A unified approach to the Darwin approximation

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There are two basic approaches to the Darwin approximation. The first involves solving the Maxwell equations in Coulomb gauge and then approximating the vector potential to remove retardation effects. The second approach approximates the Coulomb gauge equations themselves, then solves these exactly for the vector potential. There is no *a priori* reason that these should result in the same approximation. Here, the equivalence of these two approaches is investigated and a unified framework is provided in which to view the Darwin approximation. Darwin's original treatment is variational in nature, but subsequent applications of his ideas in the context of Vlasov's theory are not. We present here action principles for the Darwin approximation in the Vlasov context, and this serves as a consistency check on the use of the approximation in this setting. [DOI: 10.1063/1.2799346]

I. INTRODUCTION

The Darwin approximation of electrodynamics is the order-$(v/c)^2$ approximation to the relativistic interaction of classical charged particles and the electromagnetic field they produce. There have been several different approaches to the study of the Darwin approximation. These fall into two basic types. In the first approach one starts with the full Maxwell equations in terms of the scalar and vector potentials, solves these in Coulomb gauge to obtain exact expressions for the potentials, and then finally approximates the vector potential to remove retardation effects. In the second approach one also begins with the full Maxwell equations in Coulomb gauge, but approximates the equations themselves by removing the time derivatives of the vector potential, and then obtains expressions for the potentials by solving these approximate equations. These two procedures do not necessarily commute, and it remains to see in what sense these approaches may both be termed the Darwin approximation. A major objective of the present paper is to highlight the differences in the approaches and to demonstrate directly how they are related.

The specific setting considered here is that of charges located in an unbounded domain. This may cover a wide range of phenomena pertinent to an astrophysical plasma setting. In the laboratory setting, boundary conditions will inevitably become important.[1–3] Qin *et al.*[4] in particular simulate a system with a combination of cylindrical and periodic boundary conditions. Such situations often call for the introduction of harmonic or vacuum fields in addition to the longitudinal and transverse fields considered here.

The idea of Darwin's original approximation[5] has found application in the description of plasmas, where particle dynamics is replaced by that of a phase space density governed by a Vlasov-like dynamics. As is the case for all of the basic models of plasma physics, one expects this description to possess an action principle formulation.[6] A second objective of the present paper is to construct an action principle for the Vlasov–Darwin approach, thus returning to one of Darwin's own stated purposes: "To reduce the problem…to a Lagrangian form, so that all the well-known theorems of general dynamics may be made applicable." Along the way, we also obtain the field action for the approximated field equations and the combined particle-field action.

The paper is organized as follows. In Secs. II and III we undertake our first objective: To make clear the differences in the two basic approaches to the Darwin approximation and to show their equivalence in the specific case of an unbounded domain. Section II describes the approaches, and Sec. III demonstrates their equivalence. In Sec. IV we consider our second objective: To show that the various approaches all derive from an action principle. We conclude with Sec. V. In the remainder of the present section, we review the original Darwin approximation and its various applications.

In his original paper, Darwin derived an order-$(v/c)^2$ approximation to the fully relativistic motion of charged particles in an unbounded domain by constructing the approximate Lagrangian

$$L=\sum_{a=1}^{n}\left(\frac{m_a\dot{q}_a^2}{2}+\frac{m_a\dot{q}_a^4}{8c^2}\right)-\frac{1}{2}\sum_{a\neq b}\frac{e_ae_b}{r_{ab}}+\frac{1}{2}\sum_{a\neq b}\frac{e_ae_b}{2c^2r_{ab}}[\dot{\mathbf{q}}_a\cdot\dot{\mathbf{q}}_b+(\dot{\mathbf{q}}_a\cdot\hat{\mathbf{r}}_{ab})(\dot{\mathbf{q}}_b\cdot\hat{\mathbf{r}}_{ab})],\tag{1}$$

where $\mathbf{r}_{ab} := \mathbf{q}_a - \mathbf{q}_b$, $r_{ab} := \|\mathbf{r}_{ab}\|$, and $\hat{\mathbf{r}}_{ab} := \mathbf{r}_{ab}/r_{ab}$. (See also, e.g., Ref. 7, Sec. 65, pp. 165–169, and Ref. 8, Sec. 12.6, pp. 596–598.) The expression for the Lagrangian of Eq. (1) has a kinetic part deriving from an expansion in powers of v/c of the term $\sqrt{1-(v/c)^2}$ in the fully relativistic Lagrangian; the interaction stems from the coupling with the fields of other particles through the potential $A^{\mu} = (\phi, \mathbf{A})$, where

$$\phi(\mathbf{x},t) = \sum_b \frac{e_b}{\|\mathbf{x} - \mathbf{q}_b\|},$$
$$\mathbf{A}(\mathbf{x},t) = \sum_b \frac{e_b[\dot{\mathbf{q}}_b + (\dot{\mathbf{q}}_b \cdot \hat{\mathbf{r}}_b)\hat{\mathbf{r}}_b]}{2c\|\mathbf{x} - \mathbf{q}_b\|}, \tag{2}$$

with $\mathbf{q}_b$ the position of the bth particle at time t and $\dot{\mathbf{q}}_b := d\mathbf{q}_b/dt$ its velocity. By working to this order, one eliminates the complexities of dealing with retarded quantities, and thus calculation becomes more practicable. Darwin proceeded to apply the results to a problem of extreme interest at the time: The Bohr–Sommerfeld atom.

The Darwin approximation has found numerous applications in the area of quantum mechanics. Fewer investigations have focused on the Darwin Lagrangian from a purely classical mechanical or dynamical systems point of view. Some, such as Dettwiller,[9] have studied conserved quantities in the system.

By far the majority of literature using the Darwin system in a classical setting employs it as a simplifying approximation, in essence as it was originally intended. Several of these investigations pertain specifically to plasma physics. For example, Appel and Alastuey[10,11] have investigated the equilibrium properties of low-density, one-component plasmas governed by the Darwin interaction. Kaufman and Soda[12] applied the Darwin approximation to study the magnetic susceptibility of an imperfect gas. Kaufman and Rostler[13] in 1971 were among the first to propose the Darwin model as a self-standing basis of plasma simulation. Several have taken up the idea, and the Darwin model has found its greatest applications in particle-in-cell plasma simulation codes (cf. Nielson and Lewis[14]). Hewett[15] has given an overview of applications of the Darwin model to the simulation of low-frequency plasma phenomena: In Mirrortron simulations, simulating the fields of an ion-bunch accelerated by repulsion from a trailing tail of ions; and in simulating instabilities arising in a plasma column with stationary ions and counter-streaming electron components. Pritchett[16] points out that, although the Darwin approximation works well in two-dimensional simulations, it suffers from difficulties in three dimensions with nonperiodic boundary conditions. Weitzner and Lawson[2] take up the problem of boundary conditions, basing considerations on closeness of approximation to Maxwell's equations and on conservation of charge and energy. In the mathematics literature, Degond and Raviart,[3] Benachour *et al.*,[17] and Bauer and Kunze[18] have discussed the existence of solutions of the Darwin system.

II. APPROACHES TO THE DARWIN APPROXIMATION

In Sec. II A we discuss the approach that leads directly to Darwin's original form of the vector potential $\mathbf{A}_C^{qs}$ of Eq. (6). This method finds exact solutions to Maxwell's equations in Coulomb gauge over an unbounded region, then approximates the vector potential. We then demonstrate with hindsight that the approximated vector potential obeys Eq. (9). In Sec. II B we discuss the approach which first approximates the equations of motion, resulting in Eq. (11), with solution $\mathbf{A}_T$ of Eq. (12). Finally, in Sec. II C we show that the two approaches yield the same equations (15) for the potentials, which can be directly solved to give $\mathbf{A}_D$ in terms of the full current $\mathbf{J}$ as in Eq. (16); we show in Eq. (20) that this is in fact equivalent to Darwin's form of the interaction $\mathbf{A}_C^{qs}$.

