_0 \in \mathbb{R}^d$ , we can write

$$v(x) = \left( \frac{1}{2\pi\hbar} \right)^d \int_{\mathbb{R}^{2d}} e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0} v(x_0) dx_0 d\xi_0,$$
$$H(T, x, \frac{\hbar}{i}\partial_x)v(x) = \left( \frac{1}{2\pi\hbar} \right)^d \int_{\mathbb{R}^{2d}} e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0} H(T, x, \xi_0)v(x_0) dx_0 d\xi_0.$$

As an approximation of  $U(T, 0)v(x)$ , we may use the operator

$$I(T, 0)v(x) = \left( \frac{1}{2\pi\hbar} \right)^d \int_{\mathbb{R}^{2d}} e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0 - \frac{i}{\hbar} \int_0^T H(t, x, \xi_0) dt} v(x_0) dx_0 d\xi_0.$$

Under a suitable condition, we have the following:

Let  $\Delta_{T,0} : T = T_{J+1} > T_J > \cdots > T_1 > T_0 = 0$  be a division of  $[0, T]$ .

Set  $t_j = T_j - T_{j-1}$  and  $|\Delta_{T,0}| = \max_{1 \leq j \leq J+1} t_j$ . Then we can write

$$\begin{aligned} U(T, 0)v(x) &= \lim_{|\Delta_{T,0}| \rightarrow 0} I(T, T_J)I(T_J, T_{J-1}) \cdots I(T_2, T_1)I(T_1, 0)v(x) \\ &= \lim_{|\Delta_{T,0}| \rightarrow 0} \left( \frac{1}{2\pi\hbar} \right)^{d(J+1)} \int_{\mathbb{R}^{2d(J+1)}} e^{\frac{i}{\hbar} \sum_{j=1}^{J+1} ((x_j - x_{j-1}) \cdot \xi_{j-1} - \int_{T_{j-1}}^{T_j} H(t, x_j, \xi_{j-1}) dt)} v(x_0) \prod_{j=0}^J dx_j d\xi_j \end{aligned}$$

with  $x = x_{J+1}$ . In other words, if we consider  $U(T, 0, x, \xi_0)$  satisfying

$$U(T, 0)v(x) = \left( \frac{1}{2\pi\hbar} \right)^d \int_{\mathbb{R}^{2d}} e^{\frac{i}{\hbar} (x - x_0) \cdot \xi_0} U(T, 0, x, \xi_0) v(x_0) dx_0 d\xi_0,$$

then we can write

$$\begin{aligned} &e^{\frac{i}{\hbar} (x - x_0) \cdot \xi_0} U(T, 0, x, \xi_0) \\ &= \lim_{|\Delta_{T,0}| \rightarrow 0} \left( \frac{1}{2\pi\hbar} \right)^{dJ} \int_{\mathbb{R}^{2dJ}} e^{\frac{i}{\hbar} \sum_{j=1}^{J+1} ((x_j - x_{j-1}) \cdot \xi_{j-1} - \int_{T_{j-1}}^{T_j} H(t, x_j, \xi_{j-1}) dt)} \prod_{j=1}^J dx_j d\xi_j. \end{aligned}$$

Now we introduce the position path  $q(t)$  and the momentum path  $p(t)$  with

$q(T_j) = x_j$  for  $j = 0, 1, \dots, J + 1$  and  $p(T_j) = \xi_j$  for  $j = 0, 1, \dots, J$ .

According to Feynman(1951), we formally write  $U(T, 0, x, \xi_0)$  as

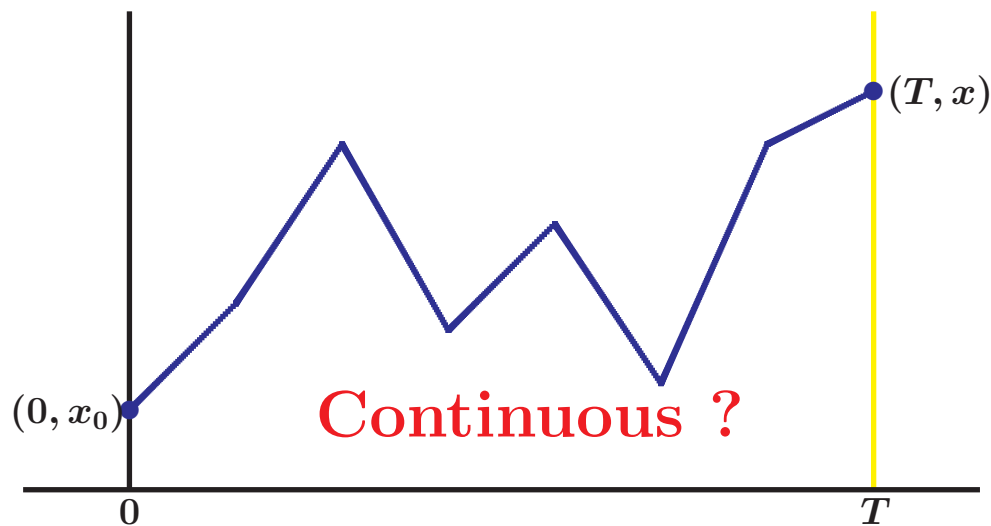
$$e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0}U(T, 0, x, \xi_0) = \int e^{\frac{i}{\hbar}\phi[q,p]}\mathcal{D}[q, p].$$

Here  $(q, p) : [0, T] \rightarrow \mathbb{R}^{2d}$  is the path with  $q(0) = x_0$ ,  $q(T) = x$  and  $p(0) = \xi_0$ ,

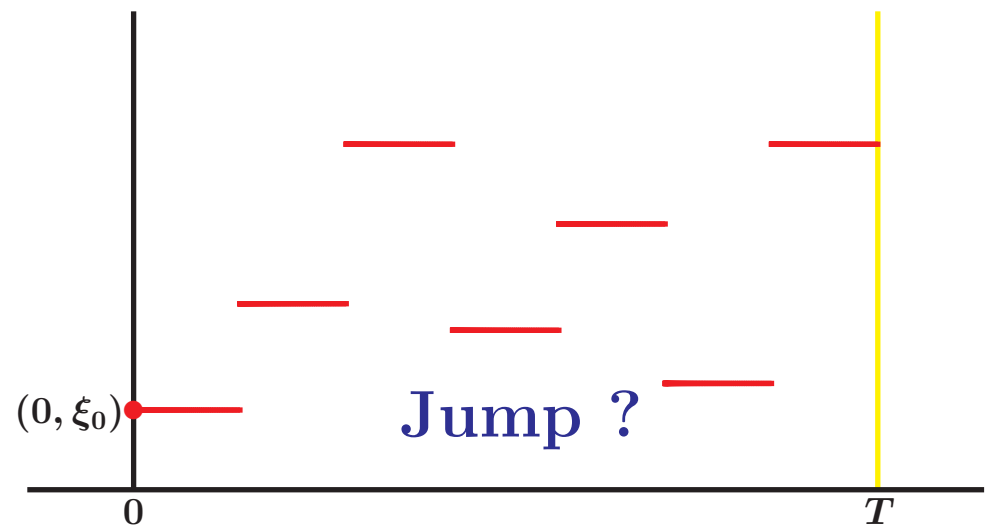
$$\phi[q, p] = \int_{[0,T)} p(t) \cdot dq(t) - \int_{[0,T)} H(t, q(t), p(t))dt,$$

and the **phase space path integral**  $\int \sim \mathcal{D}[q, p]$  is a sum over all paths  $(q, p)$ .

However, in mathematics, the measure of path integral does not exist. Even in quantum physics, we can not have  $q(t)$  and  $p(t)$  at the same time  $t$ .



The position path  $q(t)$  ?



The momentum path  $p(t)$  ?

## Mathematical definitions via Fourier integral operators of $\int e^{\frac{i}{\hbar}\phi[q,p]} \mathcal{D}[q,p]$

- H. Kumano-go-Kitada(1981)
- N. Kumano-go(1995)
- W. Ichinose(2006)
- ● ● ●

## Other mathematical definitions of $\int e^{\frac{i}{\hbar}\phi[q,p]} \mathcal{D}[q,p]$

- Daubechies-Klauder(1985)

via analytic continuation from Wiener measure.

- Albeverio-Guatterri-Mazzucchi(2002) via Fresnel integral transform.
- Smolyanov-Tokarev-Truman(2002) via Chernoff formula
- ● ● ●

Our results Using piecewise bicharacteristic paths, we prove

the existence of the phase space Feynman path integrals

$$\int e^{\frac{i}{\hbar}\phi[q,p]} F[q,p] \mathcal{D}[q,p],$$

with general functional  $F[q,p]$  as integrand. More precisely, we give a fairly general class  $\mathcal{F}$  such that for any  $F[q,p] \in \mathcal{F}$ , the time slicing approximation converges uniformly on compact subsets with respect to  $(x, \xi_0)$ .

**Remark** Using broken line paths  $q(t)$  and piecewise constant paths  $p(t)$ ,

W. Ichinose gave some examples  $F[q,p] = \prod_{k=1}^K B_k(q(\tau_k), p(\tau_k))$ ,

$0 < \tau_1 < \dots < \tau_k < T$  which **do not converge** as an operator. Note that we

will **exclude**  $F[q,p] = B(t, q(t), p(t))$  at the time  $t$  from our class  $\mathcal{F}$ .

**Remark** Piecewise bicharacteristic paths naturally lead us to the

semiclassical approximation  $\hbar \rightarrow 0$  of Hamiltonian type.

## 2. Existence of phase space Feynman path integrals

**Assumption**  $H(t, x, \xi) : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $\partial_x^\alpha \partial_\xi^\beta H(t, x, \xi)$ : continuous,

$$|\partial_x^\alpha \partial_\xi^\beta H(t, x, \xi)| \leq C_{\alpha, \beta} (1 + |x| + |\xi|)^{\max(2 - |\alpha + \beta|, 0)}.$$

### Examples

$$\begin{aligned} H(t, x, \frac{\hbar}{i} \partial_x) &= \sum_{j, k=1}^d (a_{j, k}(t) \frac{\hbar}{i} \partial_{x_j} \frac{\hbar}{i} \partial_{x_k} + b_{j, k}(t) x_j \frac{\hbar}{i} \partial_{x_k} + c_{j, k}(t) x_j x_k) \\ &+ \sum_{j=1}^d (a_j(t) \frac{\hbar}{i} \partial_{x_j} + b_j(t) x_j) + c(t, x). \end{aligned}$$

Here  $a_{j, k}(t)$ ,  $b_{j, k}(t)$ ,  $c_{j, k}(t)$ ,  $a_j(t)$ ,  $b_j(t)$  and  $\partial_x^\alpha c(t, x)$  are real-valued continuous bounded functions.

