A posteriori error estimates for Hamilton-Jacobi equations

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Motivation

Why a posteriori error estimates?

Problem: Given an arbitrary tolerance, $\tau > 0$, an approximation v to the viscosity solution u of $u + H(\nabla u) = f$ such that

$$\| u - v \|_{L^\infty} \leq \tau.$$

Idea: By using the a posteriori error estimate

$$\| u - v \|_{L^\infty} \leq \Phi(v),$$

we could devise an adaptive method that computes the approximation v satisfying the quality constraint

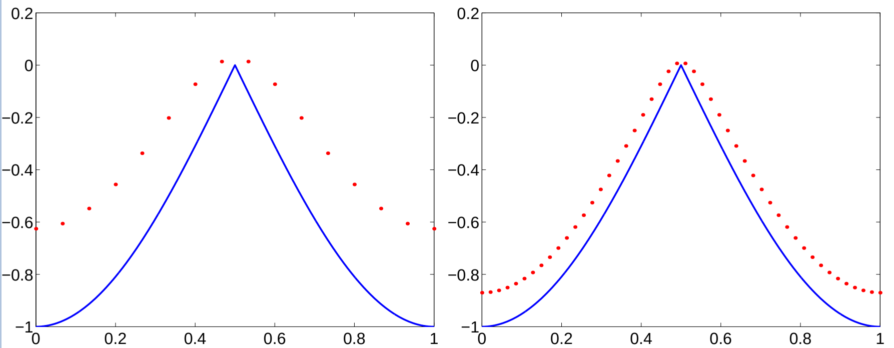
$$\Phi(v) \leq \tau,$$

with optimal complexity.

Motivation

Illustration of an adaptive algorithm

$$H(p) = \frac{1}{2}p^2 \text{ and } \tau = 0.1$$



B.C. and B. Yenikaya (2005)

Motivation

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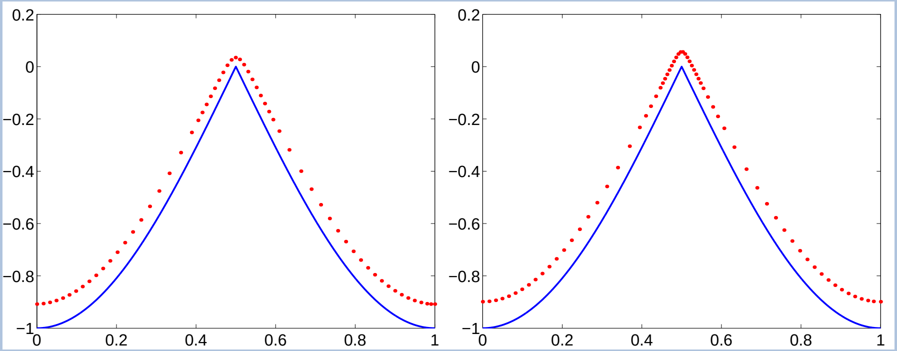


B.C. and B. Yenikaya (2005)

A posteriori error estimate for $u + H(x, \nabla u) = 0$

Definition of viscosity solution: Crandall, Evans, Lions (1984)

A viscosity solution of the Hamilton-Jacobi equation $u + H(\nabla u) = f$ is a continuous periodic function on $\mathbb{R}^d$ such that, for all x in $\mathbb{R}^d$,

$$+R(u; x, p) \leq 0 \quad \forall p \in D^{+}u(x),$$

$$-R(u; x, p) \leq 0 \quad \forall p \in D^{-}u(x),$$

where $R(u; x, p) = u(x) + H(p) - f(x)$ is the residual.

