Pullback Equation: A Survey

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Pullback

Let $0 \leqslant k \leqslant n$ and let $\Omega_1, \Omega_2 \subseteq \mathbb{R}^n$ be open. Let $g \in C^\infty(\Omega_2; \Lambda^k)$ be written as

$$g = \sum_{I \in \mathcal{T}_k} g_I d\, x_I$$

where

- $\mathcal{T}_k$ is the set of all strictly increasing k-tuples i.e.

$$\mathcal{T}_k = \{(i_1, \ldots, i_k) \in \mathbb{N}^k : 1 \leqslant i_1 < \cdots < i_k \leqslant n\}.$$

- $g_I \in C^\infty(\Omega_2)$, for all $I \in \mathcal{T}_k$ and
- $dx_I = dx_{i_1} \wedge \ldots \wedge dx_{i_k}$, where $I = (i_1, \ldots, i_k)$.

Then, the pullback of g via $\varphi \in C^\infty(\Omega_1; \Omega_2)$, is defined as, denoted by $\varphi^* g \in C^\infty(\Omega_1; \Lambda^k)$,

$$\varphi^*(g) := \sum_{I \in \mathcal{T}_k} (g_I \circ \varphi) d\, \varphi_I, \text{ in } \Omega_1.$$

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Examples

- $\boxed{k=0:}$ $\quad \varphi^*(g) = g \circ \varphi$
- $\boxed{k=1:}$ $\quad \varphi^*(g) = \sum_{i=1}^{n} \left\langle g \circ \varphi; \frac{\partial \varphi}{\partial x_i} \right\rangle dx_i.$
- $\boxed{k=2:}$

$$\varphi^*(g) = \sum_{1 \leqslant k < r \leqslant n} \left[\sum_{1 \leqslant i < j \leqslant n} (g_{ij} \circ \varphi) \left(\frac{\partial \varphi_i}{\partial x_k} \frac{\partial \varphi_j}{\partial x_r} - \frac{\partial \varphi_i}{\partial x_r} \frac{\partial \varphi_j}{\partial x_k} \right) \right] dx_k \wedge dx_r.$$

- For a k-form, pullback is a homogenous polynomial of the gradient of φ of degree k.
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Statement of the problem

Given two k-forms $f, g : \Omega \to \Lambda^k(\mathbb{R}^n)$, does there exist a diffeomorphism $\varphi : \Omega \to \Omega$ such that

$$\varphi^*(g) = f, \text{ in } \Omega.$$

- This can be seen as the change of variable formula for differential forms.
- Existence, both local and global, regularity, Dirichlet problem.

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Darboux Theorem [1882]

Theorem

In even-dimensions i.e. $n = 2m$, any closed non-degenerate two-form is locally equivalent to the standard symplectic form

$$\omega_m := \sum_{i=1}^{m} dx_i \wedge dy_i .$$

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Moser's flow method

- Introduced by Moser (1965).

$$\begin{cases} \frac{d}{dt}\varphi_t &= u_t(\varphi_t),\ t \in [0,1]\ x \in \Omega \\ \varphi_0(x) &= x. \end{cases}$$

The solution φ_1 satisfies

$$\varphi_1^*(g) = f, \text{ in } \Omega.$$

How do we construct the vector field u_t?

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Moser's flow method: Constituent Equations

The vector field $u_t : \Omega \to \mathbb{R}^n$ is recovered through the following constituent equations

$$d\alpha_t = -\frac{d}{dt} H_t,$$

and

$$\iota_{u_t} H_t = \alpha_t,$$

where

- H_t is a closed, non-degenerate homotopy joining f and g and
- ι_{u_t} denotes the *interior product* or *contraction operator*.

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Moser's flow method: Solution

Differentiation gives

$$\begin{aligned} \frac{d}{dt}\varphi_t^* H_t &= \varphi_t^* \mathcal{L}_{u_t} H_t + \varphi_t^* \frac{d}{dt} H_t \\ &= \varphi_t^* (d\iota_{u_t} H_t + \iota_{u_t} d\, H_t) + \varphi_t^* \frac{d}{dt} H_t \\ &= \varphi_t^*(d\alpha_t) - \varphi_t^*(d\alpha_t) = 0, \end{aligned}$$

Solution of Constituent Equations

How do we solve the constituent equations?

Constituent equations are "well-posed" when

$$k = 2 \text{ when } n \text{ is even and } k = n.$$

Example: (*Positive volume forms*)

$$\operatorname{div} \alpha = f - g,$$

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$$u_t = \frac{\alpha}{tg + (1-t)f}.$$

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Case of volume forms (k=n)

Reimann [1972], Banyaga [1974], Zehnder [1977], Tartar [1978] and Dacorogna [1981].