We can produce many  $F[q, p] \in \mathcal{F}$

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We will define the class  $\mathcal{F}$  in the last section. Because, even if we do not state the definition of  $\mathcal{F}$  here, we can produce many functionals  $F[q, p] \in \mathcal{F}$ .

**Examples of  $F[q, p] \in \mathcal{F}$**

(1) If  $|\partial_x^\alpha B(t, x)| \leq C_\alpha(1 + |x|)^m$ , then the functional independent of  $p$ ,

$$F[q, p] = B(t, q(t)) \in \mathcal{F}. \quad F[q, p] \equiv \mathbf{1} \in \mathcal{F}.$$

(2) If  $|\partial_x^\alpha \partial_\xi^\beta B(t, x, \xi)| \leq C_{\alpha, \beta}(1 + |x| + |\xi|)^m$ , then

$$F[q, p] = \int_{[T', T'')} B(t, q(t), p(t)) dt \in \mathcal{F}.$$

(3) If  $|\partial_x^\alpha \partial_\xi^\beta B(t, x, \xi)| \leq C_{\alpha, \beta}$ , then  $F[q, p] = e^{\int_{[T', T'')} B(t, q(t), p(t)) dt} \in \mathcal{F}$ .

**Theorem 1**  $F[q, p], G[q, p] \in \mathcal{F} \implies F[q, p] + G[q, p], F[q, p]G[q, p] \in \mathcal{F}$ .

# The bicharacteristic paths

Let  $\Delta_{T,0} : T = T_{J+1} > T_J > \dots > T_1 > T_0 = 0$  be any division of  $[0, T]$ .

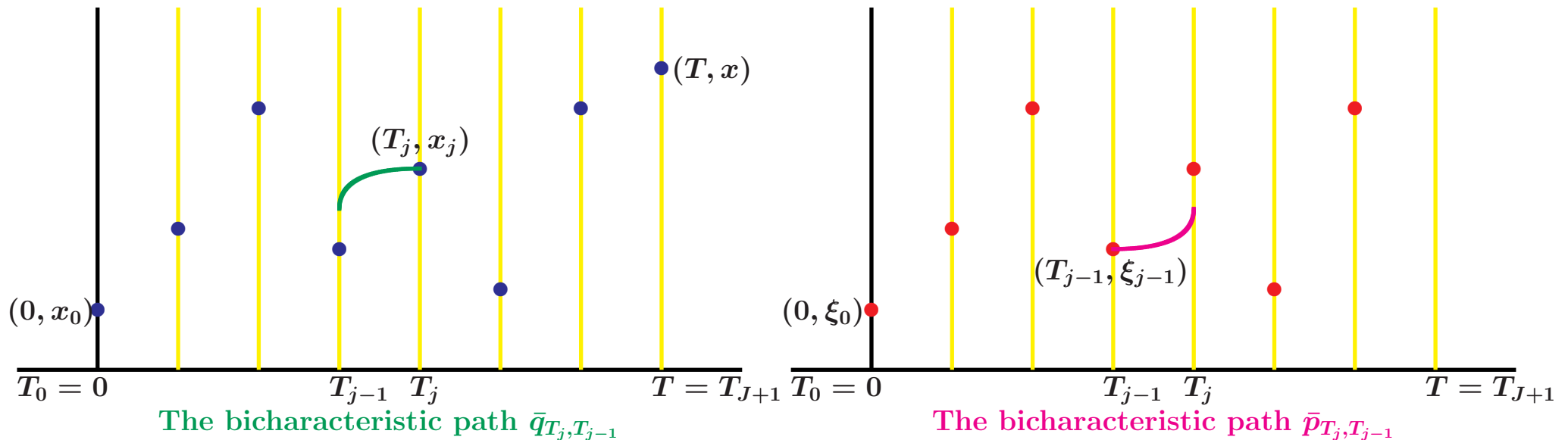
Let  $t_j = T_j - T_{j-1}$  and  $|\Delta_{T,0}| = \max_{1 \leq j \leq J+1} t_j$ . Set  $x_{J+1} = x$ . Let  $x_j \in \mathbb{R}^d$  and  $\xi_j \in \mathbb{R}^d$ . Let  $|\Delta_{T,0}|$  small. Then we can define the bicharacteristic paths

$\bar{q}_{T_j, T_{j-1}}(t, x_j, \xi_{j-1})$ ,  $\bar{p}_{T_j, T_{j-1}}(t, x_j, \xi_{j-1})$  by the Hamilton canonical equation

$$\partial_t \bar{q}_{T_j, T_{j-1}}(t) = (\partial_\xi H)(t, \bar{q}_{T_j, T_{j-1}}, \bar{p}_{T_j, T_{j-1}}),$$

$$\partial_t \bar{p}_{T_j, T_{j-1}}(t) = -(\partial_x H)(t, \bar{q}_{T_j, T_{j-1}}, \bar{p}_{T_j, T_{j-1}}), \quad T_{j-1} \leq t \leq T_j$$

with the boundary condition  $\bar{q}_{T_j, T_{j-1}}(T_j) = x_j$  and  $\bar{p}_{T_j, T_{j-1}}(T_{j-1}) = \xi_{j-1}$ .



# The piecewise bicharacteristic paths We define

$q_{\Delta_{T,0}} = q_{\Delta_{T,0}}(t, x_{J+1}, \xi_J, x_J, \dots, \xi_0, x_0)$ ,  $p_{\Delta_{T,0}} = p_{\Delta_{T,0}}(t, x_{J+1}, \xi_J, x_J, \dots, \xi_0)$  by

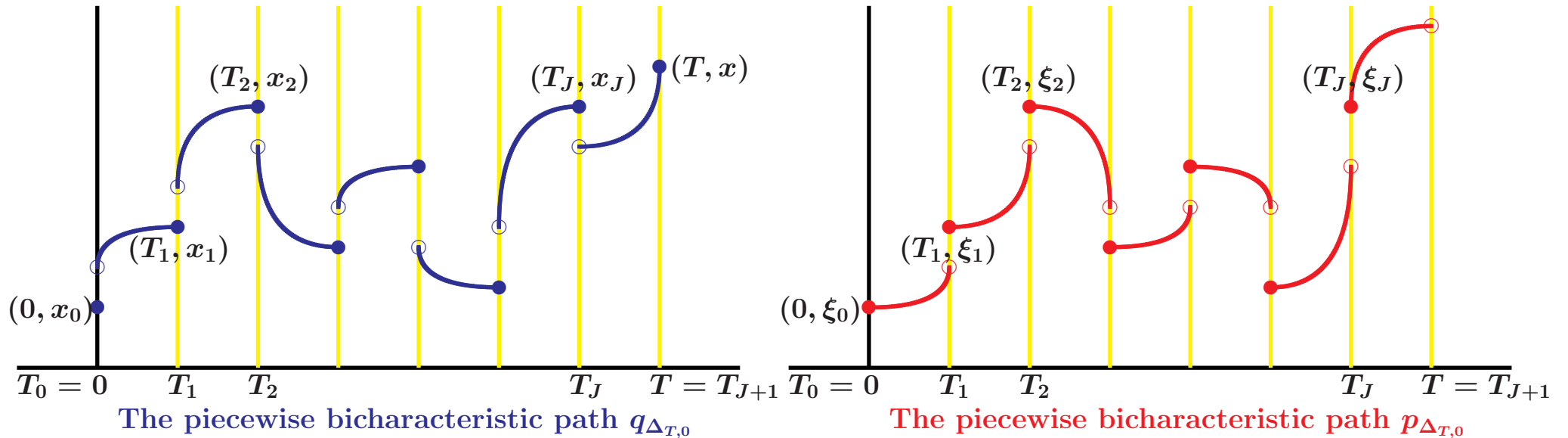
$$q_{\Delta_{T,0}}(t) = \bar{q}_{T_j, T_{j-1}}(t, x_j, \xi_{j-1}), \quad T_{j-1} < t \leq T_j, \quad q_{\Delta_{T,0}}(0) = x_0,$$

$$p_{\Delta_{T,0}}(t) = \bar{p}_{T_j, T_{j-1}}(t, x_j, \xi_{j-1}), \quad T_{j-1} \leq t < T_j, \quad j = 1, 2, \dots, J, J+1.$$

Then the functionals  $\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$ ,  $F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$  are functions, i.e.,

$$\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}] = \phi_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \dots, \xi_1, x_1, \xi_0, x_0),$$

$$F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}] = F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \dots, \xi_1, x_1, \xi_0, x_0).$$



# The phase space Feynman path integrals exist

**Theorem 2** Let  $T$  sufficiently small. Then, for any  $F[q, p] \in \mathcal{F}$ ,

$$\int e^{\frac{i}{\hbar}\phi[q,p]} F[q, p] \mathcal{D}[q, p] \\ \equiv \lim_{|\Delta_{T,0}| \rightarrow 0} \left( \frac{1}{2\pi\hbar} \right)^{dJ} \int_{\mathbb{R}^{2dJ}} e^{\frac{i}{\hbar}\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]} F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}] \prod_{j=1}^J d\xi_j dx_j \quad (\star)$$

converges uniformly on compact sets of  $(x, \xi_0)$ , i.e., well-defined.

**Remark** Even when  $F[q, p] \equiv 1$ , each integral of the right hand side of  $(\star)$  does not converge absolutely (is an oscillatory integral).

$$\int_{\mathbb{R}^{2d}} \mathbf{1} d\xi_j dx_j = \infty.$$

Furthermore, the number  $J$  of integrals (division points) tends to  $\infty$ .

$$\infty \times \infty \times \infty \times \infty \times \dots, \quad J \rightarrow \infty.$$

We treat the multiple integral  $(\star)$  directly to keep the paths  $q_{\Delta_{T,0}}, p_{\Delta_{T,0}}$ .

