Higher integrability and higher differentiability of solutions to variational problems

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T.I.F.R. August 2010

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The model problem. A solution to

$$\Delta u = 0$$

is defined as a solution to the problem of minimizing

$$\int_\Omega \frac{1}{2} |\nabla u(x)|^2 \, dx$$

on $u^0 + W^{1,2}(\Omega)$. As such, it admits only weak first derivatives. To show that, in fact, it belongs to $u^0 + W^{2,2}(\omega)$, for every $\omega \subset\subset \Omega$, one can proceed as follows.

The function u satisfies the Euler Lagrange equation

$$\int_\Omega \langle \nabla u(x), \nabla \phi(x) \rangle = 0$$

for every suitable variation ϕ.

Fix ω and let η be a smooth variation, $= 1$ on ω and with $\mathrm{supp}(\eta) \subset\subset \Omega$ and consider $\phi = \delta^i_{-h}\left(\eta^2 \delta^i_h(u)\right)$, where by $\delta^i_h(u)$ we mean the differential quotient $\frac{u(x+he_i)-u(x)}{h}$; ϕ is an admissible variation.

(ϕ is, morally, $-\frac{\partial}{\partial x_i}\left(\eta^2 \frac{\partial}{\partial x_i} u\right)$, but these derivatives need not exist.)
By "integrating by parts" the differential quotients, we obtain

$$0 = \int \langle \delta^i_h \nabla u, \nabla(\eta^2 \delta^i_h(u))\rangle$$

i.e.,

$$\begin{aligned} 0 &= \int \langle \delta^i_h \nabla u, 2\eta \nabla \eta \delta^i_h(u) + \eta^2 \delta^j_h(\nabla u))\rangle \\ &= \int 2\eta \langle \delta^i_h \nabla u, \nabla \eta \delta^i_h(u)\rangle + \eta^2 |\delta^(_h \nabla u)|^2 \end{aligned}$$

that is

$$\int \left(|\nabla \eta| \delta^i_h(u)\right)^2 = \int \left(\nabla \eta \delta^i_h(u) + \eta \delta^i_h(\nabla u)\right)^2$$

and the first term is -morally -

$$\int |\nabla\eta|^2 |\frac{\partial u}{\partial x_i}|^2$$

that exists finite and independent of h, since u is a solution to the minimization problem.

Hence, the term at the r.h.s. exists finite and independent of h. This term is, again "morally", $\left|\nabla\eta\frac{\partial u}{\partial x_i} + \eta\frac{\partial\nabla u}{\partial x_i}\right|^2$.

On ω, it reduces to $\left|\frac{\partial\nabla u}{\partial x_i}\right|^2$, so that $u \in W^{2,2}_{loc}$.

The idea of this beautiful proof is exceedingly simple: add an integrable term to both sides so as to "complete the square": in this way, an integrable term dominates the square containing the term whose integrability you want to to show. One of the points of what follows will be that the term to add will be an important choice.

1 The p-Laplacian

Ladyzenskaia e Uraltseva in a series of papers and in their book, Linear and quasilinear elliptic equations, (1968), considered the regularity problem for the solution to a more complex case, for a functional depending on the variables x and u as well, i.e., to the problem of minimizing

$$I(v) = \int_\Omega f\left(x, v(x), \nabla v(x)\right)\, dx :$$

in this case, the Euler Lagrange equation becomes

$$\int_\Omega [\langle \nabla_\xi f(x, u(x), \nabla u(x)), \nabla \eta(x)\rangle + f'_u(x, u(x), \nabla u(x))\eta(x)]dx = 0.$$

They compared the function f with the "comparison function" $\Lambda(t) = \frac{1}{p}|t|^p$, with $p \geq 2$, in the sense that the growth, with respect to the variable "gradient", of f, of $\nabla_\xi f$ and of the Hessian H_f would be akin to the functions Λ, Λ' and Λ''.