A. Quasistatic approach

The quasistatic approach (cf. Ref. 19) begins with Maxwell's equations in terms of potentials,

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -4\pi\rho,$$
$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right) = -\frac{4\pi}{c}\mathbf{J}, \tag{3}$$

and employs the Coulomb condition $\nabla \cdot \mathbf{A} = 0$, resulting in equations

$$\nabla^2\phi = -4\pi\rho,$$
$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial\nabla\phi}{\partial t}. \tag{4}$$

The solutions to these Coulomb gauge equations in the case of an unbounded region can be written as

$$\phi_C(\mathbf{x},t) = \int d^3x' \frac{\rho(\mathbf{x}',t)}{r} \tag{5}$$

and

$$\mathbf{A}_C(\mathbf{x},t) = \frac{1}{c}\int d^3x' \frac{1}{r}\{\mathbf{J}(\mathbf{x}',t') - \hat{\mathbf{r}}[\hat{\mathbf{r}} \cdot \mathbf{J}(\mathbf{x}',t')]\}_{\text{ret}}$$
$$+ c\int d^3x' \frac{1}{r}\int_0^{r/c} d\tau\,\tau\{3\hat{\mathbf{r}}[\hat{\mathbf{r}} \cdot \mathbf{J}(\mathbf{x}',t-\tau)] - \mathbf{J}(\mathbf{x}',t-\tau)\},$$

where $r := \|\mathbf{x} - \mathbf{x}'\|$ and $\hat{\mathbf{r}} = (\mathbf{x} - \mathbf{x}')/r$, "ret" denotes that the quantities in brackets are evaluated at the retarded time $t' = t - r/c$, and the spatial integration extends over $\mathbb{R}^3$. The instantaneous or "quasistatic" form of the vector potential results from the substitution $\mathbf{J}(\mathbf{x}', t-\tau) \to \mathbf{J}(\mathbf{x}', t)$ in the second integral in $\mathbf{A}_C$ and removing "ret" from the first integral; i.e., the current is evaluated at the time t. This gives

$$\mathbf{A}_C^{qs}(\mathbf{x},t) = \frac{1}{2c}\int d^3x' \frac{1}{r}\{\mathbf{J}(\mathbf{x}',t) + \hat{\mathbf{r}}[\hat{\mathbf{r}} \cdot \mathbf{J}(\mathbf{x}',t)]\}. \tag{6}$$

This is clearly the continuum analog of the vector potential in Eq. (2). This approach has the virtue of yielding explicitly the form of the potentials derived by Darwin in his original work, manifestly containing terms only up to order $(v/c)^2$.

Given the quasistatic potentials ϕ_C and $\mathbf{A}_C^{qs}$, one may turn the question around and seek the explicit equations which they satisfy. The scalar potential clearly satisfies Poisson's equation,

$$\nabla^2 \phi_C = -4\pi\rho. \tag{7}$$

A direct computation shows the quasistatic vector potential obeys

$$\nabla^2 \mathbf{A}_C^{qs} = -\frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\int d^3x' \frac{1}{r^3}[\mathbf{J}' - 3(\mathbf{J}'\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}],$$

with $\mathbf{J}' := \mathbf{J}(\mathbf{x}',t)$.

To proceed further, recall that a vector $\mathbf{J}$, under conditions of sufficient differentiability, may be decomposed into transverse and longitudinal components as $\mathbf{J}=\mathbf{J}_T+\mathbf{J}_L$, where

$$\mathbf{J}_T(\mathbf{x},t) = \frac{1}{4\pi}\nabla\times\nabla\times\int d^3x' \frac{\mathbf{J}(\mathbf{x}',t)}{r},$$
$$\mathbf{J}_L(\mathbf{x},t) = -\frac{1}{4\pi}\nabla\int d^3x' \frac{\nabla'\cdot\mathbf{J}(\mathbf{x}',t)}{r}. \tag{8}$$

The longitudinal component may be directly re-expressed as

$$\mathbf{J}_L(\mathbf{x},t) = \frac{1}{4\pi}\int d^3x' \frac{1}{r^3}[\mathbf{J}' - 3(\mathbf{J}'\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}],$$

so that $\mathbf{A}_C^{qs}$ satisfies the equation

$$\nabla^2 \mathbf{A}_C^{qs}(\mathbf{x},t) = -\frac{4\pi}{c}\mathbf{J}_T(\mathbf{x},t). \tag{9}$$

The vector potential therefore satisfies Poisson's equation, where the source is not given by the full current, but only by its transverse component $\mathbf{J}_T$.

It is worth noting, since we will have cause to return to the point in later sections, that charge conservation as given by the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0$$

follows automatically from the potential equations (3), a fact which remains true under the change of gauge giving Eq. (4). On the other hand, the equations satisfied by ϕ_C and $\mathbf{A}_C^{qs}$ [Eqs. (7) and (9)] do not enforce charge conservation.

B. Operator approximation approach

The second approach exploits the notion that the Darwin approximation neglects retardation, but, in contrast to Darwin's own derivation, it employs the approximation scheme at the level of the equations themselves. More specifically, one again begins with the full Maxwell equations for the scalar and vector field: Eq. (3). One then argues, by analogy with the case for the Lorenz gauge, that the term $(1/c^2)\partial^2\mathbf{A}/\partial t^2$ is responsible for the retardation. This term may therefore be dropped from the equations, essentially leading to a change of the differential operator in the equation. The fact that $\mathbf{A}$ scales as $1/c$, so that the term overall scales as $1/c^3$, makes this more plausible in the context of the Darwin approximation. The resulting equations are therefore

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = -4\pi\rho,$$
$$\nabla^2\mathbf{A} - \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right) = -\frac{4\pi}{c}\mathbf{J}. \tag{10}$$

The removal of the time-derivative term from the full Maxwell equations implies that the continuity equation is no longer automatically valid. In fact,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = -\frac{1}{4\pi c}\frac{\partial^2}{\partial t^2}\nabla\cdot\mathbf{A}.$$

If the sources ρ and $\mathbf{J}$ do indeed preserve charge conservation, then the ansatz $\nabla\cdot\mathbf{A}=0$ becomes natural. With this ansatz, Eq. (10) becomes

$$\nabla^2\phi = -4\pi\rho,$$
$$\nabla^2\mathbf{A} - \frac{1}{c}\frac{\partial\nabla\phi}{\partial t} = -\frac{4\pi}{c}\mathbf{J}. \tag{11}$$

We now find a solution to these equations and show that $\mathbf{A}$ does indeed satisfy the condition of the ansatz.

The solution of the first equation is the usual Coulomb potential given in Eq. (5). The solution of the second equation is

$$\mathbf{A}_T(\mathbf{x},t) = \frac{1}{c}\int d^3x' \frac{\mathbf{J}(\mathbf{x}',t)}{r} - \frac{1}{4\pi c}\frac{\partial}{\partial t}\int d^3x' \frac{\nabla'\phi_C(\mathbf{x}',t)}{r}. \tag{12}$$

We then check to see that this is divergence free. Using the equation for ϕ_C, one computes

$$\nabla\cdot\mathbf{A}_T = \frac{1}{c}\int d^3x' \frac{1}{r}\left[\nabla'\cdot\mathbf{J}(\mathbf{x}',t) + \frac{\partial\rho(\mathbf{x}',t)}{\partial t}\right].$$

Thus, $\mathbf{A}_T$ is divergence free when the sources $(\rho,\mathbf{J})$ satisfy charge conservation.

Important to note in this context is that this theory, as expressed by Eq. (11), is not gauge invariant. In terms of the fields $\mathbf{E}$ and $\mathbf{B}$, a gauge change by a function χ leaves $\nabla\cdot\mathbf{E}'=4\pi\rho$ invariant, but not the second equation,

$$\frac{4\pi}{c}\mathbf{J} = \nabla\times\mathbf{B}' + \frac{1}{c}\frac{\partial\nabla\phi'}{\partial t} = \nabla\times\mathbf{B} + \frac{1}{c}\frac{\partial\nabla\phi}{\partial t} + \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2}.$$

Thus, Eqs. (11) are invariant under gauge transformations when the function χ is of the same order in powers of $1/c$ as $\mathbf{A}$.