Example:  $d = 1$ ,  $H(t, x, \xi) = x^2/2 + \xi^2/2$ ,  $F[q, p] \equiv 1$

By the Hamiltonian canonical equation

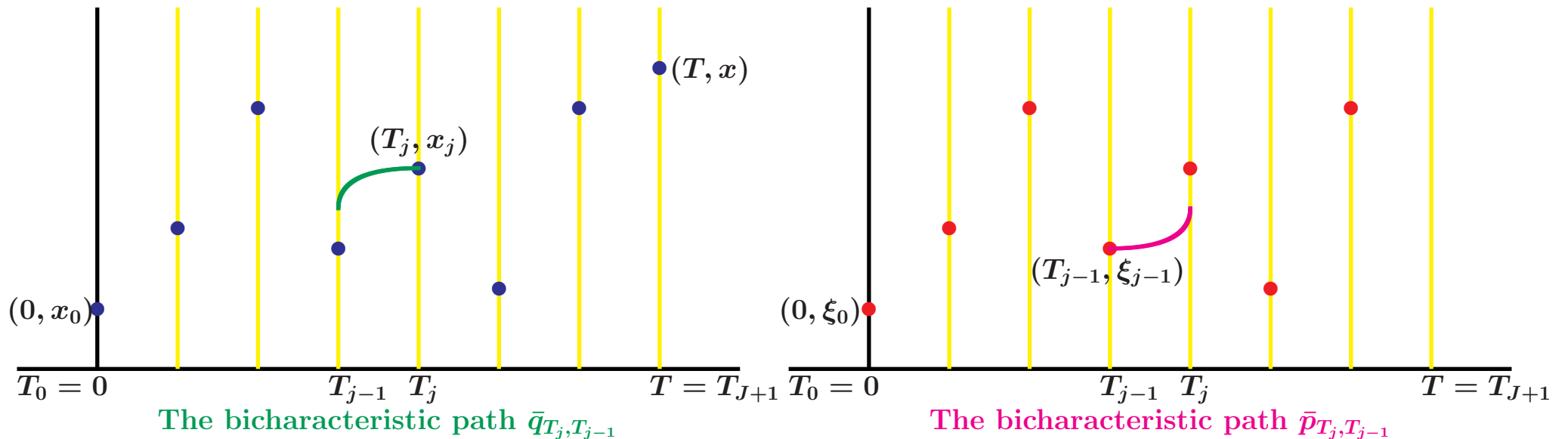
$$\partial_t \bar{q}_{T_j, T_{j-1}}(t) = \bar{p}_{T_j, T_{j-1}}(t), \quad \partial_t \bar{p}_{T_j, T_{j-1}}(t) = -\bar{q}_{T_j, T_{j-1}}(t), \quad T_{j-1} \leq t \leq T_j,$$

$$\bar{q}_{T_j, T_{j-1}}(T_j) = x_j, \quad \bar{p}_{T_j, T_{j-1}}(T_{j-1}) = \xi_{j-1},$$

we have the bicharacteristic paths

$$\bar{q}_{T_j, T_{j-1}}(t) = \frac{x_j \cos(t - T_{j-1}) - \xi_{j-1} \sin(T_j - t)}{\cos(T_j - T_{j-1})},$$

$$\bar{p}_{T_j, T_{j-1}}(t) = \frac{-x_j \sin(t - T_{j-1}) + \xi_{j-1} \cos(T_j - t)}{\cos(T_j - T_{j-1})}.$$



If we use the piecewise bicharacteristic paths  $q_{\Delta_{T,0}}$  and  $p_{\Delta_{T,0}}$ , the functional  $\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$  becomes the function

$$\begin{aligned}\phi_{\Delta_{T,0}} &= \int_{[0,T)} p_{\Delta_{T,0}} \cdot dq_{\Delta_{T,0}}(t) - \int_{[0,T)} H(t, q_{\Delta_{T,0}}, p_{\Delta_{T,0}}) dt \\ &= \sum_{j=1}^{J+1} \phi_{T_j, T_{j-1}}(x_j, \xi_{j-1}, x_{j-1}),\end{aligned}$$

where

$$\phi_{T_j, T_{j-1}}(x_j, \xi_{j-1}, x_{j-1}) = -x_{j-1} \cdot \xi_{j-1} + \frac{2x_j \cdot \xi_{j-1} - (x_j^2 + \xi_{j-1}^2) \sin(T_j - T_{j-1})}{2 \cos(T_j - T_{j-1})}.$$

First we consider the multiple integral for the division  $T = T_2 > T_1 > T_0 = 0$  when  $J = 1$ . Performing the oscillatory integration with respect to  $(\xi_1, x_1)$ , we have

$$\begin{aligned}&\left(\frac{1}{2\pi\hbar}\right) \int_{\mathbb{R}^2} e^{\frac{i}{\hbar}\phi_{T, T_1}(x_2, \xi_1, x_1) + \frac{i}{\hbar}\phi_{T_1, 0}(x_1, \xi_0, x_0)} dx_1 d\xi_1 \\ &= e^{\frac{i}{\hbar}\phi_{T, 0}(x_2, \xi_0, x_0)} \left(\frac{\cos(T_2 - T_1) \cos(T_1 - T_0)}{\cos T}\right)^{1/2}.\end{aligned}$$

Next we consider the multiple integral for the general division  $\Delta_{T,0}$ . Using this relation inductively and taking  $|\Delta_{T,0}| = \max_{1 \leq j \leq J+1} (T_j - T_{j-1}) \rightarrow 0$ , we obtain the function  $U(T, 0, x, \xi_0)$  of the fundamental solution  $U(T, 0)$ .

$$\begin{aligned}
& e^{\frac{i}{\hbar}(x-x_0) \cdot \xi_0} U(T, 0, x, \xi_0) \\
&= \int e^{\frac{i}{\hbar} \phi[q,p]} \mathcal{D}[q, p] \\
&\equiv \lim_{|\Delta_{T,0}| \rightarrow 0} \left( \frac{1}{2\pi\hbar} \right)^J \int_{\mathbb{R}^{2J}} e^{\frac{i}{\hbar} \sum_{j=1}^{J+1} \phi_{T_j, T_{j-1}}(x_j, \xi_{j-1}, x_{j-1})} \prod_{j=1}^J dx_j d\xi_j \quad (\star)' \\
&= \lim_{|\Delta_{T,0}| \rightarrow 0} e^{\frac{i}{\hbar} \phi_{T,0}(x, \xi_0, x_0)} \left( \frac{\prod_{j=1}^{J+1} \cos(T_j - T_{j-1})}{\cos T} \right)^{1/2} \\
&= \frac{1}{(\cos T)^{1/2}} \exp \frac{i}{\hbar} \left( -x_0 \cdot \xi_0 + \frac{2x \cdot \xi_0 - (x^2 + \xi_0^2) \sin T}{2 \cos T} \right). \quad \square
\end{aligned}$$

**Remark** This also implies that Theorem 2 is not valid when  $T$  is large ( $T = \pi/2$ ).

**Remark** If we regard  $(\star)'$  as composition of many operators

$$I'(T_j, T_{j-1})v(x_j) = \left( \frac{1}{2\pi\hbar} \right) \int_{\mathbb{R}^2} e^{\frac{i}{\hbar}\phi_{T_j, T_{j-1}}(x_j, \xi_{j-1}, x_{j-1})} v(x_{j-1}) dx_{j-1} d\xi_{j-1}$$

in  $L^2(\mathbb{R}^d)$ , the composed operator

$$I'(T, T_J)I'(T_J, T_{J-1}) \cdots I'(T_2, T_1)I'(T_1, 0)$$

converges to  $U(T, 0)$  in the uniform topology as  $|\Delta_{T,0}| \rightarrow 0$  even when  $T$  is large. Note that the case of  $I(T_j, T_{j-1})$  is the strong topology.

**Remark** The approach via operators is very powerful. However it treats the integrals one by one as an operator and its convergence does not seem to distinguish between the position  $x_0$  and the momentum  $\xi_0$ . On the other hand, our approach converges with respect to  $q(T) = x$  and  $p(0) = \xi_0$ .

### 3. Interchange of the order with integrals $\int \sim dt$

**Theorem 3** Let  $T$  sufficiently small. Let  $m \geq 0$ ,  $B(t, x) : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$ ,

$\partial_x^\alpha B(t, x)$ :continuous,  $|\partial_x^\alpha B(t, x)| \leq C_\alpha(1 + |x|)^m$ . Then we have

$$\int e^{\frac{i}{\hbar}\phi[q,p]} \left( \int_{[T',T'')} B(t, q(t)) dt \right) \mathcal{D}[q, p] = \int_{[T',T'')} \left( \int e^{\frac{i}{\hbar}\phi[q,p]} B(t, q(t)) \mathcal{D}[q, p] \right) dt.$$

**Remark** In order to avoid the uncertain principle, we **do not treat**

$B(t, q(t), p(t))$  at the time  $t$ .

**Remark** As a merit to treat general functional  $F[q, p]$  as integrand, we give a perturbation expansion formula. If  $|\partial_x^\alpha B(t, x)| \leq C_\alpha$ , we have

$$\begin{aligned} & \int e^{\frac{i}{\hbar}\phi[q,p] + \frac{i}{\hbar} \int_{[0,T)} B(\tau, q(\tau)) d\tau} \mathcal{D}[q, p] \\ &= \sum_{n=0}^{\infty} \left( \frac{i}{\hbar} \right)^n \int_{[0,T)} d\tau_n \int_{[0,\tau_n)} d\tau_{n-1} \cdots \int_{[0,\tau_2)} d\tau_1 \\ & \times \int e^{\frac{i}{\hbar}\phi[q,p]} B(\tau_n, q(\tau_n)) B(\tau_{n-1}, q(\tau_{n-1})) \cdots B(\tau_1, q(\tau_1)) \mathcal{D}[q, p]. \end{aligned}$$

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## 4. Semiclassical approximation $\hbar \rightarrow 0$ of Hamiltonian type

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Let  $T$  small. For any  $(x_{J+1}, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ , there exists the stationary point  $(x_J^*, \xi_J^*, \dots, x_1^*, \xi_1^*)$  of the phase function  $\phi_{\Delta_{T,0}} = \phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$ , i.e.,

$$(\partial_{(\xi_J, x_J, \dots, \xi_1, x_1)} \phi_{\Delta_{T,0}})(x_{J+1}, \xi_J^*, x_J^*, \dots, \xi_1^*, x_1^*, \xi_0) = 0.$$

We define  $D_{\Delta_{T,0}}(x_{J+1}, \xi_0)$  by

$$D_{\Delta_{T,0}}(x_{J+1}, \xi_0) = (-1)^{dJ} \det(\partial_{(\xi_J, x_J, \dots, \xi_1, x_1)}^2 \phi_{\Delta_{T,0}})(x_{J+1}, x_J^*, \xi_J^*, \dots, x_1^*, \xi_1^*, \xi_0).$$

**Lemma** We have

$$|\partial_x^\alpha \partial_{\xi_0}^\beta (D_{\Delta_{T,0}}(x, \xi_0) - 1)| \leq C_{\alpha, \beta} T^2,$$

$$|\partial_x^\alpha \partial_{\xi_0}^\beta (D_{\Delta_{T,0}}(x, \xi_0) - D(T, x, \xi_0))| \leq C_{\alpha, \beta} |\Delta_{T,0}| T.$$

We use this limit function  $D(T, x, \xi_0) = \lim_{|\Delta_{T,0}| \rightarrow 0} D_{\Delta_{T,0}}(x, \xi_0)$  as a

Hamiltonian version of the Morette-Van Vleck determinant.