For this function Λ the Euler Lagrange equation is

$$\int_\Omega |\nabla u(x)|^{p-2}\langle \nabla u(x), \nabla \phi(x)\rangle = 0$$

and, through the same variation $\phi = \delta^i_{-h}\left(\eta^2 \delta^i_h(u)\right)$, one obtains (set $b = \nabla u(x + he_i)$, $a = \nabla u(x)$ and $\nabla u^t = \nabla u(x + the_i)$)

$$0 \geq \int_{\text{spt } \eta} [\int_0^1 \eta^2 \left|\frac{b-a}{h}\right|^2 \Lambda''(|\nabla u^t|) - 2\eta|\nabla \eta| \left|\frac{b-a}{h}\right| \Lambda''(|\nabla u^t|)|\delta^j_h(u)|dt]dx$$

that can be written as

$$0 \geq \int_{\text{spt } \eta} [\eta^2 \left|\frac{b-a}{h}\right|^2 \int_0^1 \Lambda''(|\nabla u^t|)dt$$

$$-2\left(|\nabla \eta||\delta^i_h(u)|(\int_0^1 \Lambda''(|\nabla u^t|)dt)^{\frac{1}{2}}\right)\left(\eta\left|\frac{b-a}{h}\right|(\int_0^1 \Lambda''(|\nabla u^t|)dt)^{\frac{1}{2}}\right)]dx$$

and one can complete the square adding to both sides

$$\int_{\text{spt } \eta}\left(|\nabla \eta||\delta^i_h(u)|(\int_0^1 \Lambda''(|\nabla u^t|)dt)^{\frac{1}{2}}\right)^2$$

where the integrand is "of the order" of $|\nabla u|^2 \Lambda''(|\nabla u|)$, and so is the integral of this term that has to be bounded.

This offers no difficulty when $\Lambda(t)$ is a power of t, since $t^2\Lambda''(t) \sim \Lambda(t)$.

In the general case of minimizing

$$\int_\Omega f\left(x, v(x), \nabla v(x)\right)\, dx$$

there are other terms to be estimated, involving the mixed derivatives of f with respect to its variables. Through suitable assumptions, all these combinations of derivatives are related to combinations of derivatives of Λ and these combinations are chosen (in the comparison condition) in order to reduce, in the power case, to the same power of Λ.

One can have a comparison function Λ more general than a power, *as long as one can control these different combinations of derivatives to give essentially the same growth as Λ.* The case of "faster growth" has been considered by several authors: we mention Marcellini, [4], Marcellini-Papi [5] and Lieberman [3]. We believe that our method and results are different from theirs. In Lieberman [3] , for instance, to this purpose a main assumption is that

$$c_0 \leq \frac{t\Lambda''(t)}{\Lambda'(t)} \leq c_1$$

The program to show regularity by this technique rests upon a) choosing an integrable function that will represent the term to add in order to "complete the square"; b) fixing the conditions on the different terms (the mixed derivatives) in order to reduce the several different estimates to this function; and c) finding the right variation.

Consider point a): an obvious candidate is $\Lambda(|\nabla u(x)|)$, because we know that the integral of $f(x, u(x), \nabla u(x))$, that dominates the integral of $\Lambda(|\nabla u(x)|)$, is finite, and this is the obvious candidate.

How about $|\nabla u(x)|\Lambda'(|\nabla u(x)|)$ instead? For functions growing as a power, it is the same choice, but it is not so in general.

For instance, it is certainly not equivalent for functions growing exponentially.

Consider the Lagrangean $L(s) = e^{s^2}$, so that $L' = 2se^{s^2}$. For $n = 1$, the function $\xi(\cdot)$ whose derivative is

$$\xi'(t) = \sqrt{-\ln(|t|(|\ln(t)|)^{\frac{3}{2}})}$$

is such that $e^{\xi'(t)^2} = \dfrac{1}{|t||\ln(t)|^{\frac{3}{2}}}$ is integrable on $(-\frac{1}{2}, \frac{1}{2})$; however, for $|t|$ small,

$$\xi'(t)e^{\xi'(t)^2} = \frac{1}{|t|(|\ln(t)|)^{\frac{3}{2}}}\sqrt{-\ln(|t|(|\ln(t)|)^{\frac{3}{2}})}$$

$$> \frac{1}{|t|(|\ln(t)|)^{\frac{3}{2}}}\sqrt{\frac{-1}{2}|\ln(t)|} = \frac{1}{\sqrt{2}|t||\ln(t)|},$$

hence $L'(\xi'(\cdot))$ is not locally integrable.

