Regularity of HUM controls for conservative systems and convergence rates for discrete controls

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Outline of the talk

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An abstract control problem

Let $\mathcal{A}$ be the generator of a group on a Hilbert space $\mathfrak{X}$.

Consider the following model:

$$y'(t) = \mathcal{A}\, y(t) + \mathcal{B} v(t), \qquad y(0) = y_0 \in \mathfrak{X},$$

where $\mathcal{B} \in \mathfrak{L}(\mathcal{U}, \mathcal{D}(\mathcal{A})^*)$ and $v \in L^2(0, T; \mathcal{U})$.

Assumption

For all $v \in L^2(0, T; \mathcal{U})$, solutions can be defined in the sense of transposition in $C^0([0, T]; \mathfrak{X})$.

Goal : Exact controllability

Fix a time $T > 0$ and $y_0 \in \mathfrak{X}$. Can we find $v \in L^2(0, T; \mathcal{U})$ such that $y(T) = 0$?

Hypotheses

Main Assumption

$\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathfrak{X}$ generates a group.

Consequences:

- one can solve the equations forward and backward
- the same holds for the adjoint equation.

To simplify notations, we will assume that $\mathcal{A}$ is skew-adjoint:

$$\mathcal{A}^* = -\mathcal{A}$$

Examples

- Wave equation in a *bounded domain*+ BC with distributed control

$$\begin{cases} y'' - \Delta y = \chi_\omega v, & (t,x) \in \mathbb{R} \times \Omega, \\ y_{|\partial\Omega} = 0, \\ (y(0), y'(0)) = (y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega), \end{cases}$$

$$\mathcal{A} = \begin{pmatrix} 0 & Id \\ \Delta & 0 \end{pmatrix}, \quad \mathfrak{X} = H_0^1(\Omega) \times L^2(\Omega),$$

$$\mathcal{B} = \begin{pmatrix} 0 \\ \chi_\omega \end{pmatrix}, \quad \mathcal{U} = L^2(\omega).$$

- Wave equation in a *bounded domain* with boundary control
- Schrödinger equation $\mathcal{A} = -i\Delta$ + BC, Linearized KdV $\mathcal{A} = \partial_{xxx}$ + BC, Maxwell equation,...

Duality

Use the adjoint system to characterize the controls !

For all z solution of

$$z' = \mathcal{A} z, \qquad z(T) = z_T \in \mathfrak{X},$$

we have

$$\langle y(T), z_T \rangle_{\mathfrak{X}} - \langle y_0, z(0) \rangle_{\mathfrak{X}} = \int_0^T \langle v(t), \mathcal{B}^* z(t) \rangle_{\mathcal{U}} \, dt.$$

In particular, v is a control if and only if $\forall z_T \in \mathfrak{X}$

$$0 = \int_0^T \langle v(t), \mathcal{B}^* z(t) \rangle_{\mathcal{U}} \, dt + \langle y_0, z(0) \rangle_{\mathfrak{X}}.$$

Fundamental hypotheses

- $\mathcal{B}^* : \mathcal{D}(\mathcal{A}) \to \mathcal{U}$, $\mathcal{B}^* \in \mathfrak{L}(\mathcal{D}(\mathcal{A}), \mathcal{U})$.

Definition

$\mathcal{B}^*$ is admissible if $\forall T > 0, \exists K_T > 0$,

$$\int_0^T \|\mathcal{B}^* z(t)\|_{\mathcal{U}}^2 \, dt \leq K_T \, \|z_T\|_{\mathfrak{X}}^2 , \qquad \forall \, z_T \in \mathcal{D}(\mathcal{A}).$$

Definition

$\mathcal{B}^*$ is exactly observable at time $T^* > 0$ if $\exists k_* > 0$,

$$k_* \, \|z(0)\|_{\mathfrak{X}}^2 \leq \int_0^{T^*} \|\mathcal{B}^* z(t)\|_{\mathcal{U}}^2 \, dt, \qquad \forall \, z_T \in \mathfrak{X}.$$

Important remark

If $\mathcal{B}^*$ is admissible and exactly observable in some time T^*, then the following norms are equivalent:

- $\|z_T\|_{\mathfrak{X}}$, $\|z(0)\|_{\mathfrak{X}}$
- $\|z_T\|^2_{obs,1} = \int_0^{T^*} \|\mathcal{B}^* z(t)\|^2_{\mathcal{U}}\, dt$
- If $T > T^*$, $\|z_T\|^2_{obs,2} = \int_0^{T} \eta(t)\, \|\mathcal{B}^* z(t)\|^2_{\mathcal{U}}\, dt$, for $\eta \geq 0$, $\eta \geq \alpha > 0$ on some interval of length T^*.

