ADI Finite Element Methods for an Evolution Equation with a Positive Type Kernel

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Outline

Outline

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Joint work with....

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- **Amiya Pani**, Indian Institute of Technology, Bombay

PIDEs with a Positive Type Kernel

$$\begin{cases} \dfrac{\partial u}{\partial t} = \displaystyle\int_0^t \beta(t-s)\Delta u(s)\,ds + f(x,y,t), \quad (x,y,t) \in \Omega \times (0,T] \\ u(x,y,0) = u_0(x,y), \quad (x,y) \in \Omega \\ u(x,y,t) = 0, \quad (x,y,t) \in \partial\Omega \times (0,T] \end{cases}$$

- Δ – the Laplacian: $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$
- Ω – the unit square with boundary $\partial\Omega$

PIDEs with a Positive Type Kernel

The kernel β is **positive definite**: for each $T > 0$,

$$\int_0^T \phi(s) \left(\int_0^s \beta(s - \tau)\phi(\tau)\, d\tau \right) ds \geq 0 \quad \forall \phi \in C^0[0, T]$$

SOURCES:

- viscoelasticity
- heat conduction in materials with memory

Smooth Positive Type Kernels

- If the kernel $\beta \in C^2[0, T]$ and satisfies

$$(-1)^j \beta^{(j)}(t) \geq 0, \quad t > 0, \quad j = 0, 1, 2$$

then it is **positive definite**

In this case, differentiating the PIDE with respect to t gives

$$\frac{\partial^2 u}{\partial t^2} = \beta(0)\Delta u + \int_0^t \beta'(t-s)\Delta u(s)\, ds + \frac{\partial f}{\partial t}(x, y, t), \; (x, y, t) \in \Omega \times (0, T]$$

PIDES

- **Finite Element Galerkin Methods**
  - “Finite Element Methods for Integrodifferential Equations”, Chen and Shih (1998)
  - McLean and Mustapha (2007)
- **Mixed Finite Element Methods**
  - A. Pani and GF (2002, 2003)
- **Discontinuous Galerkin Methods**
  - Mustapha and McLean (2009)
- **Orthogonal Spline Collocation (OSC)**
  - Yi Yan and GF (1992, 1994)
  - A. Pani, GF and R. Fernandes (2008)

Outline

ADI OSC Methods

Reduce a multidimensional problem to sets of 1–D problems

- **Fernandes and GF**
  NMPDES 1993 (*parabolics, hyperbolics*)
- **Bialecki and Fernandes**
  Math. Comp. 1993 (*parabolics*)
  Numer. Math. 1997 (*hyperbolics*)
  IMAJNA 2003 (*hyperbolics*)
  SISC 2006 (*nonlinear parabolics on rectangular polygons*)
- **Li, GF and Bialecki**
  SINUM 1998, 2002 (*Schrödinger systems*)

Notation

- $\rho = \{x_i\}_{i=0}^{N}$ with $x_i = ih$, $Nh = 1$
  - a uniform partition of $[0, 1]$
- With $k = T/M, \quad t_n = nk, \quad 0 \leq n \leq M$
- For a sequence $\{\phi^n\}_{n=0}^{M}$:

$$\left.\begin{aligned} \partial_t \phi^n &= \frac{\phi^n - \phi^{n-1}}{k} \\ \phi^{j-1/2} &= \frac{\phi^j + \phi^{j-1}}{2} \end{aligned}\right\} \quad 1 \leq n \leq M$$

Preliminaries

- **Spaces of piecewise polynomials**:

$$\mathcal{M}_r(\rho) = \{v \in C^1[0,1] : v|_{[x_{i-1},x_i]} \in P_r, 1 \le i \le N\}$$

  - P_r – set of polynomials of degree $\le r$

$$\mathcal{M}_r^0(\rho) = \{v \in \mathcal{M}_r(\rho) : v(0) = v(1) = 0\}$$

  - $\dim(\mathcal{M}_3^0(\rho)) = N(r-1) \equiv N'$

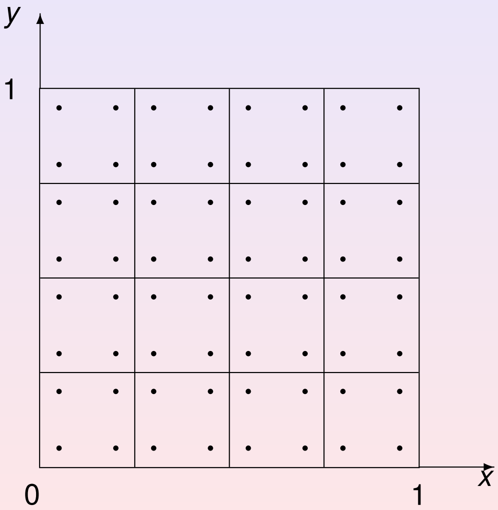
$$\mathcal{M}_r \equiv \mathcal{M}_r^0(\rho) \otimes \mathcal{M}_r^0(\rho)$$