In our regularity problem we have decided to take, as the term used to "complete the square",

$$|\nabla u(x)|\Lambda'(|\nabla u(x)|)$$

and to reduce all the estimates to the (local) integrability of this term.

But is it true that this term is integrable?

It is so for Lagrangeans growing as a p power, but what can be said for faster growth?

This is what we mean by higher integrability: a problem, of importance also for the validity of the Euler-Lagrange equation, that is far from being a completely solved problem. We know of one paper, for Lagrangeans of growth faster than exponential (A.C. & M.M, in print on J. Convex Analysis), and of a further paper not yet submitted, for the case of growth at most exponential; In the paper on J.Convex Analysis the problem of minimizing

$$\int_\Omega [e^{f(\|\nabla u(x)\|)} + g(x, u(x))]\, dx$$

where $f : \mathbb{R} \to \mathbb{R}$ is convex, symmetric, $f(0) = 0$ and such that there exist $q \geq 0$ and $\Lambda \geq 0$ such that $f'(t) \leq \Lambda t^q$ (In other words: f itself cannot grow exponentially). The function g is differentiable with respect to u, and g and g_u are Carathéodory functions. Then, one proves the following: let $\tilde{u} \in u^0 + W^{1,1}(\Omega)$ be a locally bounded solution to the problem of minimizing

$$\int_\Omega [e^{f(\|\nabla u(x)\|)} + g(x, u(x))]\, dx$$

Then, there exists a selection $S(\cdot)$ from $\partial f(\|\nabla\tilde{u}(\cdot)\|)$ such that,for every function ξ such that $\int_\Omega e^{f(\xi(x))}\,dx < \infty$, we have that

$$e^{f(\|\nabla\tilde{u}(\cdot)\|)}S(\cdot)\xi(\cdot) \in L^1_{loc}(\Omega).$$

(In particular, for $\xi(\cdot) = \|\nabla\tilde{u}(\cdot)\|$).

This result is based on a "strange" variation and on the properties of the exponential function. A similar result, tailored to the needs of the regularity result we are going to show, is at present being written down. However, a comprehensive result on higher integrability, as far as we know, is still lacking.

Before passing to the regularity result, let us further notice that the construction is based on the fact that the solution satisfies the Euler-Lagrange equation. Again, the validity of the Euler-Lagrange equation for solutions to variational problems is a question that is far from being settled.

2 Our contribution to the higher differentiability

We compare f with the "comparison function" $\Lambda(t)$, and there will be several conditions on Λ.

Our purpose is to face growth faster than polynomial growth.

We wish to consider a wider class of variations, nonlinear in the variable "i-th differential quotient", i.e., in $\delta_h^i u$:

$$\phi = \delta^i_{-h}\left(\eta^2 \gamma_\Lambda\left(\delta^i_h u\right)\right)$$

our problem will be to choose the function γ_Λ so as to make a meaningful variation.

Consider the model case, where the functional is simply

$$\int_\Omega \Lambda(|\nabla u|)dx.$$

From the Euler Lagrange equation we obtain (we set $a = \nabla u(x)$, $b = \nabla u(x + h)$)

$$0 = \int_\Omega \int_0^1 \Big[\eta^2 \gamma'_\Lambda \left(\delta_h^i u\right) \langle \frac{b-a}{h}, H_\Lambda(\nabla u^t) \frac{b-a}{h} \rangle \\ + 2\eta \langle \frac{b-a}{h}, H_\Lambda(\nabla u^t)) \nabla \eta \rangle \gamma_\Lambda \left(\delta_h^i u\right) \Big] dt dx. \qquad ((2.1))$$

(for the case $\gamma_\Lambda(t) = t$ and $\Lambda(|\xi|) = \frac{1}{2}|\xi|^{\frac{1}{2}}$, we had

$$0 = \int 2\eta \langle \delta_h^i \nabla u, \nabla \eta \delta_h^i(u) \rangle + \eta^2 |\delta_h^i(\nabla u)|^2.)$$