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  for $\eta \geq 0$, $\eta \geq \alpha > 0$ on some interval of length T^*.

The Hilbert Uniqueness Method (Lions '86)

Let $T \geq T^*$.
Define, for $z_T \in \mathfrak{X}$,

$$J(z_T) = \frac{1}{2} \int_0^T \|\mathcal{B}^* z(t)\|_{\mathcal{U}}^2 \, dt + \langle y_0, z(0) \rangle,$$

where z satisfies $z' = \mathcal{A} z$, $z(T) = z_T$.
Observability $\Rightarrow$ Existence and Uniqueness of a minimizer Z_T.
Then $V = \mathcal{B}^* Z$ is such that the solution y of

$$y' = \mathcal{A} y + \mathcal{B} V, \qquad y(0) = y_0,$$

satisfies $y(T) = 0$.
Besides, V is the control of minimal $L^2(0, T; \mathcal{U})$-norm.

On the Hilbert Uniqueness Method

Recall the following facts:

- the Hilbert Uniqueness Method does not prove observability/controllability, but prove their equivalence.
- Observability/controllability properties have to be proved first:
  $\rightsquigarrow$ several possible methods:
  - multipliers, see Lions, Komornik, Alabau...
  - non-harmonic Fourier series, see Ingham, Haraux, Tucsnak...
  - Carleman estimates, see Puel, Osses, Zhang, Fu, etc
  - pseudo-differential calculus, see Bardos, Lebeau, Rauch, Burq, Gerard,...
  - Resolvent estimates, see Burq, Zworski, Miller, Tucsnak...

A regularity problem

On the regularity

If $y_0 \in \mathcal{D}(\mathcal{A})$,

- Does the function Z_T computed that way belongs to $\mathcal{D}(\mathcal{A})$?
- Is the controlled solution (y, v) a strong solution ?

Here, strong solutions means:

$$y \in C^1([0, T]; \mathfrak{X}) \cap C([0, T]; \mathfrak{X}_1),$$

for some space $\mathfrak{X}_1$ smoother than $\mathfrak{X}$ (for instance $\simeq \mathcal{D}(\mathcal{A})$).

General Answer : NO !

Consider the wave equation

$$\begin{cases} y'' - y_{xx} = 0, & 0 < x < 1,\ 0 < t < T, \\ y(0,t) = 0,\ y(1,t) = v(t), & 0 < t < T, \\ (y(x,0), y'(x,0)) = (y_0(x), y_1(x)) & \in L^2(0,1) \times H^{-1}(0,1). \end{cases}$$

The adjoint problem is

$$z'' - z_{xx} = 0,\ \ z(0,t) = z(1,t) = 0,\ \ (z_0, z_1) \in H_0^1(0,1) \times L^2(0,1),$$

and the solutions write

$$z(x,t) = \sqrt{2} \sum_{k \geq 1} \left(\hat{z}_0^k \cos(k\pi t) + \frac{\hat{z}_1^k}{k\pi} \sin(k\pi t) \right) \sin(k\pi x),$$

Controllability in time $T = 4$, and the HUM control is explicit:
If $(y_0(x),\ y_1(x)) = \sqrt{2} \sum_{k \geq 1} (\hat{y}_0^k,\ \hat{y}_1^k) \sin(k\pi x)$,

$$\hat{Z}_0^k = \frac{\hat{y}_1^k}{4k^2\pi^2}, \quad \hat{Z}_1^k = -\frac{\hat{y}_0^k}{4}.$$

In particular, the HUM control can be computed explicitly

$$\begin{aligned} v(t) &= Z_x(1,t) \\ &= \frac{1}{4}\sum_{k\geq 1}(-1)^k k\pi \left(\frac{\hat{y}_1^k}{k^2\pi^2}\cos(k\pi t) - \frac{\hat{y}_0^k}{k\pi}\sin(k\pi t)\right). \end{aligned}$$

$$\Longrightarrow v(0) = \frac{1}{4}\sum_{k\geq 1}(-1)^k \frac{\hat{y}_1^k}{k\pi} \neq 0 \text{ !}$$

$\Longrightarrow$ If $y_0 \in H_0^1(0,1)$, the controlled solution is not a strong solution in general because of the failure of the compatibility conditions $y_0(1) = v(0) = 0$.