- **Collocation points:**

$\mathcal{G}_x = \{\xi_{i,j}^x\}_{i,j=1}^{N,r-1}$ – composite Gauss points in $[0,1]$

$$\mathcal{G} = \{\xi = (\xi^x, \xi^y) : \xi^x, \xi^y \in \mathcal{G}_x\}$$

Collocation points - $N = 4, r = 3$



Crank-Nicolson OSC Method

Definition

Given U^0, determine $\{U^n\}_{n=1}^{M} \subset \mathcal{M}_r$ such that

$$\partial_t U^n = q_{n-1/2}(\Delta U) + f(t_{n-1/2}) \quad \text{on} \quad \mathcal{G}, \quad 1 \le n \le M$$

Crank-Nicolson OSC Method

- **Modified composite midpoint rule:**

$$\begin{aligned} q_{n-1/2}(\phi) &:= k\sum_{j=1}^{n-1}\beta_{n-j}\phi^{j-1/2} + \frac{k}{2}\beta_0\phi^{n-1/2} \\ &\approx \int_0^{t_{n-1/2}} \beta(t_{n-1/2}-s)\phi(s)\,ds \end{aligned}$$

The quadrature rule $q_{n-1/2}(\phi)$ is **positive**:

$$k\sum_{n=1}^{M} q_{n-1/2}(\phi)\phi^{n-1/2} \geq 0, \quad \text{for all} \quad \phi$$

ADI Crank-Nicolson OSC Method

With $v^n = U^n - U^{n-1}$,

$$v^n - \frac{k^2\beta_0}{4}\Delta v^n = kf(t_{n-1/2}) + k^2\sum_{j=1}^{n-1}\beta_{n-j}\Delta U^{j-1/2} + \frac{k^2\beta_0}{2}\Delta U^{n-1} \equiv F^n$$

Add $\dfrac{k^4\beta_0^2}{16}\dfrac{\partial^4 v^n}{\partial x^2\partial y^2}$ to the left hand side:

Definition

Given U^0, determine $\{U^n\}_{n=1}^M \subset \mathcal{M}_r$ such that

$$\left[1 - \frac{k^2\beta_0}{4}\Delta + \frac{k^4\beta_0^2}{16}\frac{\partial^4}{\partial x^2\partial y^2}\right] v^n = F^n \quad \text{on} \quad \mathcal{G}, \quad 1 \le n \le M$$

$$U^n = U^{n-1} + v^n$$

Matrix Tensor Product

Definition

If $A = [a_{ij}]$ is $N_A \times N_A$ and B is $N_B \times N_B$, then $A \otimes B$ is the $N_A N_B \times N_A N_B$ matrix whose (i, j) block is $a_{ij} B$:

$$\begin{bmatrix} a_{11}B & a_{12}B & \ldots & a_{1N_A}B \\ a_{21}B & a_{22}B & \ldots & a_{N'_A}B \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{N_A 1}B & a_{N_A 2}B & \ldots & a_{N_A N_A}B \end{bmatrix}$$

NOTE:

- If A is diagonal, then $A \otimes B$ is block diagonal.
- $(A \otimes B)(C \otimes D) = AC \otimes BD$

Matrix Formulation

With $\{\phi_i\}_{i=1}^{N'}$ a basis for $\mathcal{M}_r^0(\rho)$, set

$$U^n(x,y) = \sum_{i=1}^{N'} \sum_{j=1}^{N'} U_{ij}^{(n)} \phi_i(x)\phi_j(y)$$

Then

$$\left[B \otimes B + \frac{k^2\beta_0}{4}(A \otimes B + B \otimes A) + \frac{k^4\beta_0^2}{16} A \otimes A \right] \boldsymbol{\nu}^{(n)} = \mathbf{F}^{(n)}$$

where

$$\boldsymbol{\nu}^{(n)} = \boldsymbol{U}^{(n)} - \boldsymbol{U}^{(n-1)}$$

$$\boldsymbol{U}^{(n)} = [U_{1,1}^{(n)}, U_{1,2}^{(n)}, \cdots, U_{1,N'}^{(n)}, U_{2,1}^{(n)}, \cdots, U_{N',N'}^{(n)}]^T$$

$$A = [-\phi_j''(\xi_i)]_{i,j=1}^{N'}, \quad B = [\phi_j(\xi_i)]_{i,j=1}^{N'}$$

Algebraic Problem

Or

$$\left[\left(B+\frac{k^2\beta_0}{4}A\right)\otimes I\right]\left[I\otimes\left(B+\frac{k^2\beta_0}{4}A\right)\right]\boldsymbol{\nu}^{(n)}=\mathbf{F}^{(n)}$$