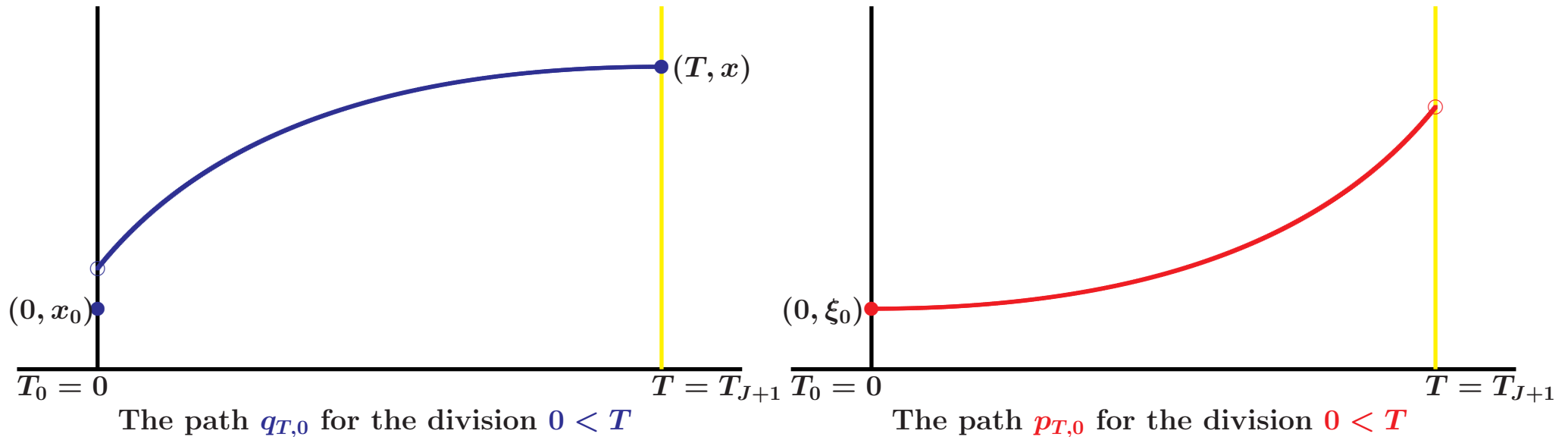
**Theorem 4** (A remainder estimate of Semiclassical approximation  $\hbar \rightarrow 0$ )

Let  $T$  sufficiently small. Then, for any  $F[q, p] \in \mathcal{F}$ ,

$$\int e^{\frac{i}{\hbar}\phi[q,p]} F[q, p] \mathcal{D}[q, p] = e^{\frac{i}{\hbar}\phi[q_{T,0}, p_{T,0}]} (D(T, x, \xi_0)^{-1/2} F[q_{T,0}, p_{T,0}] + \hbar \Upsilon(\hbar, x, \xi_0, x_0)).$$

Here  $q_{T,0} = q_{T,0}(t, x, \xi_0, x_0)$  and  $p_{T,0} = p_{T,0}(t, x, \xi_0)$  are the piecewise bicharacteristic paths for the simplest division  $0 < T$  and

$$|\partial_x^\alpha \partial_{\xi_0}^\beta \Upsilon(\hbar, x, \xi_0, x_0)| \leq C_{\alpha,\beta} (1 + |x| + |\xi_0| + |x_0|)^m.$$



## 5. Proof of Theorems 1,2 and 4

In order to prove the convergence of the multiple integral

$$(\star) \quad \lim_{|\Delta_{T,0}| \rightarrow 0} \left( \frac{1}{2\pi\hbar} \right)^{dJ} \int_{\mathbb{R}^{2dJ}} e^{\frac{i}{\hbar}\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]} F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}] \prod_{j=1}^J d\xi_j dx_j, \\ \infty \times \infty \times \infty \times \infty \times \dots, \quad J \rightarrow \infty$$

we have only to add many assumptions for

$$F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \dots, x_1, \xi_0, x_0) = F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}].$$

The assumptions should be closed under  $+$ ,  $\times$ . Then  $\mathcal{F}$  will be an algebra.

Do not consider other things. Then  $\mathcal{F}$  will be larger as a set.

If lucky,  $\mathcal{F}$  may contain at least one example  $F[q, p] \equiv 1$ .

Step 1. H. Kumano-go-Taniguchi's theorem:  $C \times C \times C \times \dots$  as  $J \rightarrow \infty$ .

Step 2. Fujiwara's stationary phase method:  $C$  independent of  $J \rightarrow \infty$ .

Step 3. My method: **Convergence** as  $|\Delta_{T,0}| \rightarrow 0$ .

## 6. Extension of H. Kumano-go-Taniguchi's theorem

We consider the multiple oscillatory integral

$$\begin{aligned} & \left( \frac{1}{2\pi\hbar} \right)^{dJ} \int_{\mathbb{R}^{2dJ}} e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}} F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \dots, \xi_1, x_1, \xi_0, x_0) \prod_{j=1}^J dx_j d\xi_j \\ & = e^{\frac{i}{\hbar}\phi_{T,0}(x, \xi_0, x_0)} q_{\Delta_{T,0}}(\hbar, x_{J+1}, \xi_0, x_0). \end{aligned}$$

**Lemma KT** Let  $m \geq 0$ . Assume that for any integer  $M \geq 0$ , there exist

$A_M, X_M > 0$  such that for any  $|\alpha_j|, |\beta_{j-1}| \leq M, j = 1, 2, \dots, J, J+1,$

$$\begin{aligned} & \left| \left( \prod_{j=1}^{J+1} \partial_{x_j}^{\alpha_j} \partial_{\xi_{j-1}}^{\beta_{j-1}} \right) F_{\Delta_{T,0}}(x_{J+1}, \xi_J, \dots, x_1, \xi_0, x_0) \right| \\ & \leq A_M (X_M)^{J+1} \left( 1 + \sum_{j=1}^{J+1} (|x_j| + |\xi_{j-1}|) + |x_0| \right)^m. \end{aligned}$$

Then we can control the multiple integral by  $C^J$  independent of  $J \rightarrow \infty,$

$$|q_{\Delta_{T,0}}(\hbar, x, \xi_0, x_0)| \leq C^J (1 + |x_{J+1}| + |\xi_0| + |x_0|)^m.$$

## Integrate by parts over and over again

For simplicity, we prove Lemma KT when  $m = 0$ .

We introduce the differential operators of the first order

$$M_j = \frac{1 - i(\partial_{\xi_j} \phi_{\Delta_{T,0}}) \partial_{\xi_j}}{1 + \hbar^{-1} |\partial_{\xi_j} \phi_{\Delta_{T,0}}|^2}, \quad N_j = \frac{1 - i(\partial_{x_j} \phi_{\Delta_{T,0}}) \partial_{x_j}}{1 + \hbar^{-1} |\partial_{x_j} \phi_{\Delta_{T,0}}|^2},$$

for  $j = 1, 2, \dots, J$ . Note that

$$M_j e^{\frac{i}{\hbar} \phi_{\Delta_{T,0}}} = e^{\frac{i}{\hbar} \phi_{\Delta_{T,0}}}, \quad N_j e^{\frac{i}{\hbar} \phi_{\Delta_{T,0}}} = e^{\frac{i}{\hbar} \phi_{\Delta_{T,0}}}.$$

Integrating by parts over and over again, we can write

$$\left( \frac{1}{2\pi\hbar} \right)^{dJ} \int_{\mathbb{R}^{2dJ}} e^{\frac{i}{\hbar} \phi_{\Delta_{T,0}}} F_{\Delta_{T,0}} \prod_{j=1}^J dx_j d\xi_j = \left( \frac{1}{2\pi\hbar} \right)^{dJ} \int_{\mathbb{R}^{2dJ}} e^{\frac{i}{\hbar} \phi_{\Delta_{T,0}}} F_{\Delta_{T,0}}^{\spadesuit} \prod_{j=1}^J dx_j d\xi_j,$$

where

$$F_{\Delta_{T,0}}^{\spadesuit} = (N_J^*)^{d+1} \cdots (N_2^*)^{d+1} (N_1^*)^{d+1} (M_J^*)^{d+1} \cdots (M_2^*)^{d+1} (M_1^*)^{d+1} F_{\Delta_{T,0}},$$

with the multi-product of the adjoint operators  $M_j^*$ ,  $N_j^*$  of  $M_j$ ,  $N_j$ .

## Expect different results

Generally speaking, we can not control the multi-product of  $J$  differential operators by  $C^J$  as  $J \rightarrow \infty$ . However

$$\begin{aligned}\partial_{\xi_j} \phi_{\Delta_{T,0}} &= -(x_j - x_{j+1}) + \partial_{\xi_j} \omega_{T_{j+1}, T_j}(x_{j+1}, \xi_j), \\ M_j^* &= a_j^1(x_{j+1}, \xi_j, x_j) \partial_{\xi_j} + a_j^0(x_{j+1}, \xi_j, x_j).\end{aligned}$$

have at most  $3d$ -variables independent of  $J$ . Therefore, only  $\partial_{x_{j+1}}, \partial_{\xi_j}, \partial_{x_j}$  differentiate  $M_j^*$ , i.e., only  $N_{j+1}^*, M_j^*, N_j^*$  differentiate  $M_j^*$ .