Main question

Goal

To design a control method, and only one, which respects the regularity of the solutions.
If $y_0 \in \mathcal{D}(\mathcal{A})$, we want

- $Z_T \in \mathcal{D}(\mathcal{A})$,
- the controlled equation $y' = \mathcal{A}y + \mathcal{B}V$ is satisfied in a strong sense.

Related result - Dehman Lebeau '09 and Lebeau Nodet '09: The wave equation with distributed control $\mathcal{B} = \chi_\omega$ where χ_ω is smooth, and where the HUM operator is modified by a weight function $\eta(t)$ vanishing at $t \in \{0, T\}$.

Remark

Remark

It is well-know that, for $y_0 \in \mathcal{D}(A)$, there exists a control function v such that the solution of the controlled equation $y' = \mathcal{A}y + \mathcal{B}v$ lies in $C([0, T]; \mathcal{D}(\mathcal{A}))$.

$\rightsquigarrow$ see Bardos Lebeau Rauch, Tucsnak-Weiss.

However, the control function computed in that case is not the one obtained from the L^2 functional !

Outline of the talk

The modified HUM method: The η-weighted HUM

Let $y_0 \in \mathfrak{X}$, and $\delta > 0$ such that $T - 2\delta \geq T^*$, where T^* is the time of observability. Define, for $z_T \in \mathfrak{X}$,

$$J(z_T) = \frac{1}{2} \int_0^T \eta(t) \, \|\mathcal{B}^* z(t)\|_{\mathcal{U}}^2 \, dt + \langle y_0, z(0) \rangle,$$

where z satisfies $z' = \mathcal{A} z$, $z(T) = z_T$ and

$$\eta = \begin{cases} 0 & \text{on } (-\infty, 0] \cup [T, \infty) \\ 1 & \text{on } [\delta, T - \delta] \end{cases} \qquad \eta \geq 0.$$

Observability $\Rightarrow$ Existence and Uniqueness of a minimizer Z_T.
Then $V = \eta \mathcal{B}^* Z$ is such that the solution y of

$$y' = \mathcal{A} y + \mathcal{B} V, \qquad y(0) = y_0,$$

satisfies $y(T) = 0$.
Besides, V is the control of minimal $L^2((0, T), dt/\eta; \mathcal{U})$-norm.

Main result

Theorem (SE Zuazua '09)

Assume that admissibility and observability property hold.
Also assume that $\eta \in C^1(\mathbb{R})$.
If $y_0 \in \mathcal{D}(\mathcal{A})$, then the minimizer Z_T and the control function $V = \eta B^* Z$ computed by the η-weighted HUM are more regular:

- $Z_T \in \mathcal{D}(\mathcal{A})$,
- $V \in H_0^1(0, T; \mathcal{U})$.

Moreover, there exists a constant C such that

$$\|Z_T\|_{\mathcal{D}(\mathcal{A})} + \|V\|_{H_0^1(0,T;\mathcal{U})} \leq C \|y_0\|_{\mathcal{D}(\mathcal{A})} .$$

Before the proof

First remark that, due to the classical observability property,

$$\|Z_T\|_{\mathfrak{X}} + \|v\|_{L^2(0,T;\mathcal{U})} \leq C \|y_0\|_{\mathfrak{X}}.$$

Also remark that admissibility and observability properties applied to $\mathcal{A}z$ yield

$$\tilde{k} \|z_T\|_{\mathcal{D}(\mathcal{A})} \leq k \|z(0)\|_{\mathcal{D}(\mathcal{A})} \leq \int_0^T \eta(t) \left\|\mathcal{B}^* z'(t)\right\|_{\mathcal{U}}^2 dt \leq K \|z_T\|_{\mathcal{D}(\mathcal{A})}$$

$\longrightarrow$ It is sufficient to prove that

$$\int_0^T \eta(t) \left\|\mathcal{B}^* Z'(t)\right\|_{\mathcal{U}}^2 dt < \infty.$$

Indeed, this implies $Z_T \in \mathcal{D}(\mathcal{A})$ and $V \in H_0^1(0, T; \mathcal{U})$.