The *seminorm* of w, $|w|_\sigma$ is

$$\begin{cases} \sup_{x\in\mathbb{R}^d}(w(x))^+ & \text{for } \sigma = -, \\ \sup_{x\in\mathbb{R}^d}(-w(x))^+ & \text{for } \sigma = +. \end{cases}$$

The *generalized residual* $R_\epsilon(u; x, p)$ is

$$u(x) + H(p) - f(x - \epsilon\, p) - \frac{\epsilon}{2}\,|\,p\,|^2.$$

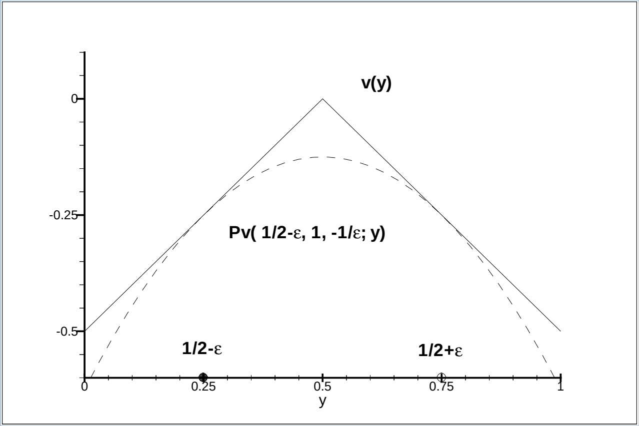
The *paraboloid* P_v is

$$v(x) + (y - x)\cdot p + \frac{\kappa}{2}|\,y - x\,|^2.$$

A posteriori error estimate for $u + H(\nabla u) = f$

The paraboloid

The parabola $y \mapsto Pv(1/2 - \epsilon, 1, -1/\epsilon; y)$ for $\epsilon = 1/4$.



A posteriori error estimate for $u + H(\nabla u) = f$

Theorem: Albert *et al.* (2001)

Let u be the viscosity solution and let v be any $\mathcal{C}^0(\mathbb{R}^d)$ function that is periodic in each coordinate with period 1. Then for $\sigma \in \{-, +\}$, we have

$$|\, u - v \,|_\sigma \leq \inf_{\epsilon > 0} \Phi_\sigma(v; \epsilon),$$

where

$$\Phi_\sigma(v; \epsilon) = \sup_{(x,p) \in \mathcal{A}_\sigma(v;\epsilon)} \left(\sigma\, R_{\sigma\epsilon}(v; x, p)\right)^+ ,$$

and the set $\mathcal{A}_\sigma(v; \epsilon)$ is the set of points $(x, p) \in \mathbb{R}^d \times \mathbb{R}^d$ satisfying

$$\sigma\, \{v(y) - P_v(x, p, \sigma/\epsilon\, ;\, y)\} \leq 0 \qquad \forall y \in \mathbb{R}^d.$$

- If v is the viscosity solution of $v + H(\nabla v) = g$, it gives the well-known $L^\infty-$contraction property, namely,

$$\|u - v\|_{L^\infty} \leq \|f - g\|_{L^\infty}.$$

- Holds for any Hamiltonian H.
- Is independent of how v has been computed.
- Its evaluation is parallelizable.

A posteriori error estimate for $u + H(\nabla u) = f$

The effectivity index for a special case

For the solution

$$v(x) := -\nu \ln \left(\exp(x/\nu) + 2 + \exp(-x/\nu) \right) =: f(x) - 1/2,$$

of the parabolic equation

$$v + \frac{1}{2}(v')^2 - \nu\, v'' = f.$$

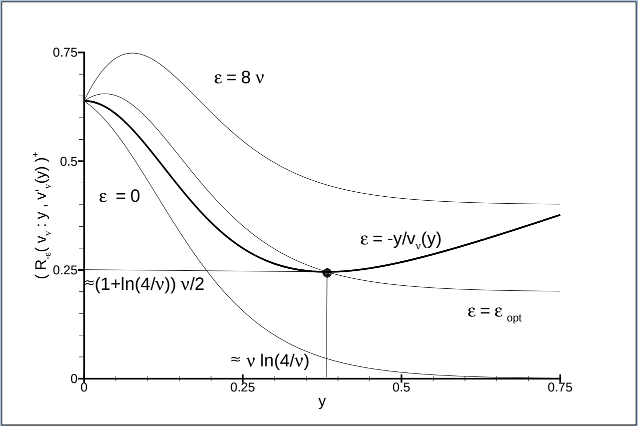
We have that

$$\frac{\inf_{\epsilon>0} \Phi_-(v;\epsilon)}{|\, u - v \,|_-} = \frac{1 + \ln(4/\nu)}{\ln 16}(1 + \mathcal{O}(\nu)).$$

A posteriori error estimate for $u + H(\nabla u) = f$

Minimization of $\Phi_\sigma(v;\epsilon)$ with respect to ϵ

The functions $y \mapsto (-R_{-\epsilon}(v; y, v'(y)))^+$.