We rewrite (2.1) as

$$0=\int_\Omega\int_0^1 [2\eta\langle\frac{b-a}{h},H_\Lambda(\nabla u^t)\frac{b-a}{h}\rangle^{\frac12}\left(\gamma'_\Lambda\left(\delta_h^i u\right)\right)^{\frac12}\cdot$$

$$\cdot\frac{\langle\frac{b-a}{h},H_\Lambda(\nabla u^t)\nabla\eta\rangle\gamma_\Lambda\left(\delta_h^i u\right)}{\langle\frac{b-a}{h},H_\Lambda(\nabla u^t)\frac{b-a}{h}\rangle^{\frac12}(\gamma'_\Lambda\left(\delta_h^i u\right))^{\frac12}}$$

$$+\langle\frac{b-a}{h},H_\Lambda(\nabla u^t)\frac{b-a}{h}\rangle\eta^2\gamma'_\Lambda\left(\delta_h^i u\right)]dtdx.$$

For the symmetric matrix H_Λ, whose larger eigenvalue is Λ'', we have

$$\frac{\left|H_\Lambda(\nabla u^t)\frac{b-a}{h}\right|}{\langle\frac{b-a}{h},H_\Lambda(\nabla u^t)\frac{b-a}{h}\rangle^{\frac12}}\le(\Lambda''(|\nabla u^t|))^{\frac12}$$

so that the inequality becomes

$$0 \geq \int_\Omega \int_0^1 [-2\eta\langle\frac{b-a}{h}, H_\Lambda(\nabla u^t)\frac{b-a}{h}\rangle^{\frac{1}{2}} \left(\gamma'_\Lambda\left(\delta_h^i u\right)\right)^{\frac{1}{2}} \cdot$$

$$\cdot|\nabla\eta|(\Lambda''(|\nabla u^t|))^{\frac{1}{2}} \frac{\gamma_\Lambda\left(\delta_h^i u\right)}{\left(\gamma'_\Lambda\left(\delta_h^i u\right)\right)^{\frac{1}{2}}}$$

$$+\eta^2\gamma'_\Lambda\left(\delta_h^i u\right)\langle\frac{b-a}{h}, H_\Lambda(\nabla u^t)\frac{b-a}{h}\rangle]dtdx$$

and we complete the square adding

$$\int_\Omega \left[\left(\int_0^1 \Lambda''(|\nabla u^t|)dt\right)^{\frac{1}{2}} |\nabla\eta| \frac{|\gamma_\Lambda\left(\delta_h^i u\right)|}{\left(\gamma'_\Lambda(\delta_h^i u)\right)^{\frac{1}{2}}}\right]^2 dx$$

to both sides.

Hence, we have to face the problem of proving the integrability of

$$\int_\Omega \left(\int_0^1 \Lambda''(|\nabla u^t|)dt \right) |\nabla \eta|^2 \frac{|\gamma_\Lambda \left(\delta_h^i u\right)|^2}{(\gamma'_\Lambda(\delta_h^i u))} dx.$$

It would be nice, instead of

$$\frac{|\gamma_\Lambda\left(\delta_h^i u\right)|^2}{\gamma_\Lambda'(\delta_h^i u)},$$

to have

$$\frac{|\delta_h^i u|\Lambda'(\delta_h^i u)}{\Lambda''(\delta_h^i u)},$$

since then we would have to prove the integrability of

$$\left(\int_0^1 \Lambda''(|\nabla u^t|)dt\right)\frac{|\delta_h^i u|\Lambda'(\delta_h^i u)}{\Lambda''(\delta_h^i u)}.$$

In order to make this true, one would like to define the function γ_Λ as a solution to the differential equation

$$\gamma_\Lambda'(t) = \frac{\Lambda''(t)}{t\Lambda'(t)}(\gamma_\Lambda(t))^2.$$

Moreover, γ_Λ could not be any solution: it would have to be "similar" to the function t, in the sense that would have to be defined on $(-\infty, +\infty)$, to be 0 at 0 and $+\infty$ at ∞; moreover, would have to be regular (for the composition to make sense) and increasing.