Idea of the proof-I

Write the characterization of the control $V = \eta \mathcal{B}^* Z$:

$$0 = \int_0^T \eta(t) \langle \mathcal{B}^* Z(t), \mathcal{B}^* z(t) \rangle_{\mathcal{U}} \, dt + \langle y_0, z(0) \rangle_{\mathfrak{X}},$$

for all z solution of $z' = \mathcal{A} z, \quad z(T) = z_T$.
Then take formally $z = Z'' = \mathcal{A}^2 Z$:

$$\int_0^T \eta(t) \left\| \mathcal{B}^* Z'(t) \right\|_{\mathcal{U}}^2 \, dt = \langle \mathcal{A} y_0, \mathcal{A} Z(0) \rangle_{\mathfrak{X}} - \int_0^T \eta'(t) \langle \mathcal{B}^* Z'(t), \mathcal{B}^* Z(t) \rangle_{\mathcal{U}} \, dt.$$

But

$$|\langle \mathcal{A} y_0, \mathcal{A} Z(0) \rangle_{\mathfrak{X}}| \leq C \left\| \mathcal{A} y_0 \right\|_{\mathfrak{X}} \left\| \mathcal{A} Z_T \right\|_{\mathfrak{X}}$$

and

$$\left\| \mathcal{A} Z_T \right\|_{\mathfrak{X}} \leq C \int_0^T \eta(t) \left\| \mathcal{B}^* Z'(t) \right\|_{\mathcal{U}}^2 \, dt.$$

Idea of the proof-II

But

$$\left| \int_0^T \eta'(t) \langle \mathcal{B}^* Z'(t), \mathcal{B}^* Z(t) \rangle_{\mathcal{U}} \, dt \right|$$
$$\leq C \left(\int_0^T \left\| \mathcal{B}^* Z'(t) \right\|_{\mathcal{U}}^2 \, dt \right)^{1/2} \left(\int_0^T \left\| \mathcal{B}^* Z(t) \right\|_{\mathcal{U}}^2 \, dt \right)^{1/2}$$
$$\leq C \left\| \mathcal{A} Z_T \right\|_{\mathfrak{X}} \left\| Z_T \right\|_{\mathfrak{X}} \leq C \left\| \mathcal{A} Z_T \right\|_{\mathfrak{X}} \left\| y_0 \right\|_{\mathfrak{X}}$$

Using observability,

$$\int_0^T \eta(t) \left\| \mathcal{B}^* Z'(t) \right\|_{\mathcal{U}}^2 \, dt \leq C \left\| y_0 \right\|_{\mathcal{D}(\mathcal{A})}^2 .$$

More general results

Theorem (SE-Zuazua 09)

Let $s \in \mathbb{N}$ and assume that $\eta \in C^s(\mathbb{R})$.
If the initial datum $y_0 \in \mathcal{D}(\mathcal{A}^s)$, then the minimizer Z_T and the control function V given by the η-weighted HUM satisfy

- $Z_T \in \mathcal{D}(\mathcal{A}^s)$
- $V \in H_0^s(0, T; \mathcal{U})$.

Besides, there exists a positive constant $C_s = C_s(\eta, k_*, K_T)$ independent of $y_0 \in \mathcal{D}(\mathcal{A}^s)$

$$\|Z_T\|^2_{\mathcal{D}(\mathcal{A}^s)} + \int_0^T \left\| V^{(s)}(t) \right\|^2_{\mathcal{U}} dt \leq C_s \|y_0\|^2_{\mathcal{D}(\mathcal{A}^s)} .$$

Comments

- If $\eta \in C^\infty(\mathbb{R})$, the η-weighted HUM map defined by

$$\Theta : \begin{cases} \mathfrak{X} \to \mathfrak{X} \times L^2(0, T; dt/\eta, \mathcal{U}) \\ y_0 \mapsto (Z_T, V) \end{cases}$$

defines by restriction a map on $\mathcal{D}(\mathcal{A}^s)$, for any $s > 0$:

$$\Theta : \mathcal{D}(\mathcal{A}^s) \to \mathcal{D}((\mathcal{A}^*)^s) \times H_0^s(0, T; \mathcal{U}).$$

- No smoothness assumption on $\mathcal{B}$, or on $[\mathcal{A}, \mathcal{B}\mathcal{B}^*]$.
$\rightsquigarrow$ generalizes Dehman Lebeau '09 to the cases $\mathcal{B} = \chi_\omega$ not smooth and boundary control cases, but less precise: pseudo-differential estimates on Θ for the wave equation with a distributed controlled.

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Corollary

if the initial datum $y_0 \in \mathcal{D}(\mathcal{A}^s)$, then the controlled trajectory y with control function V given by the η-weighted HUM satisfies

$$y(t) \in \bigcap_{k=0}^{s} C^k([0,T]; \mathcal{Z}_{s-k}),$$

where the spaces $\mathcal{Z}_j$ are defined by induction by

$$\mathcal{Z}_0 = \mathfrak{X}, \qquad \mathcal{Z}_j = \mathcal{A}^{-1}(\mathcal{Z}_{j-1} + BB^* \mathcal{D}(\mathcal{A}^j)).$$

Important Remark

In several situations, these spaces can be computed.
e.g.: if $\mathcal{B}\mathcal{B}^* \in \mathfrak{L}(\mathcal{D}(\mathcal{A}^j), \mathcal{D}(\mathcal{A}^j))$ for all j, $\mathcal{Z}_j = \mathcal{D}(\mathcal{A}^j)$ for all $j > 0$.