Hence, we can control this multi-product by  $C^J$  as  $J \rightarrow \infty$ , i.e.,

$$\begin{aligned}M_j^* \circ &\implies \frac{C}{(1 + \hbar^{-1} |\partial_{\xi_j} \phi_{\Delta_{T,0}}|^2)^{1/2}} \times, \\ N_j^* \circ &\implies \frac{C}{(1 + \hbar^{-1} |\partial_{x_j} \phi_{\Delta_{T,0}}|^2)^{1/2}} \times.\end{aligned}$$

## Change all variables at one time

Let  $z_j = \partial_{\xi_j} \phi_{\Delta_{T,0}}$  and  $\zeta_j = \partial_{x_j} \phi_{\Delta_{T,0}}$  for  $j = 1, 2, \dots, J$ . Then we have

$$|F_{\Delta_{T,0}}^\spadesuit| \leq (C')^J \prod_{j=1}^J \frac{1}{(1 + \hbar^{-1}|z_j|^2)^{(d+1)/2}} \frac{1}{(1 + \hbar^{-1}|\zeta_j|^2)^{(d+1)/2}}.$$

Furthermore, we obtain

$$\left| \det \frac{\partial(x_J, \dots, x_1, \xi_J, \dots, \xi_1)}{\partial(z_J, \dots, z_1, \zeta_J, \dots, \zeta_1)} \right| \leq (C'')^J.$$

Changing all variables at one time, we can write

$$\begin{aligned} & \left( \frac{1}{2\pi\hbar} \right)^{dJ} \int_{\mathbb{R}^{2dJ}} e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}} F_{\Delta_{T,0}}^\spadesuit \prod_{j=1}^J dx_j d\xi_j \\ &= \left( \frac{1}{2\pi\hbar} \right)^{dJ} \int_{\mathbb{R}^{2dJ}} e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}} F_{\Delta_{T,0}}^\spadesuit \left| \det \frac{\partial(x_J, \dots, x_1, \xi_J, \dots, \xi_1)}{\partial(z_J, \dots, z_1, \zeta_J, \dots, \zeta_1)} \right| \prod_{j=1}^J dz_j d\zeta_j. \end{aligned}$$

Integrating with respect to  $(z_J, \dots, z_1, \zeta_J, \dots, \zeta_1)$ , we can control the multiple integral by  $C^J$ .  $\square$

## 7. Extension of Fujiwara's stationary phase method

We consider the remainder term  $\Upsilon_{\Delta_{T,0}}(\hbar, \boldsymbol{x}, \boldsymbol{\xi}_0, \boldsymbol{x}_0)$  of the multiple integral

$$\begin{aligned} & \left( \frac{1}{2\pi\hbar} \right)^{dJ} \int_{\mathbb{R}^{2dJ}} e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}} F_{\Delta_{T,0}}(\boldsymbol{x}_{J+1}, \boldsymbol{\xi}_J, \boldsymbol{x}_J, \dots, \boldsymbol{\xi}_1, \boldsymbol{x}_1, \boldsymbol{\xi}_0, \boldsymbol{x}_0) \prod_{j=1}^J d\xi_j d\boldsymbol{x}_j \\ &= e^{\frac{i}{\hbar}\phi_{T,0}(\boldsymbol{x}, \boldsymbol{\xi}_0, \boldsymbol{x}_0)} (D_{\Delta_{T,0}}(\boldsymbol{x}, \boldsymbol{\xi}_0)^{-1/2} F_{T,0}(\boldsymbol{x}, \boldsymbol{\xi}_0, \boldsymbol{x}_0) + \hbar \Upsilon_{\Delta_{T,0}}(\hbar, \boldsymbol{x}, \boldsymbol{\xi}_0, \boldsymbol{x}_0)). \end{aligned}$$

**Lemma F** Let  $m \geq 0$ . Assume that for any integer  $M \geq 0$ , there exist  $A_M$ ,  $X_M > 0$  such that for **any**  $\Delta_{T,0}$ , any  $|\alpha_j|, |\beta_{j-1}| \leq M$ ,  $j = 1, 2, \dots, J, J+1$ ,

$$\begin{aligned} & \left| \left( \prod_{j=1}^{J+1} \partial_{x_j}^{\alpha_j} \partial_{\xi_{j-1}}^{\beta_{j-1}} \right) F_{\Delta_{T,0}}(\boldsymbol{x}_{J+1}, \boldsymbol{\xi}_J, \dots, \boldsymbol{x}_1, \boldsymbol{\xi}_0, \boldsymbol{x}_0) \right| \\ & \leq A_M (X_M)^{J+1} \left( \prod_{j=1}^{J+1} (t_j)^{\min(|\beta_{j-1}|, 1)} \right) \left( 1 + \sum_{j=1}^{J+1} (|\boldsymbol{x}_j| + |\boldsymbol{\xi}_{j-1}|) + |\boldsymbol{x}_0| \right)^m. \end{aligned}$$

Then we can control the multiple integral by  $C$ , i.e.,

$$|\Upsilon_{\Delta_{T,0}}(\hbar, \boldsymbol{x}, \boldsymbol{\xi}_0, \boldsymbol{x}_0)| \leq CT(1 + |\boldsymbol{x}_{J+1}| + |\boldsymbol{\xi}_0| + |\boldsymbol{x}_0|)^m.$$

**Remark** For  $J = 0, 1, 2$ , the condition of  $F_{\Delta_{T,0}}$  becomes the following:

For the division  $T = T_1 > T_0 = 0$  with  $J = 0$ ,

$$|\partial_{x_1}^{\alpha_1} \partial_{\xi_0}^{\beta_0} F_{T,0}(x_1, \xi_0, x_0)| \leq A_M (X_M)^1 (t_1)^{\min(|\beta_0|, 1)} (1 + |x_1| + |\xi_0| + |x_0|)^m.$$

For the division  $T = T_2 > T_1 > T_0 = 0$  with  $J = 1$ ,

$$\begin{aligned} & |\partial_{x_2}^{\alpha_2} \partial_{\xi_1}^{\beta_1} \partial_{x_1}^{\alpha_1} \partial_{\xi_0}^{\beta_0} F_{T,T_1,0}(x_2, \xi_1, x_1, \xi_0, x_0)| \\ & \leq A_M (X_M)^2 (t_2)^{\min(|\beta_1|, 1)} (t_1)^{\min(|\beta_0|, 1)} \left(1 + \sum_{j=1}^2 (|x_j| + |\xi_{j-1}|) + |x_0|\right)^m. \end{aligned}$$

For the division  $T = T_3 > T_2 > T_1 > T_0 = 0$  with  $J = 2$ ,

$$\begin{aligned} & |\partial_{x_3}^{\alpha_3} \partial_{\xi_2}^{\beta_2} \partial_{x_2}^{\alpha_2} \partial_{\xi_1}^{\beta_1} \partial_{x_1}^{\alpha_1} \partial_{\xi_0}^{\beta_0} F_{T,T_2,T_1,0}(x_3, \xi_2, x_2, \xi_1, x_1, \xi_0, x_0)| \\ & \leq A_M (X_M)^3 (t_3)^{\min(|\beta_2|, 1)} (t_2)^{\min(|\beta_1|, 1)} (t_1)^{\min(|\beta_0|, 1)} \left(1 + \sum_{j=1}^3 (|x_j| + |\xi_{j-1}|) + |x_0|\right)^m. \end{aligned}$$

**Remark** Since  $q_{\Delta_{T,0}}(t) \approx x_j - t_j \xi_{j-1}$  when  $T_{j-1} < t \leq T_j$ ,

$F[q, p] = B(t, q(t))$  satisfies the above assumption.

## Distinguish the main term from the remainder term

For simplicity, we prove Lemma F when  $m = 0$ .

We must integrate with respect to  $(\xi_1, x_1), (\xi_2, x_2), \dots, (\xi_J, x_J)$ .

First we integrate with respect to  $(\xi_1, x_1)$ . By the stationary phase method, we distinguish the main term  $(\mathcal{M}_1 F_{\Delta_{T,0}})$  from the remainder term  $(\mathcal{R}_1 F_{\Delta_{T,0}})$ .

$$\begin{aligned} & \left( \frac{1}{2\pi\hbar} \right)^d \int_{\mathbb{R}^{2d}} e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}} F_{\Delta_{T,0}}(\dots, x_2, \xi_1, x_1, \xi_0, x_0) d\xi_1 dx_1 \\ &= e^{\frac{i}{\hbar}\phi(\Delta_{T,T_2,0})} (\mathcal{M}_1 F_{\Delta_{T,0}})(\dots, \xi_2, x_2, \xi_0, x_0) \\ &+ e^{\frac{i}{\hbar}\phi(\Delta_{T,T_2,0})} (\mathcal{R}_1 F_{\Delta_{T,0}})(\dots, \xi_2, x_2, \xi_0, x_0). \end{aligned}$$

The main term is ‘simple’ for  $(\xi_2, x_2)$  and given by

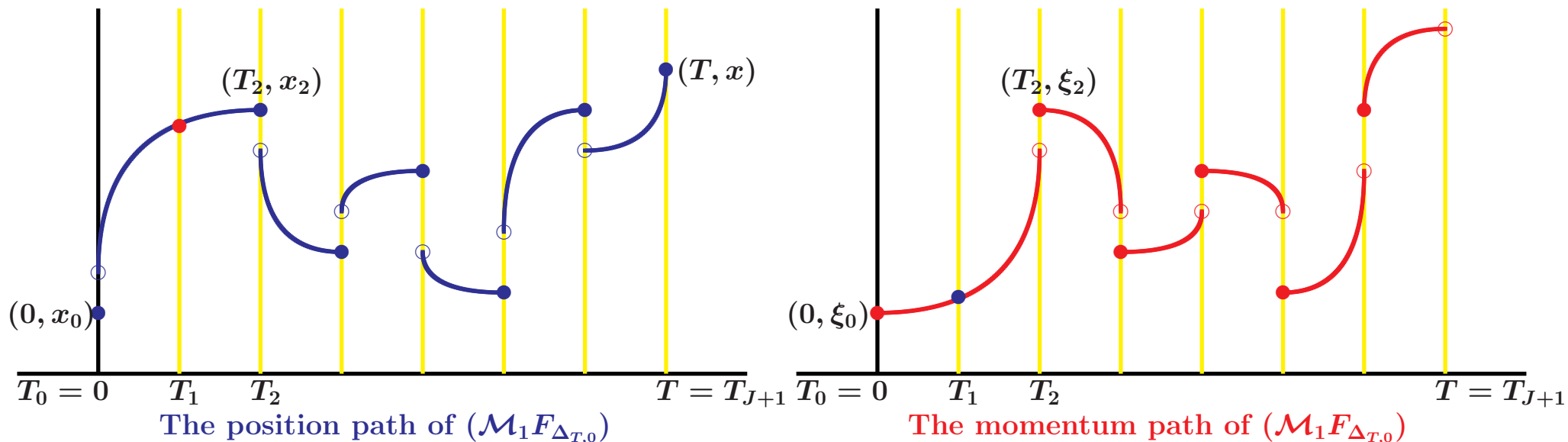
$$\begin{aligned} (\mathcal{M}_1 F_{\Delta_{T,0}})(\dots, \xi_2, x_2, \xi_0, x_0) &= D_{\Delta_{T_2,0}}(x_2, \xi_0)^{-1/2} F_{\Delta_{T,0}}(\dots, \xi_2, x_2, \xi_1^*, x_1^*, \xi_0, x_0) \\ &= D_{\Delta_{T_2,0}}(x_2, \xi_0)^{-1/2} F_{(\Delta_{T,T_2},0)}(\dots, \xi_2, x_2, \xi_0, x_0), \end{aligned}$$