3 Here a miracle occurs.

Call

$$\phi(t) = \frac{\Lambda''(t)}{t\Lambda'(t)}$$

Under the simple condition

$$\int_0^{t_0} \phi_\Lambda(t) = +\infty; \quad \int_{t_0}^{\infty} \phi_\Lambda(t) < +\infty$$

the differential equation

$$\gamma_\Lambda'(t) = \phi(t)(\gamma_\Lambda(t))^2$$

admits one and only one solution, γ_Λ, such that $\gamma_\Lambda(0) = 0$ and $\gamma_\Lambda(\infty) = \infty$ given by

$$\gamma_\Lambda(t) = \frac{1}{\int_t^\infty \phi_\Lambda(s)ds}$$

In the model case where $\Lambda(t) = \frac{1}{2}t^2$, the differential equation becomes

$$\gamma'_\Lambda(t) = \frac{1}{t^2}(\gamma_\Lambda(t))^2$$

a nonlinear equation, whose general solution is

$$\gamma_\Lambda(t) = \frac{1}{\frac{1}{\gamma_\Lambda(t^0)} - \frac{1}{t^0} + \frac{1}{t}}$$

hence, for $\gamma_\Lambda(t^0) = t^0$, we have $\gamma_\Lambda(t) = t$;

for $\gamma_\Lambda(t^0) - t^0 > 0$, $\gamma_\Lambda(t) \to 0$ as $t \to 0$

and $\gamma_\Lambda(t) \to \frac{1}{\gamma_\Lambda(t^0)} - \frac{1}{t^0}$ as $t \to +\infty$, while for $\gamma_\Lambda(t^0) - t^0 < 0$, we have that the solution exists positive only up to $\frac{1}{t^0} - \frac{1}{\gamma_\Lambda(t^0)}$:

for this equation, there is exactly one solution γ, defined on $\Re$, and such that $\gamma(0) = 0$ and $\lim_{\pm\infty} = \pm\infty$, namely $\gamma(t) = t$ and we obtain the very same variation we had started with.

If we assume that, near the origin, $\Lambda(t) = \frac{1}{2}t^2$, the condition $\int_0^{t_0} \phi_\Lambda(t) = +\infty$ is satisfied. We shall be mainly concerned with the behaviour at infinity of the function ϕ_Λ.

4 The function $\phi_\Lambda(t) = \frac{\Lambda''(t)}{t\Lambda'(t)}$

The discussion is based on the properties of the function $\frac{\Lambda''(t)}{t\Lambda'(t)}$.

This function is NOT the function $\frac{t\Lambda''(t)}{\Lambda'(t)}$ appearing e.g. in Lieberman, that we had seen.

Cases:

$$\Lambda(t) = \frac{1}{2}t^2; \quad \phi_\Lambda(t) = \frac{1}{t^2}; \quad \gamma_\Lambda = t$$

$$\Lambda(t) = \frac{1}{p}t^p, t \geq 1 : \quad \phi_\Lambda(t) = (p-1)\frac{1}{t^2}; \quad \gamma_\Lambda = \frac{1}{p-1}t$$

However, the condition

$$\int_{t_0}^{\infty} \phi_\Lambda(t) < +\infty$$

can be satisfied by functions Λ such that (for $t \geq 1$)

$$\frac{\Lambda''(t)}{\Lambda'(t)} = \frac{1}{(\log(t))^2}$$

or

$$\frac{\Lambda''(t)}{\Lambda'(t)} = \frac{1}{(\log(t))(\log(\log(t)))^2}$$

$$\cdots$$

where the corresponding γ_Λ are $\gamma_\Lambda(t) = \log(t)$ and $\gamma_\Lambda(t) = \log(\log(t))$
The condition is NOT satisfied by $\Lambda(t) = e^t$:
the integral $\int^\infty \frac{1}{t} dt$ diverges.

By this method, it seems that we can get as close to the exponential growth as we wish, but not beyond: for exponential and faster than 3exponential growth, the differential equation for γ has no solutions defined on $\Re$. I can see to possible way out for the case of very fast growth: either all solutions are (locally) Lipschitzian (and this seems unlikely) or the term used to complete the square, i.e., $|\nabla u|\Lambda'(|\nabla u|)$ is not enough.

Can anybody prove the integrability of $|\nabla u|^2\Lambda''(|\nabla u|)$?

5 A few words on the proof.

One additional difficulty.