Within this framework, we now see that, if we impose charge conservation, we may then rewrite the equation for $\mathbf{A}$ given in Eq. (11) in the form of Eq. (9). To do so, we again decompose the current as $\mathbf{J}=\mathbf{J}_T+\mathbf{J}_L$. However, in the expression for $\mathbf{J}_L$ given in Eq. (8), we may use the continuity equation to replace $\nabla'\cdot\mathbf{J}(\mathbf{x}',t)$ by $-\partial\rho(\mathbf{x}',t)/\partial t$, and then use the equation for ϕ_C. This results in

$$\frac{1}{4\pi}\frac{\partial \nabla \phi_C(\mathbf{x},t)}{\partial t} = \mathbf{J}_L(\mathbf{x},t).$$

Substituting this in the left-hand side of Eq. (11) finally results in $\nabla^2\mathbf{A}=-(4\pi/c)(\mathbf{J}-\mathbf{J}_L)=-(4\pi/c)\mathbf{J}_T$. This is exactly Eq. (9), showing that $\mathbf{A}_C^{qs}$ and $\mathbf{A}_T$ satisfy the same equation.

We may view the above system directly in terms of the fields $\mathbf{E}$ and $\mathbf{B}$.[3,17,18,20] We may decompose the electric field as $\mathbf{E}=\mathbf{E}_T+\mathbf{E}_L$, with the conditions $\nabla\cdot\mathbf{E}_T=0$ and $\nabla\times\mathbf{E}_L=\mathbf{0}$. Given this decomposition, the system

$$\frac{1}{c}\frac{\partial \mathbf{E}_L}{\partial t} - \nabla\times\mathbf{B} = -\frac{4\pi}{c}\mathbf{J}, \quad \nabla\cdot\mathbf{E}_L = 4\pi\rho, \tag{13}$$

$$\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + \nabla\times\mathbf{E}_T = \mathbf{0}, \quad \nabla\cdot\mathbf{B} = 0, \tag{14}$$

is equivalent to the Darwin system [Eq. (11)] upon writing $\mathbf{E}_L=-\nabla\phi$, $\mathbf{E}_T=-(1/c)\partial\mathbf{A}/\partial t$, and $\mathbf{B}=\nabla\times\mathbf{A}$.

C. General approach

Both the above approaches lead to field equations

$$\nabla^2\phi_D(\mathbf{x},t) = -4\pi\rho(\mathbf{x},t),$$
$$\nabla^2\mathbf{A}_D(\mathbf{x},t) = -\frac{4\pi}{c}\mathbf{J}_T(\mathbf{x},t), \tag{15}$$

for the Darwin system, where the transverse current $\mathbf{J}_T$ is given by the first expression in Eq. (8). This is supplemented by the Coulomb gauge condition, which in this setting is equivalent to imposing charge conservation.

In this form, the equations for ϕ and $\mathbf{A}$ are decoupled, and thus can be solved easily. The equation for the scalar potential is Poisson's equation, so that inversion of the ∇^2 operator yields $\phi_D=\phi_C$, with the latter given by Eq. (5). The equation for the vector potential may be solved in the same way, component by component. Using the form of $\mathbf{J}_T$ given in Eq. (8), we find

$$\mathbf{A}_D(\mathbf{x},t) = \frac{1}{4\pi c}\int \frac{d^3x'}{r}\left[\nabla'\times\nabla'\times\int d^3x''\frac{\mathbf{J}(\mathbf{x}'',t)}{r_{x'x''}}\right], \tag{16}$$

where $r_{x'x''} := \|\mathbf{x}'-\mathbf{x}''\|$ and $\nabla' := \partial/\partial\mathbf{x}'$. One shows by direct computation that $\mathbf{A}_D$ does satisfy $\nabla\cdot\mathbf{A}_D=0$.

III. EQUIVALENCE OF DARWIN FORMULATIONS

The approaches of both Secs. II A and II B may be characterized by Eq. (15), where $\mathbf{J}_T$ is the transverse component of the full current, and is thus a shorthand for the expression in Eq. (8). However, we now have two expressions $\mathbf{A}_C^{qs}$ and $\mathbf{A}_D$ for the vector potential, Eq. (6) and Eq. (16) respectively, each in some sense a valid representation of the salient features of the Darwin approximation. Since $\mathbf{A}_C^{qs}$ has the explicit form found in Darwin's original calculations, we may inquire whether $\mathbf{A}_D$ can be converted to this form. We know at the outset that it must be possible, since $\mathbf{A}_C^{qs}$ and $\mathbf{A}_D$ satisfy the same equations, and are in the same gauge. They can at most, then, differ by a constant. In the present section we show they are indeed equivalent.

Starting with $\mathbf{A}_D$ as given in Eq. (16), use of the identity $\epsilon_{ijk}\epsilon_{klm}=\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}$ and $\nabla^2(1/\|\mathbf{x}-\mathbf{x}'\|)=-4\pi\delta^{(3)}(\mathbf{x}-\mathbf{x}')$ gives

$$\mathbf{A}_D = \frac{1}{c}\int d^3x'\frac{\mathbf{J}(\mathbf{x}',t)}{\|\mathbf{x}-\mathbf{x}'\|} + \frac{1}{4\pi c}\int\frac{d^3x''d^3x'}{\|\mathbf{x}''-\mathbf{x}'\|}(\mathbf{J}'\cdot\nabla'')\nabla''\frac{1}{\|\mathbf{x}-\mathbf{x}''\|}.$$

Application of the identity $\partial_i''(1/\|\mathbf{x}-\mathbf{x}''\|)=-\partial_i(1/\|\mathbf{x}-\mathbf{x}''\|)$ allows the derivatives in the second term to be extracted from one integral,

$$\mathbf{A}_D = \frac{1}{c}\int d^3x'\frac{\mathbf{J}(\mathbf{x}',t)}{\|\mathbf{x}-\mathbf{x}'\|} + \frac{1}{4\pi c}\int d^3x'(\mathbf{J}'\cdot\nabla)\nabla\int\frac{d^3x''}{\|\mathbf{x}-\mathbf{x}''\|\|\mathbf{x}'-\mathbf{x}''\|}. \tag{17}$$

It remains at this point to perform the integral over $\mathbf{x}''$. Before doing so, however, we note that the gradient with respect to $\mathbf{x}$ acts on this integral. Given this, adding a term independent of $\mathbf{x}$ to the integrand will not change the result. We thus add the term $-1/\|\mathbf{x}''\|^2$ to the integrand $1/\|\mathbf{x}-\mathbf{x}''\|\|\mathbf{x}'-\mathbf{x}''\|$. This gives

$$\mathbf{A}_D = \frac{1}{c}\int d^3x'\frac{\mathbf{J}(\mathbf{x}',t)}{\|\mathbf{x}-\mathbf{x}'\|} + \frac{1}{4\pi c}\int d^3x'(\mathbf{J}'\cdot\nabla)\nabla\left[\int\frac{d^3x''}{\|\mathbf{x}-\mathbf{x}''\|\|\mathbf{x}'-\mathbf{x}''\|} - \int\frac{d^3x''}{\|\mathbf{x}''\|^2}\right]. \tag{18}$$

The addition of the term $-1/\|\mathbf{x}''\|^2$ has the effect of canceling the divergence as $\|\mathbf{x}''\|\to\infty$. Postponing the calculation for a moment and simply using the result that the term in square brackets is equal to $-2\pi\|\mathbf{x}-\mathbf{x}'\|$ [see Eq. (21) below], this becomes

$$\mathbf{A}_D = \frac{1}{c}\int d^3x'\frac{\mathbf{J}(\mathbf{x}',t)}{\|\mathbf{x}-\mathbf{x}'\|} - \frac{1}{2c}\int d^3x'(\mathbf{J}'\cdot\nabla)\nabla\|\mathbf{x}-\mathbf{x}'\|, \tag{19}$$

which finally gives

$$\mathbf{A}_D = \frac{1}{2c}\int d^3x'\frac{\mathbf{J}(\mathbf{x}',t)+\hat{\mathbf{r}}'[\hat{\mathbf{r}}'\cdot\mathbf{J}(\mathbf{x}',t)]}{\|\mathbf{x}-\mathbf{x}'\|} = \mathbf{A}_C^{qs}. \tag{20}$$

Thus, we find that the two forms $\mathbf{A}_C^{qs}$ and $\mathbf{A}_D$ of the vector potential are equivalent.