where  $(\Delta_{T,T_2}, 0)$  is the division given by

$$(\Delta_{T,T_2}, 0) : T = T_{J+1} > T_J > \dots > T_2 > T_0 = 0.$$

The remainder term  $(\mathcal{R}_1 F_{\Delta_{T,0}})$  is ‘complicated’ for  $(\xi_2, x_2)$  but small, i.e.,

$$|(\mathcal{R}_1 F_{\Delta_{T,0}})| \leq C(t_2 \hbar).$$



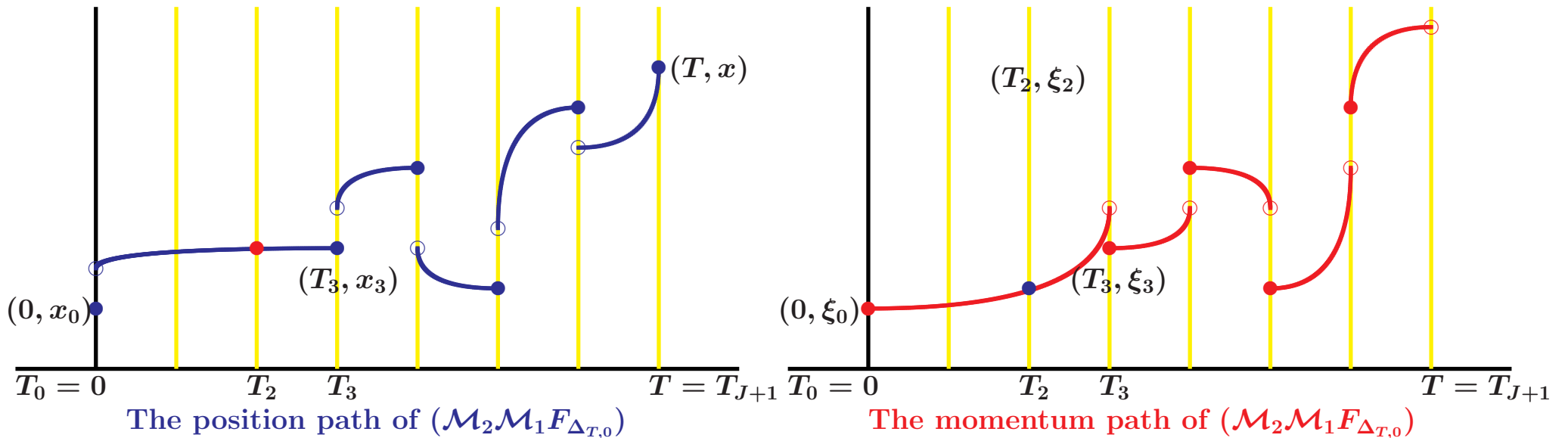
# Do only simple integrals

Since  $(\mathcal{M}_1 F_{\Delta_{T,0}})$  is ‘simple’ for  $(\xi_2, x_2)$ , we integrate it with respect to  $(\xi_2, x_2)$ .

$$\begin{aligned} & \left( \frac{1}{2\pi\hbar} \right)^d \int_{\mathbb{R}^{2d}} e^{\frac{i}{\hbar}\phi(\Delta_{T,T_2,0})} (\mathcal{M}_1 F_{\Delta_{T,0}})(\dots, x_3, \xi_2, x_2, \xi_0, x_0) d\xi_2 dx_2 \\ &= e^{\frac{i}{\hbar}\phi(\Delta_{T,T_3,0})} (\mathcal{M}_2 \mathcal{M}_1 F_{\Delta_{T,0}})(\dots, \xi_3, x_3, \xi_0, x_0) \\ &+ e^{\frac{i}{\hbar}\phi(\Delta_{T,T_3,0})} (\mathcal{R}_2 \mathcal{M}_1 F_{\Delta_{T,0}})(\dots, \xi_3, x_3, \xi_0, x_0), \end{aligned}$$

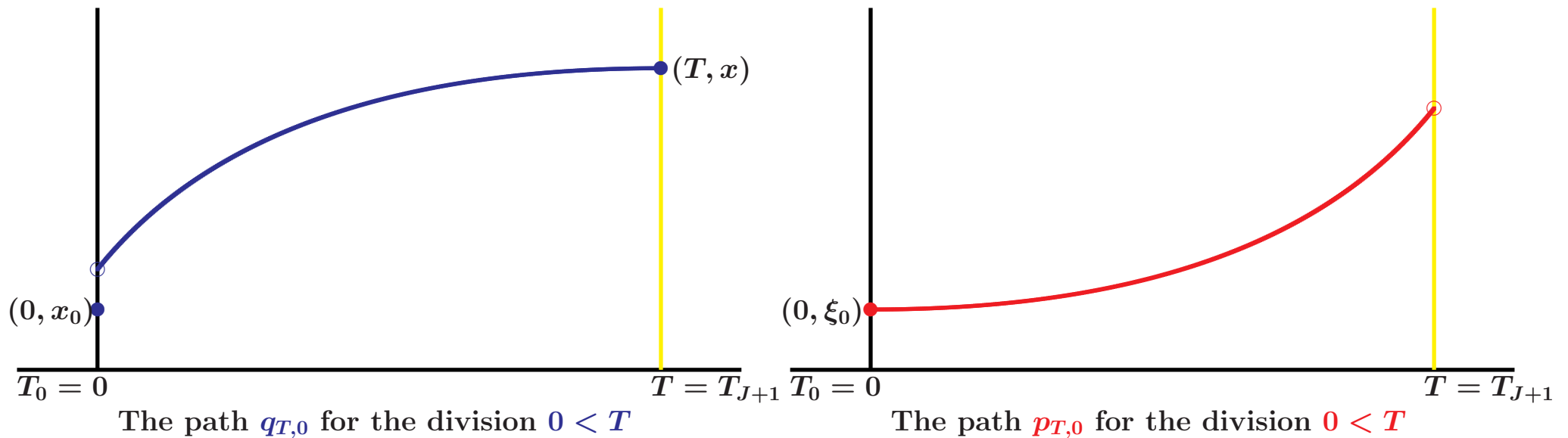
$$(\mathcal{M}_2 \mathcal{M}_1 F_{\Delta_{T,0}}) = D_{\Delta_{T_3,0}}(x_3, \xi_0)^{-1/2} F_{(\Delta_{T,T_3,0})}(\dots, \xi_3, x_3, \xi_0, x_0),$$

$$|(\mathcal{R}_2 \mathcal{M}_1 F_{\Delta_{T,0}})| \leq C(t_3 \hbar).$$



Since the main term  $(\mathcal{M}_2\mathcal{M}_1F_{\Delta_{T,0}})$  is ‘simple’ for  $(\xi_3, x_3)$ , we integrate it further with respect to  $(\xi_3, x_3)$ . Repeating this simple process, we get the main term for Theorem 4,

$$\begin{aligned} (\mathcal{M}_J\mathcal{M}_{J-1}\dots\mathcal{M}_1F_{\Delta_{T,0}}) &= D_{\Delta_{T,0}}(x, \xi_0)^{-1/2}F_{T,0}(x, \xi_0, x_0) \cdot \\ &= D_{\Delta_{T,0}}(x, \xi_0)^{-1/2}F[q_{T,0}, p_{T,0}]. \end{aligned}$$



## Skip all complicated integrals

Now, since  $(\mathcal{R}_1 F_{\Delta_{T,0}})$  is ‘complicated’ for  $(\xi_2, x_2)$ , we skip the integration with respect to  $(\xi_2, x_2)$  and integrate it with respect to  $(\xi_3, x_3)$  beforehand.

$$\begin{aligned} & \left( \frac{1}{2\pi\hbar} \right)^d \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}\phi(\Delta_{T,T_2,0})} (\mathcal{R}_1 F_{\Delta_{T,0}})(\dots, x_4, \xi_3, x_3, \xi_2, x_2, \xi_0, x_0) d\xi_3 dx_3 \\ &= e^{\frac{i}{\hbar}\phi(\Delta_{T,T_4,T_2,0})} (\mathcal{M}_3 \mathcal{R}_1 F_{\Delta_{T,0}})(\dots, \xi_4, x_4, \xi_2, x_2, \xi_0, x_0) \\ &+ e^{\frac{i}{\hbar}\phi(\Delta_{T,T_4,T_2,0})} (\mathcal{R}_3 \mathcal{R}_1 F_{\Delta_{T,0}})(\dots, \xi_4, x_4, \xi_2, x_2, \xi_0, x_0), \end{aligned}$$

$$|(\mathcal{R}_3 \mathcal{R}_1 F_{\Delta_{T,0}})| \leq C(t_4\hbar)C(t_2\hbar).$$

Since the main term  $(\mathcal{M}_3 \mathcal{R}_1 F_{\Delta_{T,0}})$  is ‘simple’ for  $(\xi_4, x_4)$ , we integrate it further with respect to  $(\xi_4, x_4)$ . But since the remainder term  $(\mathcal{R}_3 \mathcal{R}_1 F_{\Delta_{T,0}})$  is ‘complicated’ for  $(\xi_4, x_4)$ , we skip the integration with respect to  $(\xi_4, x_4)$  and integrate it with respect to  $(\xi_5, x_5)$  beforehand.

Fujiwara's rule is the following.

Integrate with respect to  $(\xi_j, x_j)$ . By the stationary phase method, we distinguish the main term from the remainder term.

The main term is 'simple' for  $(\xi_{j+1}, x_{j+1})$ .

Therefore, we integrate it further with respect to  $(\xi_{j+1}, x_{j+1})$ .

However, the remainder term is 'complicated' for  $(\xi_{j+1}, x_{j+1})$ .

Therefore, we skip the integration with respect to  $(\xi_{j+1}, x_{j+1})$  and integrate it with respect to  $(\xi_{j+2}, x_{j+2})$  beforehand.

Carry out the rule until the end.