Further remarks

If

$$\mathcal{B}\mathcal{B}^* \in \mathfrak{L}(\mathcal{D}(\mathcal{A}^j), \mathcal{D}(\mathcal{A}^j)), \quad \forall j \in \mathbb{N},$$

taking η as the characteristic function of $[0, T]$, the same results hold.

Indeed, in that case, the boundary terms in the integration by parts can be bounded easily.

Outline of the talk

Boundary control

Let $\Omega \subset \mathbb{R}^N$, $\Gamma = \partial\Omega$, $\Gamma_0 \subset \Gamma$ satisifying GCC in time T^*.

Let ξ_{Γ_0} be defined on $\partial\Omega$, non-vanishing on $\overline{\Gamma_0}$.

We now consider the following wave equation:

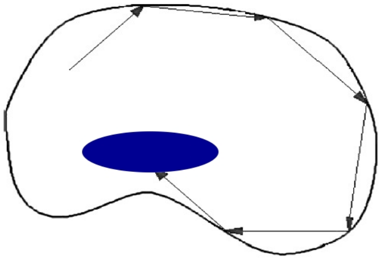
$$\begin{cases} y'' - \Delta y = 0, & \text{in } \Omega \times (0, \infty), \\ y = \xi_{\Gamma_0} v, & \text{on } \Gamma \times (0, \infty), \\ (y(0), y'(0)) = (y_0, y_1) & \in L^2(\Omega) \times H^{-1}(\Omega). \end{cases}$$

Geometric Control Condition

Assume GCC in time ω, T^* (resp. Γ_0, T^*):

(Bardos Lebeau Rauch '92 & Burq Gerard '96)

All the rays of Geometric Optics enter in the control region ω (resp. Γ_0) before the time $T^* > 0$.



Take $T - 2\delta > T^*$, and $\eta = \begin{cases} 0 & \text{on } (-\infty, 0] \cup [T, \infty) \\ 1 & \text{on } [\delta, T - \delta] \end{cases}$ $\eta \geq 0$.

Our method

The functional J reads:

$$J(z_0, z_1) = \frac{1}{2}\int_0^T \int_\Gamma \eta(t)\xi_{\Gamma_0}(x)^2 |\partial_n z(x,t)|^2 \, d\Gamma dt + \int_0^1 y_0(x)\, z'(x,0)\, dx - \langle y_1, z(\cdot,0)\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)},$$

where z is the solution of

$$\begin{cases} z'' - \Delta z = 0, & \text{in } \Omega \times (0,\infty), \\ z = 0, & \text{on } \partial\Omega \times (0,\infty), \\ (z(T), z'(T)) = (z_0, z_1) & \in H_0^1(\Omega) \times L^2(\Omega). \end{cases}$$

Results

Theorem

Assume that Γ_0, T^* satisfies GCC.
Given any $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists a unique minimizer (Z_0, Z_1) of J over $H_0^1(\Omega) \times L^2(\Omega)$. The function

$$V(x,t) = \eta(t)\xi_{\Gamma_0}(x)\partial_n Y(x,t)_{|\Gamma_1}$$

is the control of minimal $L^2(0,T;dt/\eta;L^2(\Gamma))$-norm, defined by

$$\|v\|^2_{L^2(0,T;dt/\eta;L^2(\Gamma))} = \int_0^T \int_\Gamma |v(x,t)|^2 d\Gamma \frac{dt}{\eta(t)}.$$

If $(y_0, y_1) \in \mathcal{D}(\mathcal{A}^s)$ for some $s \in \mathbb{N}$, then
$(Z^0, Z^1) \in \mathcal{D}((\mathcal{A}^*)^s) = \mathcal{D}(\mathcal{A}^{s+1})$ and $V \in H_0^s(0,T;L^2(\Gamma))$.