Thus

$$\begin{cases}\left[\left(B+\frac{k^2\beta_0}{4}A\right)\otimes I\right]\hat{\boldsymbol{\nu}}^{(n)}=\mathbf{F}^{(n)}\\ \left[I\otimes\left(B+\frac{k^2\beta_0}{4}A\right)\right]\boldsymbol{\nu}^{(n)}=\hat{\boldsymbol{\nu}}^{(n)}\end{cases}$$

and

$$\boldsymbol{U}^{(n)}=\boldsymbol{U}^{(n-1)}+\boldsymbol{\nu}^{(n)}$$

Algebraic Problem

Step I. Solve

$$\left(B+\frac{k^2\beta_0}{4}A\right)\hat{\boldsymbol{\nu}}_j^{(n)}=\mathbf{F}_j^{(n)},\quad 1\le j\le N'$$

where

$$\mathbf{v}_j=[v_{1,j},v_{2,j},\ldots,v_{N',j}]^T$$

Step II. Solve

$$\left(B+\frac{k^2\beta_0}{4}A\right)\boldsymbol{\nu}_i^{(n)}=\hat{\boldsymbol{\nu}}_i^{(n)},\quad 1\le i\le N'$$

where

$$\mathbf{v}_i=[v_{i,1},v_{i,2},\ldots,v_{i,N'}]^T$$

cf., solving two-point BVPs in the x-direction followed by two-point BVPS in the y-direction.

Algebraic problem

Since

$$B + \frac{k^2\beta_0}{4}A$$

is nonsingular, there exists a unique ADI Crank–Nicolson OSC approximation.

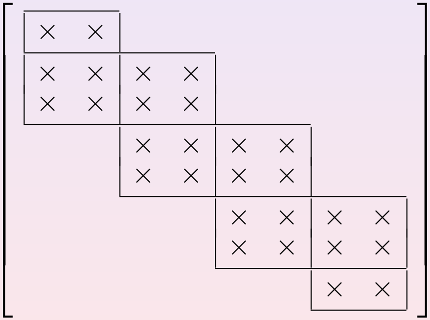
* * * * * *

With standard choices of bases for $\mathcal{M}_r^0(\rho)$, linear systems are **almost block diagonal**

- P. Amodio et al., NLAA (2000)
- GF and I. Gladwell, SIAM Rev. (2004)

Almost Block Diagonal Matrix – an Example

- For $r = 3$ and $N = 3$:



Alternate Row and Column Elimination

Alternate....

- **Row elimination with row pivoting**
  - partial pivoting with row interchanges followed by elimination by rows
  - *the standard procedure in Gaussian elimination*

with

- **Column elimination with column pivoting**
  - partial pivoting by columns with column interchanges, followed by elimination by columns

switching from one to the other when fill-in would occur otherwise.

Alternate Row and Column Elimination

$$\begin{bmatrix} \times & * & & & & & & \\ \times & \times & \times & \times & & & & \\ \times & \bullet & \times & * & & & & \\ & & \times & \times & \times & \times & & \\ & & \times & \bullet & \times & * & & \\ & & & & \times & \times & \times & \times \\ & & & & \times & \bullet & \times & * \\ & & & & & & \times & \times \end{bmatrix}$$

$*$ - annihilated by column elimination

$\bullet$ - annihilated by row elimination

$$\mathcal{A} \leftarrow PL\mathcal{B}UQ$$

Error Analysis

For ϕ and ψ defined on $\mathcal{G}$,

$$\langle \phi, \psi \rangle = h^2 \sum_{i,j=1}^{N} \sum_{k,l=1}^{r-1} \omega_k \omega_l (\phi\psi)(\xi_{i,k}, \xi_{j,l}), \qquad |\phi| = \langle \phi, \phi \rangle^{1/2}$$

Then, for $V \in \mathcal{M}_r, \quad 1 \leq n \leq M$,

$$\langle \partial_t U^n, V \rangle + \frac{k^4 \beta_0^2}{16} \left\langle \frac{\partial^4 [\partial_t U^n]}{\partial x^2 \partial y^2}, V \right\rangle = \langle q_{n-1/2}(\Delta U), V \rangle + \langle f(t_{n-1/2}), V \rangle$$

OSC Elliptic Projection

Define $W : [0, T] \to \mathcal{M}_r$ by

$$\Delta(u - W) = 0 \quad \text{on} \quad \mathcal{G} \times [0, T]$$

If u is sufficiently smooth, then

$$\left\| \frac{\partial^i (u - W)}{\partial t^i} \right\|_{H^j} \leq C h^{r+1-j}, \quad i = 0, 1, 2, \quad j = 0, 1, 2$$