We now return to the calculation of the integral

$$I_D := \int d^3x''\left(\frac{1}{\|\mathbf{x}''-\mathbf{x}\|\|\mathbf{x}''-\mathbf{x}'\|} - \frac{1}{\|\mathbf{x}''\|^2}\right) \tag{21}$$

in Eq. (18). We break the integral into two parts as $I_D = I_1 - I_2$, with

$$I_1 := \int \frac{d^3x''}{\|\mathbf{x}''-\mathbf{x}\|\|\mathbf{x}''-\mathbf{x}'\|} \quad \text{and} \quad I_2 := \int \frac{d^3x''}{\|\mathbf{x}''\|^2}.$$

Note that both I_1 and I_2 are divergent because of insufficient decay at large $\|\mathbf{x}''\|$, but the divergence is removed upon subtraction.

A simple approach to the calculation of I_1 involves the use of Legendre polynomials P_l. We first make the substitutions $\mathbf{y} := \mathbf{x}'' - \mathbf{x}'$ and $\mathbf{z} := \mathbf{x} - \mathbf{x}'$. We also employ the orthogonality condition

$$\int_{-1}^{1} P_k(\xi)P_l(\xi)d\xi = \frac{2\delta_{kl}}{2l+1}$$

and the expansion

$$\frac{1}{\|\mathbf{y}-\mathbf{z}\|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma),$$

with $r_< := \min\{\|\mathbf{y}\|, \|\mathbf{z}\|\}$, $r_> := \max\{\|\mathbf{y}\|, \|\mathbf{z}\|\}$, and γ the angle between $\mathbf{y}$ and $\mathbf{z}$. Recalling that $P_0(\xi) = 1$, a straightforward calculation shows

$$\begin{aligned} I_1 &= \int \frac{d^3y}{\|\mathbf{y}\|\|\mathbf{y}-\mathbf{z}\|} \\ &= 2\pi \sum_{l=0}^{\infty} \int_0^{\infty} \frac{r_<^l}{r_>^{l+1}} y\,dy \int_{-1}^{1} P_0(\xi)P_l(\xi)d\xi \\ &= 4\pi \int_0^{\infty} \frac{y}{r_>} dy, \end{aligned}$$

where $y := \|\mathbf{y}\|$.

For the integral I_2, we change the variable $\mathbf{x}''$ to $\mathbf{y}$ to find $I_2 = \int d^3x''/\|\mathbf{x}''\|^2 = \int d^3y/y^2 = 4\pi\int_0^{\infty} dy$. Putting the integrals together and writing $z := \|\mathbf{z}\|$, we find

$$\begin{aligned} I_D = I_1 - I_2 &= 4\pi \int_0^{\infty} \left[\frac{y}{r_>} - 1\right] dy \\ &= 4\pi \int_0^{z} \left[\frac{y}{z} - 1\right] dy + 4\pi \int_z^{\infty} [1-1] dy \\ &= -2\pi z \\ &= -2\pi\|\mathbf{x} - \mathbf{x}'\|, \end{aligned}$$

which is the result used in Eq. (19).

IV. ACTION PRINCIPLES

Given the various approaches to the Darwin approximation, it is natural to seek a unified viewpoint via action principles. This is facilitated by the analysis of Secs. II and III, culminating in the equivalence demonstrated in Eq. (20). Essential motivation comes from the fact that an action principle provides a natural method by which to ensure consistency. In approximation schemes such as the Darwin approximation, lack of a top-down viewpoint can open the door to inconsistency in the order of approximation, and therefore to the rise of artifacts of calculation not properly due to physical processes.

Moreover, Darwin's original formulation focused on the interaction between particles, while the discussion of Secs. II and III focused on the field equations. In Sec. IV A we demonstrate an action for the Darwin field equations (10). In Sec. IV B we describe a combined action principle for particles interacting with the fields of other particles all subject to the Darwin approximation. Both the field and particle equations follow from a single action principle. The particle equation of motion produced from this action differs somewhat from that found elsewhere in the literature. This is then generalized in Sec. IV C, where we discuss a noncanonical action principle and include a particle distribution function.

A. Darwin field action

By analogy with the Maxwell field action

$$S_{\mathrm{Mf}}[\phi,\mathbf{A}] = \int dt d^3x \left[-\rho\phi + \frac{1}{c}\mathbf{J}\cdot\mathbf{A}\right] + \frac{1}{8\pi}\int dt d^3x [E^2 - B^2]$$

giving the Maxwell equations (3) for the electromagnetic potentials coupled to sources, we may look for an action that yields the Darwin field equations as given in Eq. (10). If we decompose $\mathbf{E}$ into an irrotational part $\mathbf{E}_L := -\nabla\phi$ and a divergence-free part $\mathbf{E}_T := -(1/c)\,\partial\mathbf{A}/\partial t$, we can then see that the term E_T^2 must be dropped since, by an argument similar to the one at the beginning of Sec. II B, it scales as $1/c^3$ and is responsible for the retardation. Thus, we obtain the following action:

$$S_{\mathrm{Df}}[\phi,\mathbf{A}] = \int dt d^3x \left[-\rho\phi + \frac{1}{c}\mathbf{J}\cdot\mathbf{A}\right] + \frac{1}{8\pi}\int dt d^3x [E^2 - E_T^2 - B^2],$$

whose variation with respect to its arguments yields Eq. (10).

B. Darwin particle-field and particle actions

The initial impetus of Darwin's original work was to describe the motion of charged particles in the field of other charged particles. This remains the main thrust of most applications of the Darwin approximation, where one seeks to evolve many-particle systems without the computationally expensive complication of retardation. With this in mind, we formulate a combined particle-field action for the Darwin system, by analogy with the full Maxwell particle-field action. Expanding the kinetic term $\sqrt{1-(\dot{q}_a/c)^2}$ to second order in $\dot{q}_a/c$ and dropping the constant, we find

$$S_D[\mathbf{q};\phi,\mathbf{A}] = \int dt \Bigg\{ \sum_a \left[\frac{m_a \dot{q}_a^2}{2} + \frac{m_a \dot{q}_a^4}{8c^2} \right] + \sum_a e_a \int d^3x\, \delta^{(3)}(\mathbf{x}-\mathbf{q}_a) \left[-\phi(\mathbf{x},t) + \frac{\dot{\mathbf{q}}_a}{c}\cdot \mathbf{A}(\mathbf{x},t) \right] + \frac{1}{8\pi} \int d^3x [E^2(\mathbf{x},t) - E_T^2(\mathbf{x},t) - B^2(\mathbf{x},t)] \Bigg\}.$$

Variation with respect to the field variables ϕ and $\mathbf{A}$ reproduces Eq. (10) as before, with the imposition of the Coulomb condition giving (15). The density ρ and current $\mathbf{J}$ are given in terms of particles by

$$\rho(\mathbf{x},t) = \sum_b e_b \delta^{(3)}[\mathbf{x}-\mathbf{q}_b(t)],$$
$$\mathbf{J}(\mathbf{x},t) = \sum_b e_b \dot{\mathbf{q}}_b(t) \delta^{(3)}[\mathbf{x}-\mathbf{q}_b(t)]. \tag{22}$$

We may solve the resulting field equations for the fields ϕ and $\mathbf{A}$ due to the particles. Solution in terms of ρ and $\mathbf{J}$ yields the usual Coulomb scalar potential ϕ_C, and use of the continuity equation gives the equation for the vector potential the form in Eq. (15). The solution $\mathbf{A}_D$ is therefore given by Eq. (16), which may be rewritten as $\mathbf{A}_C^{qs}$. Insertion of the expressions (22) then produces the expressions given in Eq. (2), so that substitution back into S_D yields an action in terms of particles alone. This action has the form $S_D[\mathbf{q}]=\int dt\, L$, with L given by Eq. (1), bringing the procedure full circle.