Theorem (Dacorogna-Moser (1990))

Let $\Omega \subset \mathbb{R}^n$ be a bounded, smooth domain. Let $m \in \mathbb{N} \cup \{0\}$ and $0 < \alpha < 1$. Let $f, g \in C^{m,\alpha}(\overline{\Omega})$ with $f, g > 0$ in $\overline{\Omega}$. There exists $\varphi \in \operatorname{Diff}^{m+1,\alpha}(\overline{\Omega})$ satisfying

$$\begin{aligned} \varphi^*(g) &= f \text{ in } \Omega \\ \varphi &= \text{Id on } \partial\Omega \end{aligned}$$

if and only if

$$\int_\Omega f = \int_\Omega g.$$

Technique: Moser's flow method combined with fixed point theorem.

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  - When $f \in BMO(\overline{\Omega})$, we have $u, u^{-1} \in C^{0,\beta}(\overline{\Omega})$, where

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$$0 < \beta < c_1 \left(\frac{\inf f}{1 + \|f\|_{BMO}} \right)^{c_2},$$

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Case of volume forms (k=n)

- Ye [1994] (on $W^{m,p}(\Omega)$, for $\infty > p > \max\left\{1, \frac{n}{m}\right\}$),
- Rivière-Ye [1996] (on $C(\overline{\Omega})$, $BMO(\overline{\Omega})$, $L^{\infty}(\Omega)$).
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Non-degenerate case: Global result

Theorem (Bandyopadhyay-Dacorogna I)

Let $n > 2$ be even, $r \geqslant 3$, $0 < \alpha < 1$ and let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain. Let $f, g \in C^{r,\alpha}(\overline{\Omega}; \Lambda^2) \cap C^{r+2,\alpha}(\partial\Omega; \Lambda^2)$ be two closed non-degenerate forms. Then, there exists $\varphi \in \mathrm{Diff}^{r+1,\alpha}(\overline{\Omega})$ satisfying

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Let $n = 2m > 2$, $r \in \mathbb{N} \cup \{0\}$ and $0 < \alpha < 1$. Let $\Omega \subseteq \mathbb{R}^n$ be open and let $p \in \Omega$. Let $f \in C^{r,\alpha}(\Omega; \Lambda^2)$ be a closed two-form satisfying

$$\operatorname{rank} f(p) = n.$$

Then, there exist a neighbourhood V of p and $\varphi \in \operatorname{Diff}^{r+1,\alpha}(V; \mathbb{R}^n)$ such that

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- No improvement in regularity of solutions in general.
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Rank of an exterior form

Definition (Rank of an exterior k-form)

Let $n \in \mathbb{N}$, let $1 \leqslant k \leqslant n$ and let $\omega \in \Lambda^k(\mathbb{R}^n)$. Then,

$$\operatorname{rank} \omega := \dim \left\{ a \in \Lambda^1(\mathbb{R}^n) : \text{there exists } b \in \Lambda^{k-1}(\mathbb{R}^n) \text{ satisfying } \iota_b \, \omega = a \right\}.$$

Theorem (Basic properties of rank of an exterior k-form)

Let $n \in \mathbb{N}$, let $1 \leqslant k \leqslant n$ and let $\omega \in \Lambda^k(\mathbb{R}^n)\{0\}$. Then,

- *$\operatorname{rank} T^*\omega = \operatorname{rank} \omega$, for all $T \in GL_n(\mathbb{R})$. In other words, the notion of rank is natural.*
- *When $k = 1$, $\operatorname{rank} \omega = 1$.*
- *$\operatorname{rank} \omega \in \{k, k+2, \ldots, n\}$, when $k \geqslant 3$. In particular, when $k = n-1$, $\operatorname{rank} \omega = n-1$ and when $k = n$, $\operatorname{rank} \omega = n$.*

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Degenerate Darboux Theorem: Hölder regularity

Theorem (Bandyopadhyay-Dacorogna-Kneuss)

Let $n \geq 3$, $r, l \in \mathbb{N} \cup \{0\}$ *and* $0 < \alpha < 1$. *Let* $\Omega \subseteq \mathbb{R}^n$ *be open, let* $p \in \Omega$ *and let* ω_l *be the standard symplectic form of rank* $2l < n$, *namely*

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Let $f \in C^{r,\alpha}(\Omega; \Lambda^2)$ *be a closed two-form satisfying*