In the general case, i.e. for the minimization of

$$\int_{\Omega} f\left(x, v(x), \nabla v(x)\right)\, dx$$

as we have noticed, there are various different combinations of derivatives of the comparison function Λ, arising from the mixed derivatives of f with respect to its variables. For instance, we have to consider expressions as

$$\Lambda'(a)\ b; \quad \frac{\Lambda'(a)}{a} b^2; \quad \Lambda''(a)\frac{b\Lambda'(b)}{\Lambda''(b)};$$

$$\left(a\Lambda'(a)\Lambda''(a)\right)^{\frac{1}{2}} \left(\frac{\Lambda'(b)\ b}{\Lambda''(b)}\right)^{\frac{1}{2}}$$

They are the same in the case of a power, but they are NOT the same in our case: it is a difficulty one has to face. (We do impose conditions, but mainly of qualitative type, as that Λ' be convex and that ϕ be non-increasing.)

As in the model case, eventually we prove that there exists a constant H such that, for every $\omega \subset\subset \Omega$, and every h sufficiently small, bounds a square of a different quotient of derivatives. In our present case we obtain

$$\int_\omega \left\{ \left| \frac{u_{x_j}(x+he_i) - u_{x_j}(x)}{h} \right|^2 \gamma'_\Lambda(\delta_h^i u) \cdot \right.$$

$$\left. \cdot \int_0^1 \frac{\Lambda'(|(1-s)\nabla u(x) + (s)\nabla u(x+he_i)|)}{(|(1-s)\nabla u(x) + (s)\nabla u(x+he_i)|)} ds \right\} dx \le H.$$

and here the problem is that γ'_Λ need not be bounded away from 0: for instance, when $\Lambda'(t) = e^{\frac{1}{e}\int_e^t \frac{1}{\log^2 s} ds}$ for $t \ge e$, we have that

$$\gamma'_\Lambda(t) = \frac{1}{t}$$

When Λ grows faster, $\phi(t)$ increases, and our intuition tells us that the solution γ_Λ to the differential equation

$$\gamma_\Lambda'(t) = \phi(t)(\gamma_\Lambda(t))^2$$

should increase.

Just the opposite is true:

γ_Λ is a special solution, a separatrix, issuing from a singular point.

To obtain the final higher differentiability result, one has still to impose some condition giving a way to make explicit the implicit previous result. The easist way is to require that Λ grows strictly faster than $\frac{1}{2}t^2$:

Additional assumption: there exist s^* and $\varepsilon > 0$, such that, for $s \geq s^*$, we have

$$\Lambda''(s) \geq (s\Lambda'(s))^{\varepsilon}$$

Under this condition we have that

a) there exists $p > 1$ such that $u_{x_j} \in W^{1,p}(\omega)$

and, setting $G(t) = \int^t \left[\frac{\Lambda'(s)}{s}\gamma'_{\Lambda}(s)\right]^{\frac{1}{2}} ds$,

b) $G(|u_{x_j}|) \in W^{1,2}$.

Cases of regularity for problems of "faster growth" have already been considered by several authors: we mention Marcellini, [4], Marcellini-Papi [5] and Lieberman [3]. Our method and results are different from theirs.

References

ACERBI, E, FUSCO, N *Regularity for minimizers of non-quadratic functionals: the case* $1 < p < 2$*.* J. Math. Anal. Appl. 140 (1989), 115-135 CELLINA, A.; MAZZOLA,M. *Higher integrability for solutions to variational problems.* J. Convex Analysis, to appear. LADYZHENSKAYA, OLGA A.; URAL'TSEVA, NINA N. Linear and quasilinear elliptic equations. Translated from the Russian. Academic Press, New York-London 1968

LIEBERMAN, G *The natural generalization of the natural condition of Ladyzhenskaya and Ural'tseva for elliptic equations* Comm. Partial Diff. Equat. 16 (1991), 311-361

MARCELLINI, P *Regularity for elliptic equations with general growth conditions* Journal Diff. Equat. 105 (1993), 296-333

MARCELLINI, P.; PAPI, G.*Nonlinear elliptic systems with general growth.* J. Differential Equations 221 (2006), 412–443