We may use this action formulation to derive the equation of motion for the particles. It is convenient to write the solutions of the field equations as

$$\phi(\mathbf{x},t) = \int d^3x' K(\mathbf{x}|\mathbf{x}')\rho(\mathbf{x}',t),$$
$$A_i(\mathbf{x},t) = \frac{1}{c}\int d^3x' K_{ij}(\mathbf{x}|\mathbf{x}') J_j(\mathbf{x}',t), \tag{23}$$

with kernels

$$K(\mathbf{x}|\mathbf{x}') \coloneqq \frac{1}{\|\mathbf{x}-\mathbf{x}'\|},$$
$$K_{ij}(\mathbf{x}|\mathbf{x}') \coloneqq \frac{1}{2\|\mathbf{x}-\mathbf{x}'\|}\left[\delta_{ij} + \frac{(x_i - x_i')(x_j - x_j')}{\|\mathbf{x}-\mathbf{x}'\|^2} \right]. \tag{24}$$

This makes explicit the fact that the kernels of the two potentials are symmetric in their arguments. Insertion of the relations (22) yields the potentials centered on the particles: $\phi(\mathbf{x},t)=\Sigma_b e_b K(\mathbf{x}|\mathbf{q}_b)$ and $A_i(\mathbf{x},t)=(1/c)\Sigma_b e_b K_{ij}(\mathbf{x}|\mathbf{q}_b)\dot{q}_{bj}$. In this form the action becomes

$$S_D[\mathbf{q}] = \int dt \Bigg\{ \sum_a T(\dot{\mathbf{q}}_a) + \frac{1}{2}\sum_{a\neq b} e_a e_b \times \left[K(\mathbf{q}_a|\mathbf{q}_b) + \frac{\dot{q}_{ai}\dot{q}_{bj}}{c^2} K_{ij}(\mathbf{q}_a|\mathbf{q}_b) \right] \Bigg\},$$

with $T(\dot{\mathbf{q}}_a) \coloneqq m_a \dot{q}_a^2/2 + m_a \dot{q}_a^4/8c^2$. Using the symmetry of the kernels, so that $\delta q_{bl}\partial K(\mathbf{q}_a|\mathbf{q}_b)/\partial q_{bl} = \delta q_{bl}\partial K(\mathbf{q}_b|\mathbf{q}_a)/\partial q_{bl} = \delta q_{al}\partial K(\mathbf{q}_a|\mathbf{q}_b)/\partial q_{al}$ and $K_{ij}=K_{ji}$, then the condition of stationary variation $dS_D[\mathbf{q}_a+\epsilon\delta\mathbf{q}_a]/d\epsilon|_{\epsilon=0}=0$ gives

$$\frac{d}{dt}\frac{\partial T(\dot{\mathbf{q}}_a)}{\partial \dot{\mathbf{q}}_a} = -e_a \frac{\partial \phi(\mathbf{q}_a,t)}{\partial \mathbf{q}_a} + \frac{e_a}{c}\frac{\partial}{\partial \mathbf{q}_a}[\dot{\mathbf{q}}_a\cdot\mathbf{A}(\mathbf{q}_a,t)] - \frac{e_a}{c}\frac{d}{dt}\mathbf{A}(\mathbf{q}_a,t).$$

Using standard vector identities and the definitions of $\mathbf{E}_L$, $\mathbf{E}_T$, and $\mathbf{B}$ in terms of potentials, this gives

$$\frac{d}{dt}\frac{\partial T(\dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}} = e(\mathbf{E}_L + \mathbf{E}_T) + \frac{e}{c}\dot{\mathbf{q}}\times\mathbf{B},$$

where we have dropped the particle label a. It remains to compute the derivatives of the kinetic term T. We find

$$\frac{d}{dt}\frac{\partial T(\dot{\mathbf{q}})}{\partial \dot{q}_j} = \frac{d}{dt}\left[\left(1 + \frac{\dot{q}^2}{2c^2}\right) m\dot{q}_j \right] = m\ddot{q}_i A_{ij},$$

where $A_{ij} \coloneqq \delta_{ij} + (\dot{q}^2/c^2)a_{ij}$ and $a_{ij} = \delta_{ij}/2 + \dot{q}_i\dot{q}_j/\dot{q}^2$. Searching for the inverse of A_{ij} to second order in $\dot{q}/c$ gives $A_{ij}^{-1} = \delta_{ij} - (\dot{q}^2/c^2)a_{ij}$. The equation of motion for the ith component of any particle then becomes

$$m\ddot{q}_i = eE_i + \frac{e}{c}(\dot{\mathbf{q}}\times\mathbf{B})_i - e\frac{\dot{q}^2}{c^2}a_{ij}E_{Lj} \tag{25}$$

to order $\dot{q}^2/c^2$. This expression differs from some[21] so-called nonrelativistic implementations of Darwin by the presence of a term proportional to $(\dot{q}/c)^2 E_{Lj}$.

C. Vlasov–Darwin action

Following Low[22] and Ye and Morrison,[23] we formulate the action principle for both Vlasov–Maxwell and Vlasov–Darwin systems by modifying the usual action for Maxwell's equations coupled to particles to account for a continuum of particles; i.e., the characteristics of Vlasov's theory. In these actions, the particles and fields are conventionally varied independently. However, in the case of the Vlasov–Darwin system, because the fields are instantaneously determined by particle states, there is a fairly simple action principle in terms of particle variations alone, and we present this also. This latter action is a continuum action based on the idea of Darwin's original Lagrangian. Lastly, we present the noncanonical Hamiltonian field description of the Vlasov–Darwin system.

Because the Darwin equations of motion [Eq. (25)], as well as the corresponding fully relativistic equations,

$$m\ddot{q}_i = \sqrt{1-\frac{\dot{q}^2}{c^2}}\left(\delta_{ij}-\frac{\dot{q}_i\dot{q}_j}{c^2}\right)\left[eE_j+\frac{e}{c}(\dot{\mathbf{q}}\times\mathbf{B})_j\right], \tag{26}$$

require an initial position and velocity, these may be used to label particles. However, the volume in these velocity phase space coordinates $d^3qd^3\dot{q}$ is not conserved. For convenience, we seek volume preserving coordinates for these dynamics. This is most naturally achieved by starting, in the relativistic case, from the standard fully relativistic Lagrangian for particles in specified fields ϕ and $\mathbf{A}$,

$$L_R = -mc^2\sqrt{1-\frac{\dot{q}^2}{c^2}} - e\phi + \frac{e}{c}\mathbf{A}\cdot\dot{\mathbf{q}},$$

and from a corresponding Lagrangian in the case of the Darwin approximation with an expansion that parallels that of Eq. (1). Using a standard Legendre transformation, we obtain the respective Hamiltonians

$$H_R(\mathbf{q},\mathbf{p},t) = mc^2\sqrt{1+\frac{1}{m^2c^2}\left[\mathbf{p}-\frac{e}{c}\mathbf{A}(\mathbf{q},t)\right]^2} + e\phi(\mathbf{q},t) \tag{27}$$

and

$$H_D(\mathbf{q},\mathbf{p},t) = \frac{1}{2m}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2 - \frac{1}{8m^3c^2}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^4 + e\phi(\mathbf{q},t). \tag{28}$$