Results

Theorem

Assume that ξ_{Γ_0} is smooth.
If $(y_0, y_1) \in \mathcal{D}(\mathcal{A}^s)$ for some $s \in \mathbb{N}$, the η- weighted HUM yields:

$$V \in H_0^s(0, T; L^2(\Gamma)) \cap \bigcap_{k=0}^{s} C^k([0, T]; H^{s-k-1/2}(\Gamma))$$

and $(Z_0, Z_1) \in \mathcal{D}((\mathcal{A}^*)^s) = \mathcal{D}(\mathcal{A}^{s+1})$. In particular, the controlled trajectory y satisfies

$$y \in \bigcap_{k=0}^{s} C^k([0, T]; H^{s-k}(\Omega) \times H^{s-1-k}(\Omega)).$$

Outline of the talk

The 1d wave equation

$$\begin{cases} y'' - y_{xx} = 0, & 0 < x < 1,\ 0 < t < T, \\ y(0,t) = 0,\ y(1,t) = v(t), & 0 < t < T, \\ (y(x,0), y'(x,0)) = (y^0(x), y^1(x)) & \in L^2(0,1) \times H^{-1}(0,1). \end{cases}$$

The adjoint problem is

$$z'' - z_{xx} = 0,\ \ z(0,t) = z(1,t) = 0,\ \ (z^0, z^1) \in H_0^1(0,1) \times L^2(0,1),$$

Controllability is OK for $T \geq T^* = 2$.

The 1-d discrete case

Space semi-discretization (finite difference, $h = \frac{1}{N+1}$)

$$\begin{cases} y_j'' - \frac{1}{h^2}(y_{j-1} + y_{j+1} - 2y_j) = 0, & j \in \{1, \cdots, N\}, t \geq 0, \\ y_0(t) = 0, \quad y_{N+1}(t) = v(t), & t \geq 0. \end{cases}$$

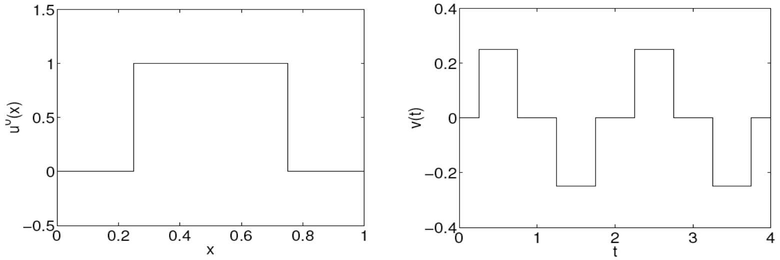


Figure: Left, the initial data $y(0)$. Right, the HUM control for the continuous system for initial data $(y(0), 0)$.

Numerical experiments

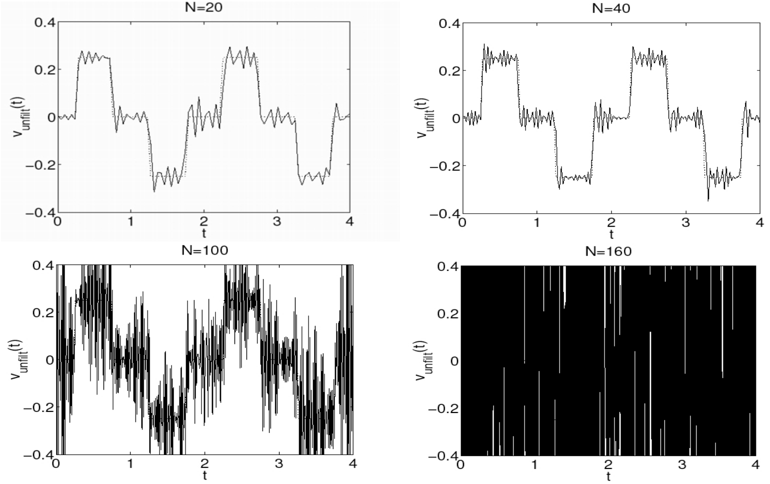
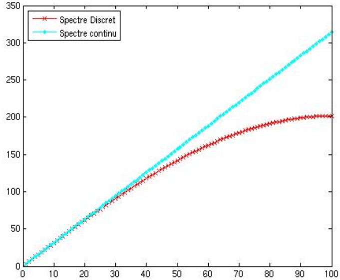


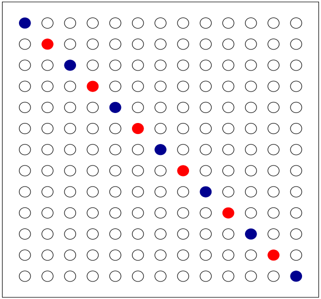
Figure: Discrete controls for different values of N.

Spectral explanation

Discrete schemes are not uniformly observable



Discrete spectrum vs Continuous spectrum. *Cf Zuazua.*



A localized eigenfunction of the discrete Laplacian on $[0,1]^2$. *Cf Kavian.*

$\rightsquigarrow$ Relaxation is needed.