A posteriori error estimate for $u + H(\nabla u) = f$

Applications to adaptivity

- 1D, LF: Appl. Numer. Math., B. Yenikaya, vol. 52 (2005).
- 2D, LF: J.C.P., B. Yenikaya, vol. 209 (2005).
- 1D, DG: J.C.P., Y. Chen, vol. 226 (2007).

A posteriori error estimate for $u_t + H(x, \nabla u) = 0$

Definition of the viscosity solution: Crandall, Evans, Lions (1984)

A viscosity solution u of the initial-value problem for the Hamilton-Jacobi equation $u_t + H(x, \nabla u) = 0$, is a continuous function on $\mathbb{R}^d \times (0, T)$ satisfying $u(x, t = 0) = u_0(x)$ such that, for all (x, t) in $\mathbb{R}^d \times (0, T)$,

$$\sigma R(x, p) \leq 0, \qquad \forall p \in D^\sigma u(x, t), \, \sigma \in \{+, -\},$$

where $R(x, p) = p_t + H(x, p_x)$ is the residual.

For any given any sub-domain Ω of $\mathbb{R}^d$ and any time $T > 0$, we are going to obtain upper bounds for

$$
\begin{aligned}
|\, u - v \,|_{-,\Omega,T} &= \sup_{x \in \Omega} \left(u(x,T) - v(x,T) \right)^+ , \\
|\, u - v \,|_{+,\Omega,T} &= \sup_{x \in \Omega} \left(v(x,T) - u(x,T) \right)^+ .
\end{aligned}
$$

We consider the behavior of the function v on the set

$$Q_T = \cup_{t \in (0,T)} \Omega_{V(T-t)} \times \{t\},$$

where

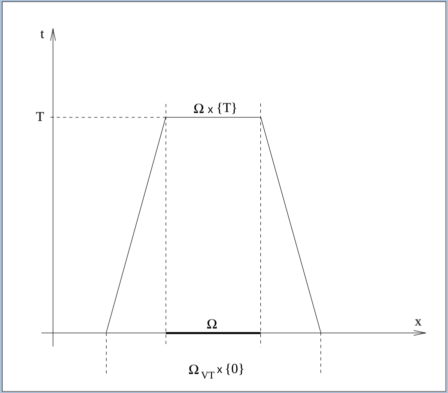
$$\Omega_{Vt} = \{x : \text{ dist}(x, \Omega) \leq Vt\},$$

and

$$V = \sup_{x \in \mathbb{R}^d} |\, H(x, \cdot)\,|_{\text{Lip}},$$

A posteriori error estimate for $u_t + H(x, \nabla u) = 0$

Preliminaries: The region Q_T



The region enclosed by the trapezoid Q_T.

The generalized residual R_ϵ is

$$p_t + H(x - \epsilon_x p_x, p_x).$$

The paraboloid P_v is

$$v(x,t) + (y-x, s-t)\cdot p + \frac{\kappa_t}{2}|\, s-t\,|^2 + \frac{\kappa_x}{2}|\, y-x\,|^2,$$

The modulus of continuity $\omega_\epsilon(v; x, t, p)$ is

$$v(x,t) - v(x - \epsilon_x p_x, 0) - \frac{\epsilon_x}{2}|\, p_x\,|^2 - \frac{t^2}{2\epsilon_t}.$$