The equations of motion for $(\mathbf{q},\mathbf{p})$ derived from these Hamiltonians obscure the elementary form of either the Lorentz law or its approximate Darwin expression. However, we can perform the following noncanonical yet volume preserving transformation:

$$\mathbf{P} = \mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{q},t), \tag{29}$$

and obtain the equations of motion in the following familiar form:

$$\dot{\mathbf{q}} = \frac{\mathbf{P}}{m}\left(1+\frac{P^2}{m^2c^2}\right)^{-1/2},$$
$$\dot{\mathbf{P}} = e\left[\mathbf{E} + \left(1+\frac{P^2}{m^2c^2}\right)^{-1/2}\frac{\mathbf{P}}{mc}\times\mathbf{B}\right], \tag{30}$$

in the Lorentz case, and in the Darwin case

$$\dot{\mathbf{q}} = \frac{\mathbf{P}}{m}\left(1-\frac{P^2}{2m^2c^2}\right),$$
$$\dot{\mathbf{P}} = e\left[\mathbf{E} + \left(1-\frac{P^2}{2m^2c^2}\right)\frac{\mathbf{P}}{mc}\times\mathbf{B}\right]. \tag{31}$$

It can readily be verified that the volume in these phase space coordinates $Z := (\mathbf{q},\mathbf{P})$ [as well as in the $z := (\mathbf{q},\mathbf{p})$ coordinates] is conserved by the above dynamics. However, Eqs. (30) and (31) are not in canonical Hamiltonian form because they arise from an action that is not of the standard phase space action form, but rather of the type

$$S[Z] = \int_{t_0}^{t_1}\left[\theta_\mu(Z,t)\dot{Z}^\mu - H(Z,t)\right]dt,$$

where S is defined on paths $Z(t)$ with appropriately fixed end points. Variation of this general form yields

$$\omega_{\mu\nu}\dot{Z}^\nu = H_{,\mu} + \partial_t\theta_\mu,$$

where $\omega_{\mu\nu} := \theta_{\nu,\mu} - \theta_{\mu,\nu}$ satisfies $\omega_{\nu\mu,\sigma}+\omega_{\mu\sigma,\nu}+\omega_{\sigma\nu,\mu}=0$. If $\omega_{\mu\nu}$ is invertible, then the equations of motion become

$$\dot{Z}^\nu = J^{\nu\mu}(H_{,\mu} + \partial_t\theta_\mu), \tag{32}$$

where $J^{\nu\sigma}\omega_{\sigma\mu} = \delta^\nu_\mu$.

In the present setting, the noncanonical relativistic action takes the form

$$S_R[\mathbf{q},\mathbf{P}] = \int dt\left[\left(\mathbf{P}+\frac{e}{c}\mathbf{A}\right)\cdot\dot{\mathbf{q}} - mc^2\sqrt{1+\frac{P^2}{m^2c^2}} - e\phi\right],$$

while the noncanonical Darwin action becomes

$$S_D[\mathbf{q},\mathbf{P}] = \int dt\left[\left(\mathbf{P}+\frac{e}{c}\mathbf{A}\right)\cdot\dot{\mathbf{q}} - \frac{P^2}{2m} + \frac{P^4}{8m^3c^2} - e\phi\right].$$

In both cases, we identify $Z := (\mathbf{q},\mathbf{P})$ and $\theta(Z,t) := (\mathbf{P} + e\mathbf{A}/c, \mathbf{0})$, and we note that $\phi = \phi(\mathbf{q},t)$ and $\mathbf{A} = \mathbf{A}(\mathbf{q},t)$. The form ω, and hence J, is the same in both cases,

$$\omega = \left(\begin{array}{c|c} \frac{e}{c}B_{ij} & -\mathbb{1}_{3\times3} \\ \hline \mathbb{1}_{3\times3} & \mathbb{0}_{3\times3}\end{array}\right), \quad J := \left(\begin{array}{c|c} \mathbb{0}_{3\times3} & \mathbb{1}_{3\times3} \\ \hline -\mathbb{1}_{3\times3} & \frac{e}{c}B_{ij}\end{array}\right)$$

with $B_{ij} := \epsilon_{ijk}B_k$. Such symplectic two-forms and the cosymplectic dual J were used by Littlejohn in the context of guiding center perturbation theory.[24] With the above expression for J, Eq. (32) is

$$\begin{pmatrix}\dot{\mathbf{q}}\\ \dot{\mathbf{P}}\end{pmatrix} = \left(\begin{array}{c|c} \mathbb{0}_{3\times3} & \mathbb{1}_{3\times3} \\ \hline -\mathbb{1}_{3\times3} & \frac{e}{c}B_{ij}\end{array}\right)\begin{pmatrix} e\dfrac{\partial\phi}{\partial\mathbf{q}} + \dfrac{e}{c}\dfrac{\partial\mathbf{A}}{\partial t} \\ \left(1+\dfrac{P^2}{m^2c^2}\right)^{-1/2}\dfrac{\mathbf{P}}{m}\end{pmatrix}$$

in the relativistic case, reproducing Eq. (30); and

$$\begin{pmatrix}\dot{\mathbf{q}}\\ \dot{\mathbf{P}}\end{pmatrix} = \left(\begin{array}{c|c} \mathbb{0}_{3\times3} & \mathbb{1}_{3\times3} \\ \hline -\mathbb{1}_{3\times3} & \frac{e}{c}B_{ij}\end{array}\right)\begin{pmatrix} e\dfrac{\partial\phi}{\partial\mathbf{q}} + \dfrac{e}{c}\dfrac{\partial\mathbf{A}}{\partial t} \\ \left(1-\dfrac{P^2}{2m^2c^2}\right)\dfrac{\mathbf{P}}{m}\end{pmatrix}$$

in the Darwin case, reproducing Eq. (31).

Now, as alluded to above, instead of a discrete particle label a, we envision a particle trajectory passing through every point of a six-dimensional phase space; i.e., a continuum of particles. Natural continuum particle labels are the initial position and momentum, $Z_0 := (\mathbf{q}_0, \mathbf{P}_0)$; i.e., the trajectories $\mathbf{q}(\mathbf{q}_0, \mathbf{P}_0, t)$ and $\mathbf{P}(\mathbf{q}_0, \mathbf{P}_0, t)$, which we write compactly as $Z(Z_0, t)$, define an invertible map of the phase space onto itself. We assume that associated with each trajectory is a phase space number density (distribution function) $f_0(Z_0)$, which, like the initial conditions, is constant on the trajectory. To find the phase space density at a phase space observation point $\Xi := (\mathbf{x}, \mathbf{\Pi})$ at time t, one needs to know the phase space density associated with the trajectory that will be there at that instant in time. Setting $\mathbf{x} = \mathbf{q}(\mathbf{q}_0, \mathbf{P}_0, t)$ and $\mathbf{\Pi} = \mathbf{P}(\mathbf{q}_0, \mathbf{P}_0, t)$, or more succinctly, $\Xi = Z(Z_0, t)$, inverting, and substituting into f_0, gives

$$f(\Xi,t)d^6\Xi = f_0(Z_0(\Xi,t))d^6 Z_0, \tag{33}$$

which implies

$$f(\Xi,t) = \left.\frac{f_0(Z_0)}{\left|\dfrac{\partial Z}{\partial Z_0}\right|}\right|_{Z_0(\Xi,t)}, \tag{34}$$

where $|\partial Z/\partial Z_0|$ is the Jacobian determinant, which will be seen *a posteriori* to be unity.