Results

Let us consider the following wave equation:

$$\begin{cases} y'' - \Delta y = \chi_\omega v, & (t,x) \in \mathbb{R} \times \Omega, \\ y_{|\partial\Omega} = 0, \\ (y(0), y'(0)) = (y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega), \end{cases}$$

Then it is well-known that a control can be computed when ω, T satisfies GCC.
Assume $Y_0 = (y^0, y^1) \in \mathcal{D}(\mathcal{A}^{3/2}) = H^3_{(0)} \times H^2 \cap H_0^1(\Omega)$, where $H^3_{(0)}$ denotes the set of functions $y^0 \in H^3(\Omega)$ satisfying $\Delta y^0 = y^0 = 0$ on $\partial\Omega$.

An approximate control

Minimize

$$J(z_0^T, z_1^T) = \frac{1}{2}\int_0^T \eta(t) \int_\omega \chi_\omega^2 |\partial_t z|^2 \, dxdt + \int_\Omega \nabla z(0) \cdot \nabla y^0 + \int_\Omega z'(0) y^1$$

over solutions z of the (adjoint) wave equation with initial data $(z(T), z'(T)) = (z_0^T, z_1^T) \in H_0^1(\Omega) \times L^2(\Omega)$.

According to the previous results, the minimizer $Z^T = (Z_0^T, Z_1^T)$ belongs to $\mathcal{D}(\mathcal{A}^{3/2})$ provided η is smooth and vanishes at $t \in \{0, T\}$.

Approximation of Z

Using P1 finite elements, one can approximate Z solution of the wave operator $Z' = \mathcal{A}Z$ with initial data Z^T by Z_h solution of $Z_h' = \mathcal{A}_h Z_h$, where $\mathcal{A}_h$ is the finite element approximation of $\mathcal{A}$, with

$$\|Z_h - Z\|_{L^\infty(0,T;\mathfrak{X})} \leq Ch \left\|Z^T\right\|_{\mathcal{D}(\mathcal{A}^{3/2})} \leq Ch \|Y_0\|_{\mathcal{D}(\mathcal{A}^{3/2})}.$$

Thus, setting $V_h = \eta(t)\mathcal{B}_h^* Z_h$,

$$\|V_h - V\|_{L^2(0,T;dt/\eta;\mathcal{U})} \leq Ch \|Y_0\|_{\mathcal{D}(\mathcal{A}^{3/2})}$$

Moreover, the projection of Y_0 in the finite element space satisfies

$$\|Y_{0h} - Y_0\|_{\mathfrak{X}} \leq Ch \|Y_0\|_{\mathcal{D}(\mathcal{A}^{3/2})}.$$

Thus, Y_h solution of $Y_h' = \mathcal{A}_h Y_h + \mathcal{B}_h V_h, \quad Y_h(0) = Y_{0h}$ satisfies

$$\|Y_h(T)\|_{\mathfrak{X}} \leq Ch \|Y_0\|_{\mathcal{D}(\mathcal{A}^{3/2})}$$

Consequences

Corollary

If $Y_0 \in \mathcal{D}(\mathcal{A}^{3/2})$, the finite element approximation of the trajectory of Z immediately yields an $\varepsilon(h)$-approximate control for

$$\varepsilon(h) = Ch \| Y_0 \|_{\mathcal{D}(\mathcal{A}^{3/2})},$$

where C is independent of h and $Y_0 \in \mathcal{D}(\mathcal{A}^{3/2})$.

$\rightarrow \varepsilon(h)$ approximate controls $\hat{V}_h$ of minimal $L^2(0, T; dt/\eta; \mathcal{U})$-norms are bounded. They can be computed by minimizing

$$J_h(\varphi_h^T) = \frac{1}{2} \int_0^T \eta(t) \|\mathcal{B}_h^* \varphi_h\|_{\mathcal{U}}^2 \, dt + \langle \varphi_h(0), Y_{0h} \rangle_{\mathfrak{X}} + \varepsilon(h) \left\| \varphi_h^T \right\|_{\mathfrak{X}},$$

where φ_h is the solution of $\varphi_h' = \mathcal{A}_h \varphi_h$ with $\varphi_h(T) = \varphi_h^T$.
Then, if ψ_h^T is the minimizer of J_h, $\hat{V}_h(t) = \eta(t) \mathcal{B}_h^* \psi_h(t)$.