Then we have

$$q_{\Delta_{T,0}}(\hbar, x_{J+1}, \xi_0, x_0) = q_0(x_{J+1}, \xi_0, x_0) + \sum' q_{j_K, j_{K-1}, \dots, j_1}(x_{J+1}, \xi_0, x_0), \cdot$$

Here  $q_0(x_{J+1}, \xi_0, x_0) = D_{\Delta_{T,0}}(x, \xi_0)^{-1/2} F_{T,0}(x_{J+1}, \xi_0, x_0)$  is the main term for Theorem 4,  $\sum'$  means the summation over all the sequence of integers  $(j_{s+1}, j_s, \dots, j_1)$  such that

$$0 = j_0 < j_1 - 1 < j_1 < j_2 - 1 < j_2 < \dots < j_K - 1 < j_K \leq J + 1,$$

and the summand  $q_{j_K, j_{K-1}, \dots, j_1}(x_{J+1}, \xi_0, x_0)$  is the complicated integrals

$$\begin{aligned} & e^{\frac{i}{\hbar} \phi_{T,0}(x_{J+1}, \xi_0, x_0)} q_{j_K, j_{K-1}, \dots, j_1}(x_{J+1}, \xi_0, x_0) \\ &= \int_{\mathbb{R}^{dK}} e^{\frac{i}{\hbar} \phi_{T, T_{j_K}, \dots, T_1, 0}} b_{j_K, j_{K-1}, \dots, j_1}(x_{J+1}, \xi_{j_K}, x_{j_K}, \dots, \xi_{j_1}, x_{j_1}, \xi_0, x_0) \prod_{k=1}^K d\xi_{j_k} dx_{j_k}, \end{aligned}$$

where

$$\begin{aligned}
& b_{j_K, j_{K-1}, \dots, j_1}(x_{J+1}, \xi_{j_K}, x_{j_K}, \dots, \xi_{j_1}, x_{j_1}, \xi_0, x_0) \\
& = (\mathcal{Q}_J \cdots \mathcal{Q}_3 \mathcal{Q}_2 \mathcal{Q}_1 F_{\Delta T, 0})(x_{J+1}, \xi_{j_K}, x_{j_K}, \dots, \xi_{j_1}, x_{j_1}, \xi_0, x_0),
\end{aligned}$$

with

$$\mathcal{Q}_j = \begin{cases} \text{Identity} & \text{if } j = j_K, j_{K-1}, \dots, j_1 \\ \mathcal{R}_j & \text{if } j = j_K - 1, j_{K-1} - 1, \dots, j_1 - 1 \text{ .} \\ \mathcal{M}_j & \text{otherwise} \end{cases}$$

Therefore the integrand has many small terms  $(t_{j_k} \hbar)$ , i.e.,

$$\begin{aligned}
& |b_{j_K, j_{K-1}, \dots, j_1}(x_{J+1}, \xi_{j_K}, x_{j_K}, \dots, \xi_{j_1}, x_{j_1}, \xi_0, x_0)| \\
& \leq C^K \left( \prod_{k=1}^K (t_{j_k} \hbar) \right).
\end{aligned}$$

## Force all complicated integrals on others.

We force on H. Kumano-go-Taniguchi's theorem all the complicated integrals which we skipped. Then we have

$$|q_{j_K, j_{K-1}, \dots, j_1}(x_{J+1}, \xi_0, x_0)| \leq (C')^K \left( \prod_{k=1}^K (t_{j_k} \hbar) \right).$$

The remainder term is the sum of  $q_{j_K, j_{K-1}, \dots, j_1}(x_{J+1}, \xi_0, x_0)$ , i.e.,

$$\Upsilon_{\Delta_{T,0}}(\hbar, x_{J+1}, \xi_0, x_0) = \frac{1}{\hbar} \sum' q_{j_K, j_{K-1}, \dots, j_1}(x_{J+1}, \xi_0, x_0).$$

Using  $\sum_{j=1}^{J+1} t_j = T$ , we take a sum. Note  $0 < \hbar < 1$ . Then we have

$$\begin{aligned} |\Upsilon_{\Delta_{T,0}}(\hbar, x_{J+1}, x_0)| &\leq \frac{1}{\hbar} \sum' \left( (C')^K \prod_{k=1}^K (t_{j_k} \hbar) \right) \\ &\leq \frac{1}{\hbar} \left( \prod_{j=1}^{J+1} (1 + C' t_j \hbar) - 1 \right) \\ &\leq (C'')T. \quad \square \end{aligned}$$

## 8. Assumption of $F[q, p] \in \mathcal{F}$

**Assumption of  $F[q, p] \in \mathcal{F}$**  Let  $m \geq 0$ ,  $u_j \geq 0$ ,  $\sum_{j=1}^{J+1} u_j \equiv U < \infty$ . For any

integer  $M \geq 0$ , there exist  $A_M, X_M > 0$  such that for any  $\Delta_{T,0}$ , any  $|\alpha_j|$ ,

$|\beta_{j-1}| \leq M$ ,  $j = 1, 2, \dots, J, J+1$  and **any  $1 \leq k \leq J$** ,

$$\begin{aligned}
 & \left| \left( \prod_{j=1}^{J+1} \partial_{x_j}^{\alpha_j} \partial_{\xi_{j-1}}^{\beta_{j-1}} \right) F_{\Delta_{T,0}}(x_{J+1}, \xi_J, \dots, x_1, \xi_0, x_0) \right| \\
 & \leq A_M (X_M)^{J+1} \left( \prod_{j=1}^{J+1} (t_j)^{\min(|\beta_{j-1}|, 1)} \right) \left( 1 + \sum_{j=1}^{J+1} (|x_j| + |\xi_{j-1}|) + |x_0| \right)^m, \\
 & \left| \left( \prod_{j=1}^{J+1} \partial_{x_j}^{\alpha_j} \partial_{\xi_{j-1}}^{\beta_{j-1}} \right) \partial_{x_k} F_{\Delta_{T,0}}(x_{J+1}, \xi_J, \dots, x_1, \xi_0, x_0) \right| \\
 & \leq A_M (X_M)^{J+1} u_k \left( \prod_{j \neq k} (t_j)^{\min(|\beta_{j-1}|, 1)} \right) \left( 1 + \sum_{j=1}^{J+1} (|x_j| + |\xi_{j-1}|) + |x_0| \right)^m.
 \end{aligned}$$

## Consider integrals by paths

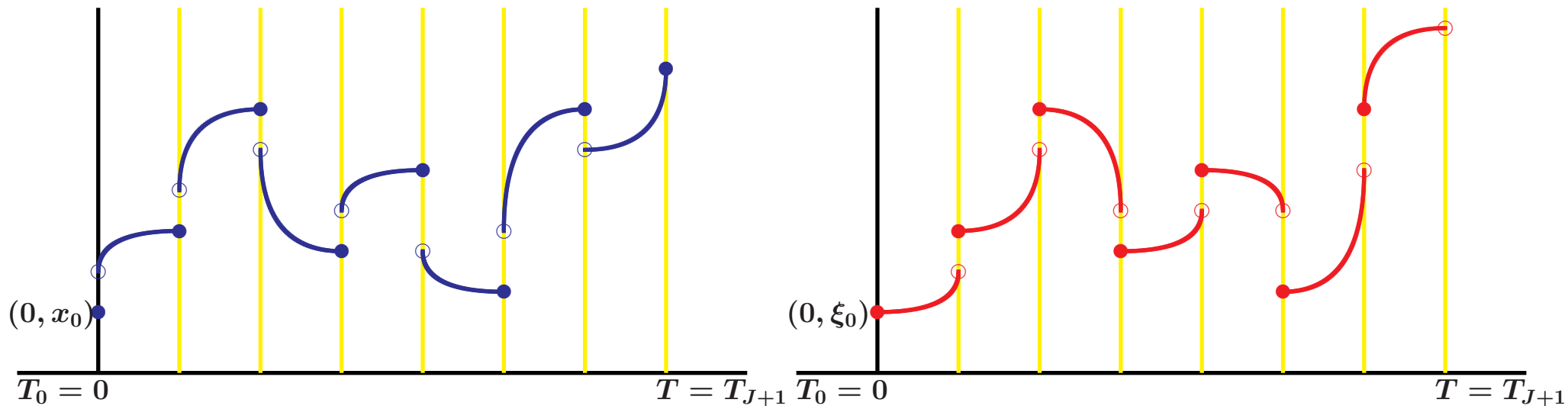
Then the estimates for the multiple integral become the following:

$$\begin{aligned} & \left( \frac{1}{2\pi\hbar} \right)^{dJ} \int_{\mathbb{R}^{2dJ}} e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}} F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \dots, \xi_1, x_1, \xi_0, x_0) \prod_{j=1}^J d\xi_j dx_j \\ &= e^{\frac{i}{\hbar}\phi_{T,0}(x, \xi_0, x_0)} q_{\Delta_{T,0}}(\hbar, x, \xi_0, x_0) \\ &= e^{\frac{i}{\hbar}\phi_{T,0}(x, \xi_0, x_0)} \left( D_{\Delta_{T,0}}(x, \xi_0)^{-1/2} F_{T,0}(x, \xi_0, x_0) + \hbar \Upsilon_{\Delta_{T,0}}(\hbar, x, \xi_0, x_0) \right). \end{aligned}$$

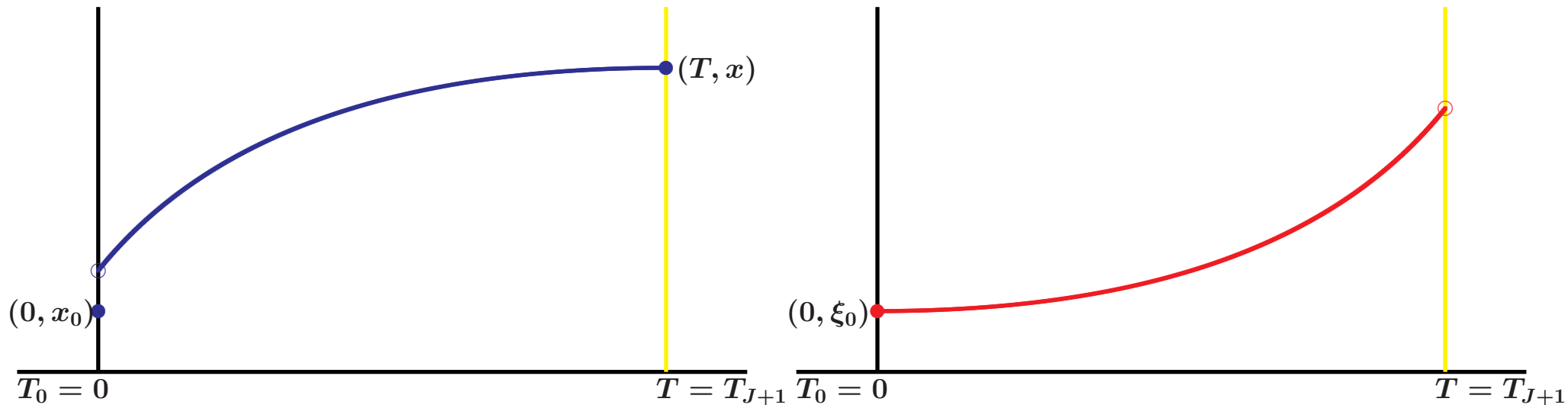
### Lemma F'

$$\begin{aligned} |\Upsilon_{\Delta_{T,0}}(\hbar, x, \xi_0, x_0)| &\leq CT(T+U)(1 + |x_{J+1}| + |\xi_0| + |x_0|)^m, \\ |q_{\Delta_{T,0}}(\hbar, x, \xi_0, x_0)| &\leq C'(1 + |x_{J+1}| + |\xi_0| + |x_0|)^m. \end{aligned}$$