The quantity $f(\Xi, t) = f(\mathbf{x}, \mathbf{\Pi}, t)$ is the usual phase space density of Vlasov's theory, and Eq. (33) or (34) establishes the connection between the Lagrangian phase space trajectories $Z(Z_0, t)$ and the Eulerian distribution function $f(\Xi, t)$. Later we will find it convenient to use the canonical Lagrangian phase space trajectories and their Eulerian counterparts, so we record these here. The canonical trajectories are

$$z(z_0,t) := [\mathbf{q}(\mathbf{q}_0,\mathbf{p}_0,t),\mathbf{p}(\mathbf{q}_0,\mathbf{p}_0,t)], \tag{35}$$

where we recall that $\mathbf{p} = \mathbf{P} + e\mathbf{A}/c$; corresponding to these trajectories is the Eulerian phase space observation point $\zeta := (\mathbf{x}, \boldsymbol{\pi})$, whence the Eulerian phase space density is given by

$$f(\zeta,t)d^6\zeta = f_0[z_0(\zeta,t)]d^6 z_0. \tag{36}$$

The phase space action for the relativistic Vlasov–Maxwell system has three parts:

$$S_{VR}[\mathbf{q},\mathbf{P};\phi,\mathbf{A}] = S_{VRp}[\mathbf{q},\mathbf{P}] + S_{VRc}[\mathbf{q},\mathbf{P};\phi,\mathbf{A}] + S_{VRf}[\phi,\mathbf{A}]; \tag{37}$$

the first contains only the particle or the trajectory degrees of freedom, the second provides the coupling, and the third depends only on the fields. Specifically, these are

$$S_{VRp} = \int dt \int d^6 Z_0 f_{R0}(Z_0)\left[\mathbf{P}\cdot\dot{\mathbf{q}} - mc^2\sqrt{1+\frac{P^2}{m^2c^2}}\right],$$

$$S_{VRc} = \int dt \int d^6 Z_0 f_{R0}(Z_0)\left[-e\phi(\mathbf{q},t) + \frac{e}{c}\mathbf{A}(\mathbf{q},t)\cdot\dot{\mathbf{q}}\right] = \int dt \int d^6 Z_0 f_{R0}(Z_0)\int d^6\Xi\,\delta(\Xi - Z)\left[-e\phi(\mathbf{q},t) + \frac{e}{c}\mathbf{A}(\mathbf{q},t)\cdot\dot{\mathbf{q}}\right]$$

$$= \int dt \int d^6\Xi f_R(\Xi,t)\left[-e\phi(\mathbf{x},t) + \frac{e}{c}\mathbf{A}(\mathbf{x},t)\cdot\dot{\mathbf{q}}\big|_{Z_0(\Xi,t)}\right], \tag{38}$$

$$S_{VRf} = \frac{1}{8\pi}\int dt \int d^3x[E^2 - B^2].$$

Observe that the coupling term is written first in a form suitable for variation with respect to $\mathbf{q}(Z_0, t)$ and $\mathbf{P}(Z_0, t)$, and last in a form suitable for variation with respect to $\phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$. The middle form uses the Dirac delta function on the trajectories to show the equivalence of the two. After deriving the particle equations of motion from this action, we can replace $\dot{\mathbf{q}}|_{Z_0(\Xi,t)}$, which occurs in the current, by an expression in terms of the Eulerian variable Ξ by using the first of Eqs. (30), and we can also show that the Jacobian $|\partial Z/\partial Z_0|$ of Eq. (34) is unity, as mentioned earlier.

Similarly, for the Vlasov–Darwin action

$$S_{VD}[\mathbf{q},\mathbf{P};\phi,\mathbf{A}] = S_{VDp}[\mathbf{q},\mathbf{P}] + S_{VDc}[\mathbf{q},\mathbf{P};\phi,\mathbf{A}] + S_{VDf}[\phi,\mathbf{A}], \tag{39}$$

we write

$$S_{VDp} = \int dt \int d^6 Z_0 f_{R0}(Z_0)\left[\mathbf{P}\cdot\dot{\mathbf{q}} - \frac{P^2}{2m} + \frac{P^4}{8m^3c^2}\right],$$

$$S_{VDc} = S_{VRc}, \tag{40}$$

$$S_{VDf} = \frac{1}{8\pi}\int dt \int d^3x[E^2 - E_T^2 - B^2].$$

In S_{VDf}, the $E^2 - E_T^2 - B^2$ is shorthand for an expression in terms of $(\phi, \mathbf{A})$, using the usual definitions given in Sec. IV A.

Variation of S_{VR} and S_{VD} with respect to $\mathbf{q}$ and $\mathbf{P}$, with the usual condition of $\mathbf{q}$ being fixed at initial and final times, yields the equations of motion (30) and (31), respectively. These equations together with Eq. (33) imply the distribution functions satisfy the Vlasov equations

$$\frac{\partial f_R}{\partial t}+\left(1+\frac{\Pi^2}{m^2c^2}\right)^{-1/2}\frac{\mathbf{\Pi}}{m}\cdot\frac{\partial f_R}{\partial \mathbf{x}} + e\left[\mathbf{E}+\left(1+\frac{\Pi^2}{m^2c^2}\right)^{-1/2}\frac{\mathbf{\Pi}}{mc}\times\mathbf{B}\right]\cdot\frac{\partial f_R}{\partial \mathbf{\Pi}}=0 \tag{41}$$

in the relativistic case, and

$$\frac{\partial f_D}{\partial t}+\left(1-\frac{\Pi^2}{2m^2c^2}\right)\frac{\mathbf{\Pi}}{m}\cdot\frac{\partial f_D}{\partial \mathbf{x}}+e\left[\mathbf{E}+\left(1-\frac{\Pi^2}{2m^2c^2}\right)\times\frac{\mathbf{\Pi}}{mc}\times\mathbf{B}\right]\cdot\frac{\partial f_D}{\partial \mathbf{\Pi}}=0 \tag{42}$$

in the Darwin approximation.

Variation of S_{VR} and S_{VD} with respect to ϕ and $\mathbf{A}$ with the usual endpoint conditions leads to the Maxwell equations in the relativistic case, and in the Darwin approximation to Eq. (10); imposing Coulomb gauge leads to the Darwin field equations (15). In both cases,

$$\rho(\mathbf{x},t)=e\int d^3\Pi f(\mathbf{x},\mathbf{\Pi},t) \tag{43}$$

expresses the density, while the current takes the form

$$\mathbf{J}_R=e\int d^3\Pi f_R(\mathbf{x},\mathbf{\Pi},t)\frac{\mathbf{\Pi}/m}{\sqrt{1+(\Pi/mc)^2}} \tag{44}$$

in the relativistic system, and

$$\mathbf{J}_D=e\int d^3\Pi f_D(\mathbf{x},\mathbf{\Pi},t)\left(1-\frac{\Pi^2}{2m^2c^2}\right)\frac{\mathbf{\Pi}}{m} \tag{45}$$

in the Darwin system.

Alternatively, we could have written the action in terms of the canonical coordinates $z=(\mathbf{q},\mathbf{p})$ and introduced the corresponding phase space observation point $\zeta=(\mathbf{x},\boldsymbol{\pi})$, yielding straightforward equations of motion and the Vlasov equations for the distribution function. In particular, the fully relativistic current in these coordinates becomes

$$\mathbf{J}_R(\mathbf{x},t)=e\int d^3\pi f_R(\mathbf{x},\boldsymbol{\pi},t)\frac{(\boldsymbol{\pi}-e\mathbf{A}/c)/m}{\sqrt{1+\frac{1}{m^2c^2}\left(\boldsymbol{\pi}-\frac{e}{c}\mathbf{A}\right)^2}}, \tag{46}$$

and the Darwin expression becomes

$$\mathbf{J}_D(\mathbf{x},t)=e\int d^3\pi f_D(\mathbf{x},\boldsymbol{\pi},t)\left[1-\frac{1}{2m^2c^2}\times\left(\boldsymbol{\pi}-\frac{e}{c}\mathbf{A}\right)^2\right]\cdot\frac{1}{m}\left(\boldsymbol{\pi}-\frac{e}{c}\mathbf{A}\right). \tag{47}$$