Thm (SE-Zuazua 2010)

For $Y_0 \in \mathcal{D}(\mathcal{A}^{3/2})$, the above method satisfies

$$\left\| \hat{V}_h - V \right\|_{L^2(0,T;dt/\eta;\mathcal{U})} \leq C\sqrt{h} \, \| Y_0 \|_{\mathcal{D}(\mathcal{A}^{3/2})}.$$

Sketch of the proof:

- $\| V_h - V \|_{L^2(0,T;dt/\eta;\mathcal{U})} \leq Ch \, \| Y_0 \|_{\mathcal{D}(\mathcal{A}^{3/2})}$.
- $\left\| \hat{V}_h - V_h \right\|^2_{L^2(dt/\eta;\mathcal{U})} = \left\| \hat{V}_h \right\|^2 - \| V_h \|^2 + 2\langle V_h, V_h - \hat{V}_h \rangle_{L^2(dt/\eta;\mathcal{U})}$

By minimality, $\left\| \hat{V}_h \right\|^2 - \| V_h \|^2 \leq 0$. Then we only need to estimate

$$\langle V_h, V_h - \hat{V}_h \rangle_{L^2(0,T;dt/\eta;\mathcal{U})} = \int_0^T \eta(t) \langle \mathcal{B}_h^* Z_h, \mathcal{B}_h^* (Z_h - \psi_h) \rangle_{\mathcal{U}} \, dt.$$

Using the equation satisfied by Y_h,

$$\int_0^T \eta(t)\langle \mathcal{B}_h^* Z_h, \mathcal{B}_h^*(Z_h - \psi_h)\rangle_{\mathcal{U}}\, dt = -\langle Y_{0h}, Z_h(0) - \psi_h(0)\rangle_{\mathfrak{X}} + \langle Y_h(T), Z_h(T) - \psi_h(T)\rangle_{\mathfrak{X}}$$

But, using the equation of Y_h and $\hat{y}_h$ yields the two identities:

$$\begin{aligned}\int_0^T \eta(t)\langle \mathcal{B}_h^* Z_h, \mathcal{B}_h^* \psi_h\rangle_{\mathcal{U}}\, dt &= \langle Y_h(T), \psi_h(T)\rangle_{\mathfrak{X}} - \langle Y_{0h}, \psi_h(0)\rangle_{\mathfrak{X}} \\ &= \langle \hat{y}_h(T), Z_h(T)\rangle_{\mathfrak{X}} - \langle Y_{0h}, Z_h(0)\rangle_{\mathfrak{X}}.\end{aligned}$$

Hence

$$\begin{aligned}\int_0^T \eta(t)\langle \mathcal{B}_h^* Z_h, \mathcal{B}_h^*(Z_h - \psi_h)\rangle_{\mathcal{U}}\, dt &= \langle Z_h(T), Y_h(T) - \hat{y}_h(T)\rangle_{\mathfrak{X}} \\ &\leq 4\varepsilon(h)\, \|Z_h(T)\|_{\mathfrak{X}}.\end{aligned}$$

Further comments

• Uniform observability does not occur: Hence nothing indicates that $(\psi_h(T))_{h>0}$ is bounded. Only $\mathcal{B}_h^*\psi_h$ is *a priori*.

• General method for deriving convergence rates for discrete controls, obtained by approximate controllability, filtering techniques, Tychonoff regularizations, bi-grid, (etc...) whatever the dimension is ! (see Zuazua SIAM Rev for extensive references)

$\rightsquigarrow$ Can also be extended to boundary controls, but more technical (requires uniform hidden regularity properties).

$\rightsquigarrow$ Recent results by Cindae Micu Tucsnak '10 :
Similar convergence results (with rates) of the discrete controls by Russell's stabilizability method. Hence convergence to Russell's control, which is not the one of minimal L^2-norm.

HUM-regularity issues

• What happens for the exact controllability when $\mathcal{A}$ does not generate a group ?
$\rightsquigarrow$ Typical example: Linearized KdV equation

- $\rightsquigarrow$ Can we design a HUM type method which yields smooth controlled trajectory for smooth initial data ?
- $\rightsquigarrow$ Non linear KdV equation ?

• Controllability to trajectories for heat type equations :
$\rightsquigarrow$ Typical example: The heat equation.

- $\rightsquigarrow$ Can we design a HUM type method which yields smooth controlled trajectory for smooth initial data ?
- $\rightsquigarrow$ Use it to design well-conditioned numerical control methods ? (see Münch Zuazua 2010).

Recent results on the heat controls by Micu-Zuazua '10, SE-Zuazua '10 by transmutation (under GCC).

Thank you for your attention !

Articles available on

`http://www.math.univ-toulouse.fr/~ervedoza/`