A posteriori error estimate for $u_t + H(x, \nabla u) = 0$

Theorem: C., Merev, Qian

Let u be the viscosity solution of the problem under consideration and let v be any continuous function on $\mathbb{R}^d \times [0, T]$. Let Ω be a sub-domain of $\mathbb{R}^d$. Then, for $\sigma \in \{-, +\}$, we have that

$$|\, u - v \,|_{\sigma,\Omega,T} \leq |\, u - v \,|_{\sigma,\Omega_{VT},0} + \inf_{\epsilon_x,\epsilon_t > 0} \Phi_\sigma(v;\epsilon),$$

where

$$\Phi_\sigma(v;\epsilon) = \sup_{((x,t),p)\in\mathcal{A}_\sigma(v;\epsilon)} \left\{ \Big(\sigma\, \omega_{\sigma\epsilon}(v; x, t, p)\Big)^+ + \Big(\sigma\, T\, R_{\sigma\epsilon}(x, p)\Big)^+ \right\}.$$

The set $\mathcal{A}_\sigma(v;\epsilon)$ is the set of points $((x,t),p)$ in $\overline{Q_T} \times \mathbb{R}^{d+1}$ such that

$$\sigma\, \{v(y,s) - P_v(x, t, p, (\sigma/\epsilon_x, \sigma/\epsilon_t)\,;\, y, s)\} \leq 0 \qquad \forall (y,s) \in \overline{Q_T}.$$

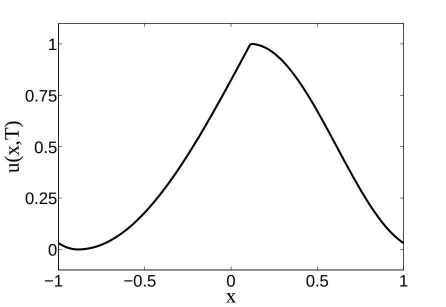
- For an Hamiltonian independent on x and if v is the viscosity solution of $v_t + H(\nabla v) = 0$, we recover the well-known $L^\infty-$contraction property, namely,

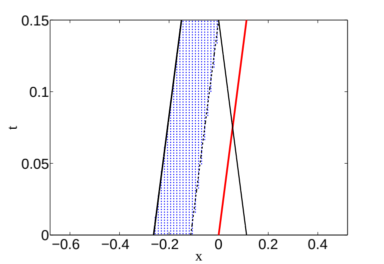
$$\|u(T) - v(T)\|_{L^\infty(\Omega)} \leq \|u(0) - v(0)\|_{L^\infty(\Omega_{VT})}.$$

- Holds for any Hamiltonian H.
- Is independent of how v has been computed.
- Its evaluation is parallelizable.

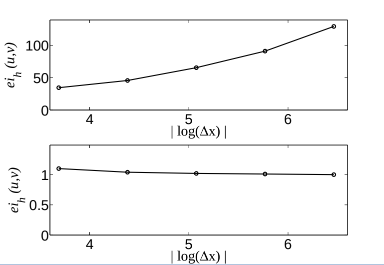
Numerical experiments

Linear Hamiltonian: Exact solution





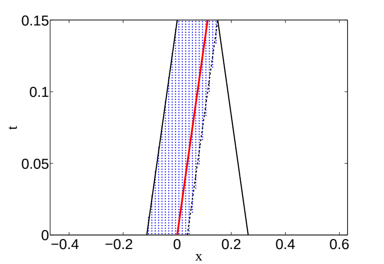
(a) $\Omega = [-0.15, 0.00]$



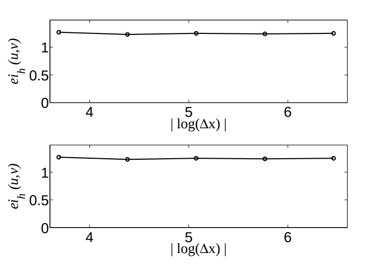
(b) top: $Q_{h,T}$; bottom: $\Gamma_{h,\Omega,T}$