Given the top-down nature of the present unified approach to Darwin approximations, accounting for the order of approximation at the level of the action, we should check that the above expressions, Eq. (45) or equivalently Eq. (47), do not introduce terms of order higher than $(v/c)^2$. The current expressions enter the action in the form of the coupling term $(1/c)\int d^3x\mathbf{J}_D(\mathbf{x},t)\cdot\mathbf{A}(\mathbf{x},t)$. We have seen that $\mathbf{A}$ is already of order $1/c$. Using number conservation in the form

$$d^6\Xi f_D(\Xi,t)=d^3xd^3v\tilde{f}_D(\mathbf{x},\mathbf{v},t),$$

and employing the relation

$$\mathbf{v}=\frac{\mathbf{\Pi}}{m}\left(1-\frac{\Pi^2}{2m^2c^2}\right),$$

where $\mathbf{v}$ is the Eulerian observation variable associated with $\dot{\mathbf{q}}$ [cf. Eq. (31)], we find from Eq. (45) that

$$\int d^3x\mathbf{J}_D\cdot\mathbf{A}=\int d^6\Xi f_D(\Xi,t)\left(1-\frac{\Pi^2}{2m^2c^2}\right)\frac{\mathbf{\Pi}}{m}\cdot\mathbf{A} =\int d^6\Xi f_D(\Xi,t)\mathbf{v}\cdot\mathbf{A} =\int d^3xd^3v\tilde{f}_D(\mathbf{x},\mathbf{v},t)\mathbf{v}\cdot\mathbf{A}.$$

Thus, given that $\mathbf{A}$ is of order v/c, the coupling term $(1/c)\int d^3x\mathbf{J}_D\cdot\mathbf{A}$ has overall order $(v/c)^2$, as expected.

Because the Vlasov–Darwin model employs instantaneous action at a distance, one can parallel the procedure outlined in Ye and Morrison[23] and write an action in terms of $\mathbf{q}$ alone or in terms of $\mathbf{q}$ and $\mathbf{P}$ alone. We work this out for the latter case.

Inserting Eq. (43) into Eq. (23) gives

$$\phi(\mathbf{x},t)=e\int d^6\Xi' f_D(\Xi',t)K(\mathbf{x}|\mathbf{x}'), \tag{48}$$

which can be written as

$$\phi(\mathbf{q})=e\int d^6Z_0'f_{D0}(Z_0')K(\mathbf{q}|\mathbf{q}'), \tag{49}$$

using Eq. (33) and $\Xi=Z(Z_0,t)$. Here, $\mathbf{q}':=\mathbf{q}(Z_0',t)$. Similarly, Eq. (45) gives

$$A_i(\mathbf{x},t)=\frac{e}{c}\int d^6\Xi' f_D(\Xi',t)K_{ij}(\mathbf{x}|\mathbf{x}')\cdot\left(1-\frac{\Pi'^2}{2m^2c^2}\right)\frac{\Pi_j'}{m}, \tag{50}$$

and together with Eq. (31), this gives

$$A_i(\mathbf{q})=\frac{e}{c}\int d^6Z_0'f_{D0}(Z_0')K_{ij}(\mathbf{q}|\mathbf{q}')\dot{q}_j'. \tag{51}$$

Inserting Eqs. (49) and (51) into S_{VD} and rearranging gives

$$S[\mathbf{q},\mathbf{P}]=\int dt\left\{\int d^6Z_0f_{D0}(Z_0)\left[\mathbf{P}\cdot\dot{\mathbf{q}}-\frac{P^2}{2m}+\frac{P^4}{8m^3c^2}\right] -\frac{e^2}{2}\int d^6Z_0\int d^6Z_0'f_{D0}(Z_0)f_{D0}(Z_0')K(\mathbf{q}|\mathbf{q}') +\frac{e^2}{2c^2}\int d^6Z_0\int d^6Z_0'f_{D0}(Z_0)f_{D0}(Z_0')K_{ij}(\mathbf{q}|\mathbf{q}')\dot{q}_i\dot{q}_j'\right\}. \tag{52}$$

The functional derivatives $\delta S/\delta\mathbf{q}=0$ and $\delta S/\delta\mathbf{P}=0$ produce

Eq. (31), and again these equations together with Eq. (34) are equivalent to the Vlasov equation.

The existence of the above action principles ensures that the Vlasov–Darwin system possesses a Hamiltonian field description in terms of a noncanonical Poisson bracket.[25,26] As is the case for the Vlasov-Poisson system,[27] the theory can be represented in terms of the sole dynamical variable f_D with a bracket of Lie–Poisson form

$$\{F,G\}_D = \int d^6\zeta f_D\left[\frac{\delta F}{\delta f_D},\frac{\delta G}{\delta f_D}\right], \tag{53}$$

where F and G are arbitrary functionals of $f_D(\mathbf{x},\boldsymbol{\pi},t)$, $\delta F/\delta f_D$ denotes the functional derivative, and the "inner bracket" $[,]$ is the usual Poisson bracket in terms of $\mathbf{x}$ and $\boldsymbol{\pi}$. Finally, the Hamiltonian

$$\begin{aligned} H[f_D] = &\int d^6\zeta f_D(\zeta,t)\left[\frac{\pi^2}{2m} - \frac{\pi^4}{8m^3c^2}\right] \\ &+ \frac{e^2}{2}\int d^6\zeta \int d^6\zeta' f_D(\zeta,t) f_D(\zeta',t) K(\mathbf{x}|\mathbf{x}') \\ &- \frac{e^2}{2m^2c^2}\int d^6\zeta \int d^6\zeta' f_D(\zeta,t) f_D(\zeta',t) K_{ij}(\mathbf{x}|\mathbf{x}')\pi_i\pi_j', \end{aligned} \tag{54}$$

allows us to write compactly a Vlasov equation equivalent to Eq. (42) within the same order of v/c,

$$\frac{\partial f_D}{\partial t} = \{f_D, H\}. \tag{55}$$

This Hamiltonian formulation opens the portal for a rigorous investigation of stability of equilibria by means of the energy-Casimir method (see, e.g., Refs. 26 and 28), but this is beyond the scope of the present work.

V. CONCLUSIONS

This work has focused on the search for a direct method of establishing the equivalence of the Darwin field equations and the quasistatic solutions in an unbounded domain. In the quasistatic approach one transforms to Coulomb gauge, solves the equations exactly, and then takes the instantaneous, or quasistatic, limit of this solution. This yields $\mathbf{A}_C^{qs}$. In the operator approximation approach, by contrast, one argues the form of the equations before obtaining a solution. Thus, one drops the term $(1/c^2)(\partial^2\mathbf{A}/\partial t^2)$ and solves the resulting equation. This results in $\mathbf{A}_D$. There is, however, no *a priori* reason that the two methods should lead to the same result. We have shown directly the equivalence of $\mathbf{A}_C^{qs}$ and $\mathbf{A}_D$.

Another novel feature of the present work is a self-consistent treatment of the Darwin system, starting with an action that includes particles and fields in an inclusive whole. Here we began with the action for particles coupled to the electromagnetic field, the equations for the fields being derived from the action by variation on the fields themselves. Thus, with this action, the fields themselves are initially as dynamic as the particles. Given this starting point, we then studied the system in the situation where the fields are those of the particles themselves, and thus we could express the potentials in terms of the particle variables. This procedure of particlization leads us eventually to an action $S_D[\mathbf{q}]$ solely in terms of particles. This ultimately returns us to Darwin's original action. We have shown that the equations of motion derived from this particle action contain a term proportional to the longitudinal electric field. We finally obtained an action principle for the Vlasov–Darwin system by using the volume-preserving phase space coordinates in the Hamiltonian formulation. The expression of this system solely in terms of the particles provided the groundwork for a Hamiltonian field description in terms of a noncanonical Poisson bracket.

A historical motivation for the action principles of the present work is to distinguish which formulations truly conform to Darwin's original work, and which do not. A more pragmatic motivation recognizes that the above action principles have provided a clear vantage point for considering orders of approximation. Specifically, approximating at the level of the action ensured that all ensuing equations of motion are consistent to the desired level of approximation. Without the above unified formulation, this is not *a priori* clear in the disparate approaches to the Darwin system.

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