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Degenerate Darboux Theorem for k-forms

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Let $2 \leq k \leq n$, $r \in \mathbb{N}$ and let $0 < \alpha < 1$. Let $\Omega \subseteq \mathbb{R}^n$ be open and let $p \in \Omega$. Let $f, g \in C^{r,\alpha}(\Omega; \Lambda^k)$ be closed k-forms satisfying

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Then there exist a neighbourhood V of p and $\varphi \in \operatorname{Diff}^{r,\alpha}(V; \mathbb{R}^n)$ such that

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Let $2 \leq k \leq n$, $r \in \mathbb{N}$ and let $0 < \alpha < 1$. Let $\Omega \subseteq \mathbb{R}^n$ be open and let $p \in \Omega$. Let $f, g \in C^{r,\alpha}(\Omega; \Lambda^k)$ be closed k-forms satisfying

$$\operatorname{rank} f = \operatorname{rank} g = k, \text{ in a neighbourhood of } p.$$

Then there exist a neighbourhood V of p and $\varphi \in \operatorname{Diff}^{r,\alpha}(V; \mathbb{R}^n)$ such that

$$\varphi^*(g) = f, \text{ in } V \quad \text{and} \quad \varphi(p) = p.$$

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Local structure of zero divergence maps

Theorem (Bandyopadhyay-Dacorogna-Kneuss)

Let $r \in \mathbb{N}$ and let $0 < \alpha < 1$. Let $\Omega \subseteq \mathbb{R}^n$ be open, let $p \in \Omega$ and let f be a $C^{r,\alpha}(\Omega; \mathbb{R}^n)$ satisfy

$$f(p) \neq 0 \quad \text{and} \quad \operatorname{div} f = 0 \text{ in a neighbourhood of } p\,.$$

Then there exist a neighbourhood V of p and $\varphi \in \operatorname{Diff}^{r,\alpha}(V; \mathbb{R}^n)$ such that,

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Proof in a nutshell

Look for solutions

$$\varphi(x) = x + u(x), \text{ i.e. } \varphi = \mathrm{Id} + u.$$

The problem is

$$(\mathrm{Id} + u)^*(g) = f.$$

Linearize around the Identity to obtain

$$(L) \quad d(\iota_u g) = f - g + Q(u),$$

where Q is quadratic and higher. Choosing $\|f - g\|$ small, in a suitable lower order hölder norm, we solve (L) by fixed point method.
Finally,

iterate the process to remove $\|f - g\|$ small.

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Schematic presentation of the proof

Schematically, the proof can be presented in the following way

$$\begin{array}{ccccc} & g^{\epsilon} & \xrightarrow[\text{solution obtained by } \textit{Moser's flow method}]{\varphi^{\epsilon}} & f^{\epsilon} & \\ \text{solution obtained by } \textit{fixed point method} & \downarrow \eta^{\epsilon} & & \psi^{\epsilon} \downarrow & \text{solution obtained by } \textit{fixed point method} \\ & g & \xrightarrow[\varphi := (\eta^{\epsilon})^{-1} \circ \varphi^{\epsilon} \circ \psi^{\epsilon}]{\textbf{(final solution obtained by } \textit{composition}\textbf{)}} & f & \end{array}$$

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Fixed-point argument

Theorem

Let $n > 2$ be even and $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain. Let $r \in \mathbb{N} \cup \{0\}$ and $0 < \alpha \leqslant \beta < 1$. Then, there exists $\gamma = \gamma(r, \alpha, \beta, \Omega) > 0$ such that for every $g \in C^{r+2,\alpha}(\overline{\Omega}; \Lambda^2)$ and $f \in C^{r,\alpha}(\overline{\Omega}; \Lambda^2)$ satisfying the following hypotheses

$$\operatorname{rank}[g] = n, \; df = dg = 0 \text{ in } \Omega, \; f \wedge \nu = g \wedge \nu \text{ on } \partial\Omega,$$

$$\int_\Omega \langle f; \psi \rangle \, dx = \int_\Omega \langle g; \psi \rangle \, dx, \; \forall \psi \in H_T^2\left(\overline{\Omega}\right),$$

and

$$\|f - g\|_{C^{0,\beta}(\overline{\Omega})} \leqslant \frac{\gamma}{\|(\overline{g})^{-1}\|_{C^{1,\beta}(\overline{\Omega})} \max\left\{\|g\|_{C^{r+2,\alpha}(\overline{\Omega})} \|(\overline{g})^{-1}\|_{C^{r+1,\alpha}(\overline{\Omega})}, 1\right\}}$$

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