Numerical experiments

Linear Hamiltonian



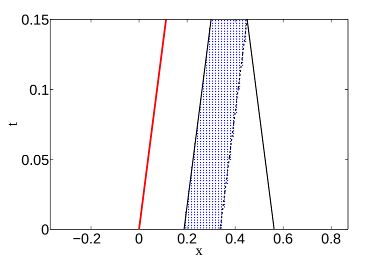
(c) $\Omega = [0.00, 0.15]$



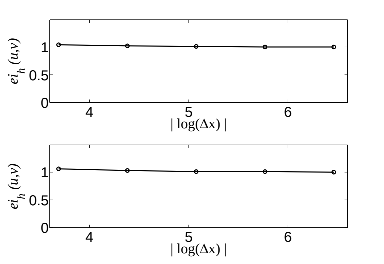
(d) top: $Q_{h,T}$; bottom: $\Gamma_{h,\Omega,T}$

Numerical experiments

Linear Hamiltonian



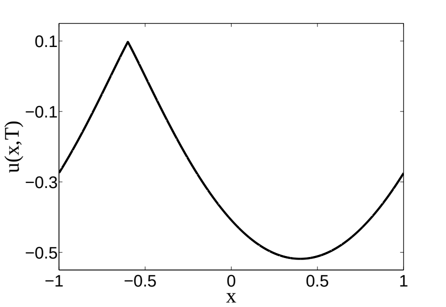
(e) $\Omega = [0.30, 0.45]$

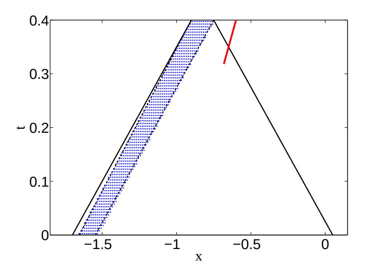


(f) top: $Q_{h,T}$; bottom: $\Gamma_{h,\Omega,T}$

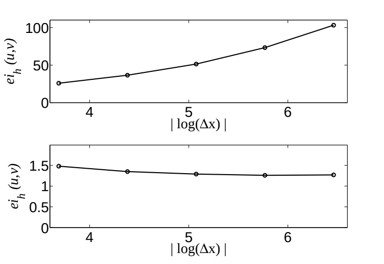
Numerical experiments

Quadratic Hamiltonian: Exact solution

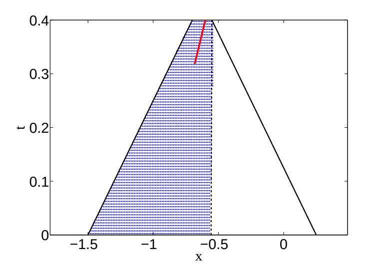




(g) $\Omega = [-0.15, 0.00]$



(h) top: $Q_{h,T}$; bottom: $\Gamma_{h,\Omega,T}$



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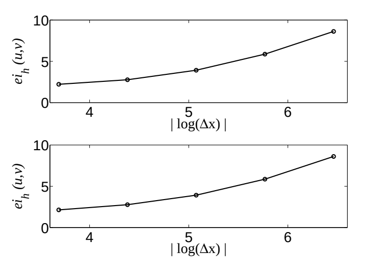
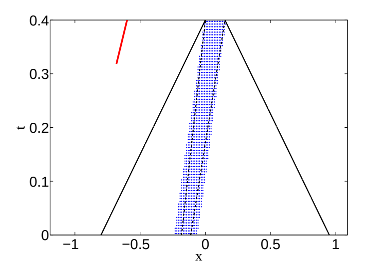


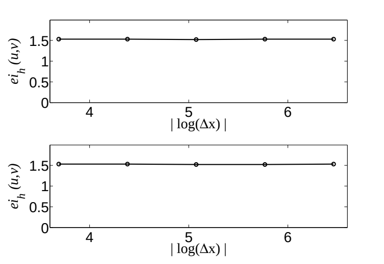
(j) top: $Q_{h,T}$; bottom: $\Gamma_{h,\Omega,T}$

Numerical experiments

Quadratic Hamiltonian



(k) $\Omega = [0.30, 0.45]$



(l) top: $Q_{h,T}$; bottom: $\Gamma_{h,\Omega,T}$

Open problems

- How to obtain lower bounds?
- How to deal with bounded domains?
- How to extend these type of results to second-order equations?