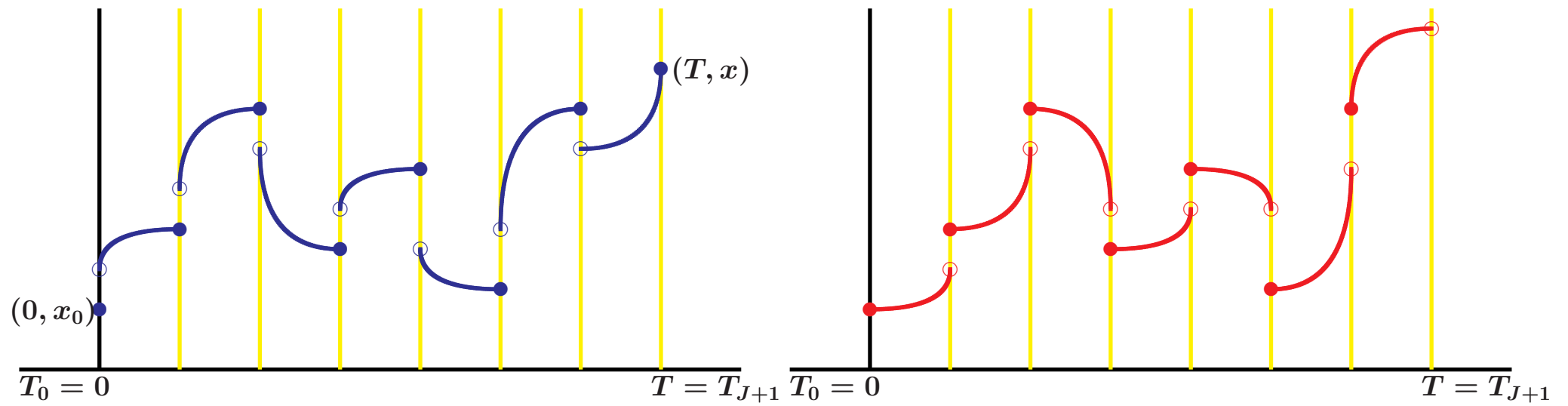
Using paths, we consider this Lemma F'. For simplicity, we set  $m = 0$ .



|



$$\leq CT(T + U)$$



$$\leq C'$$

In order to show the existence of the limit, we have only to make a Cauchy sequence of multiple integrals  $(\star)$  with respect to the division  $\Delta_{T,0}$ .

# Compare two integrals by two paths

For the two divisions

$$\Delta_{T,0} : T = T_{J+1} > T_J > \cdots \cdots \cdots > T_1 > T_0 = 0,$$

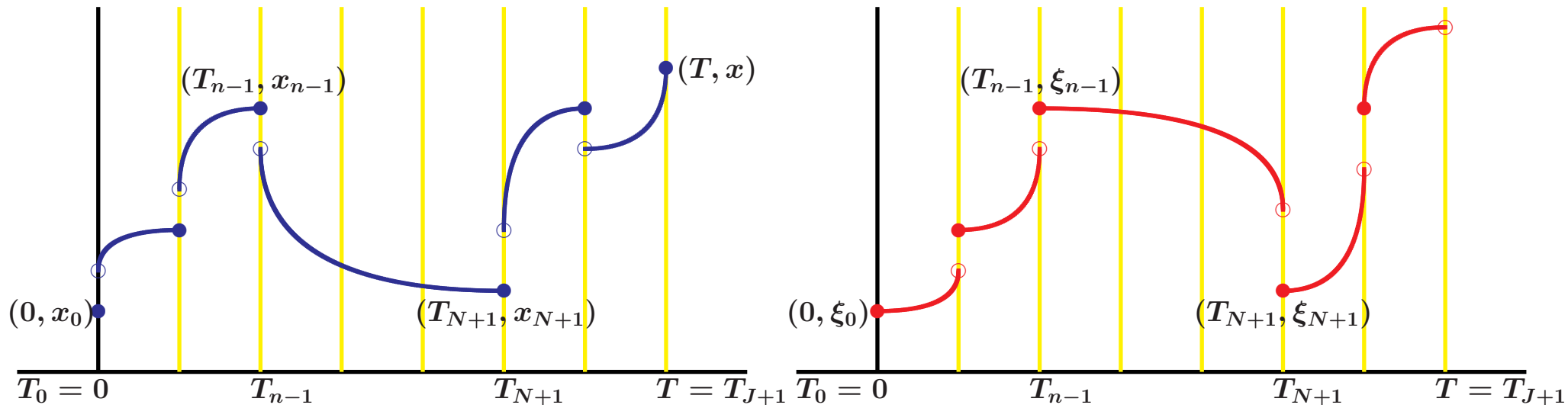
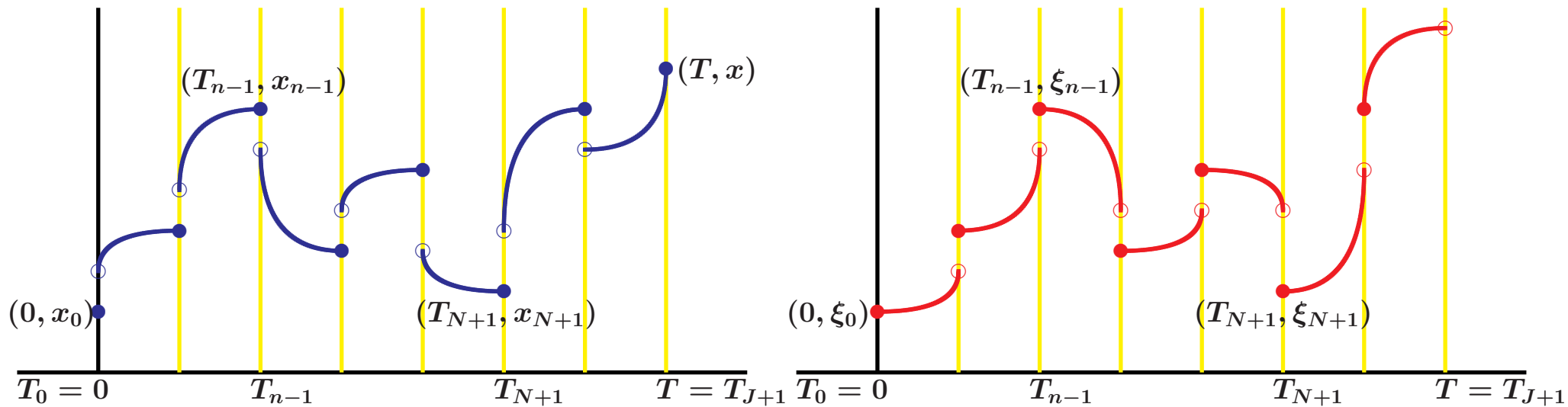
$$(\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0}) : T = T_{J+1} > \cdots > T_{N+1} > T_{n-1} > \cdots > T_0 = 0,$$

we compare the multiple integral

$$\begin{aligned} & \int \cdots \cdots \int \cdots \cdots \int \cdots \cdots \int \prod_{j=1}^J d\xi_j dx_j \\ & = e^{\frac{i}{\hbar} \phi_{T,0}(x, \xi_0, x_0)} q_{\Delta_{T,0}}(\hbar, x, \xi_0, x_0), \end{aligned}$$

and the multiple integral

$$\begin{aligned} & \int \cdots \cdots \int \int \cdots \cdots \int \prod_{j=N+1}^J d\xi_j dx_j \prod_{j=1}^{n-1} d\xi_j dx_j \\ & = e^{\frac{i}{\hbar} \phi_{T,0}(x, \xi_0, x_0)} q_{(\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0})}(\hbar, x, \xi_0, x_0). \end{aligned}$$



$$\leq C' \times C(T_{N+1} - T_{n-1})(T_{N+1} - T_{n-1} + U_{N+1} - U_{n-1}) \times C'.$$

This implies the difference of two multiple integrals.

$$\begin{aligned}
& |q_{\Delta_{T,0}}(\hbar, \mathbf{x}, \boldsymbol{\xi}_0, \mathbf{x}_0) - q_{(\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0})}(\hbar, \mathbf{x}, \boldsymbol{\xi}_0, \mathbf{x}_0)| \\
& \leq C(T_{N+1} - T_{n-1})(T_{N+1} - T_{n-1} + U_{N+1} - U_{n-1})(1 + |\mathbf{x}| + |\boldsymbol{\xi}_0| + |\mathbf{x}_0|)^m.
\end{aligned}$$

**Theorem 2'** (The phase space Feynman path integrals exist)

$$\begin{aligned}
& |q_{\Delta_{T,0}}(\hbar, \mathbf{x}, \boldsymbol{\xi}_0, \mathbf{x}_0) - q(T, \hbar, \mathbf{x}, \boldsymbol{\xi}_0, \mathbf{x}_0)| \\
& \leq C|\Delta_{T,0}|(U + T)(1 + |\mathbf{x}| + |\boldsymbol{\xi}_0| + |\mathbf{x}_0|)^m,
\end{aligned}$$

with a limit function  $q(T, \hbar, \mathbf{x}, \boldsymbol{\xi}_0, \mathbf{x}_0) = \lim_{|\Delta_{T,0}| \rightarrow 0} q_{\Delta_{T,0}}(\hbar, \mathbf{x}, \boldsymbol{\xi}_0, \mathbf{x}_0)$ , i.e., the multiple integral  $(\star)$  converges on compact subsets of  $\mathbb{R}^{3d}$  as  $|\Delta_{T,0}| \rightarrow 0$ .

**Remark** We added assumptions closed under  $+$  and  $\times$ . Therefore  $\mathcal{F}$  is an algebra. Furthermore, by accident,  $\mathcal{F}$  contains the previous examples.