- Percell and Wheeler, SINUM 1980

Error Estimates

Theorem

Assume that $\beta \in C^2[0, T]$ *and* $U^0 = W^0$. *Then if* u *is sufficiently smooth,*

$$\max_{1 \le n \le M} \left\| u(t_n) - U^n \right\| \le C(k^2 + h^{r+1})$$

Error Estimate

Theorem

Assume that $\beta \in C^2[0,T]$ *and* $U^0 = W^0$. *Then*

$$\begin{aligned}
&\max_{1\le n\le M} \|u(t_n) - U^n\| \\
\le\; & C\left(h^{r+1}\int_0^T [\|u\|_{H^{r+3}} + \|u_t\|_{H^{r+3}}]\,ds\right. \\
&+ k^4 h^{r-3}\int_0^T [h^2\|u\|_{H^{r+3}} + \|u_t\|_{H^{r+3}}]\,ds \\
&+ k^2\int_0^T [\|\Delta u\|_{L^\infty} + \|\Delta u_t\|_{L^\infty} + \|\Delta u_{tt}\|_{L^\infty} + \|u_{ttt}\|_{L^\infty}]\,ds \\
&\left.+ k^4\int_0^T \left\|\frac{\partial^4 u_t}{\partial x^2 \partial y^2}\right\|_{L^\infty} ds\right)
\end{aligned}$$

Numerical Results: $r = 3$

- Confirm predicted optimal L^2 error estimate
- Demonstrate optimal accuracy in L^∞ and H^1 norms
- Demonstrate superconvergence ($O(h^4)$ accuracy) in maximum norm error at the partition nodes in U_x and U_y

Outline

ADI FEG Methods

- Douglas and Dupont (1971)
- Dendy and Fairweather (1975)
- Dryja (1978)
- Hayes (1980)-(1986)
- Fernandes and Fairweather (1991)

And “Laplace-modified methods”

PIDEs with a Positive Type Kernel

$$\begin{cases} \dfrac{\partial u}{\partial t} = \displaystyle\int_0^t \beta(t-s)\Delta u(s)\,ds + f(x,y,t), & (x,y,t) \in \Omega \times (0,T] \\ u(x,y,0) = u_0(x,y), & (x,y) \in \Omega \\ u(x,y,t) = 0, & (x,y,t) \in \partial\Omega \times (0,T] \end{cases}$$

- Δ – the Laplacian: $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$
- Ω – the unit square with boundary $\partial\Omega$

Preliminaries

- S_h – a finite dimensional subspace of $H_0^1(\Omega)$
  - $\dim(S_h) = N'$ (say)
- $(\,\cdot\,,\,\cdot\,)$ – usual L^2 inner product
- $\mathcal{A}(\upsilon, \chi) = \int_\Omega \left(\frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \psi}{\partial y} \right) dxdy, \quad \phi, \ \psi \in H_0^1$

Elliptic Projection

Theorem

For given u, define $W \in S_h$ as the solution of

$$\mathcal{A}(u - W, \chi) = 0, \quad \chi \in S_h$$

If u is sufficiently smooth, then

$$\left\| \frac{\partial^i (u - W)}{\partial t^i} \right\|_{H^j} \leq C h^{r+1-j} \left\| \frac{\partial^i u}{\partial t^i} \right\|_{H^{r+1-j}}, \quad i = 0, 1, 2, \quad j = 0, 1, 2$$

Continuous–Time FEG Scheme

Definition

Given $U(0)$, determine $U(t) \in S_h, \quad t \in (0, T]$, such that

$$\left(\frac{\partial U}{\partial t}, \chi\right) + \int_0^t \beta(t-s)\mathcal{A}\left(U(s), \chi\right) ds = (f, \chi), \quad \chi \in S_h$$

Crank–Nicolson FEG Method

Definition

Given U^0, determine $\{U^n\}_{n=1}^M \subset S_h$ such that

$$(\partial_t U^n, \chi) + Q_{n-1/2}(U)(\chi) = (f(t_{n-1/2}), \chi), \quad \chi \in S_h$$

where

$$Q_{n-1/2}(\varphi)(\chi) = k \sum_{j=1}^{n} \beta(t_{n-j}) \mathcal{A}(\varphi^{j-1/2}, \chi) + \frac{k\beta_0}{2} \mathcal{A}(\varphi^{n-1/2}, \chi)$$

– modified composite midpoint rule

ADI Crank–Nicolson FEG Method

Add the term

$$\frac{k^4\beta_0^2}{16}\left(\frac{\partial^2[\partial_t U^n]}{\partial x \partial y}, \frac{\partial^2 \chi}{\partial x \partial y}\right)$$

to the left hand side to obtain

Definition

Given U^0, determine $\{U^n\}_{n=1}^M \subset S_h$ such that

$$(\partial_t U^n, \chi) + Q^{n-1/2}(U)(\chi) + \frac{k^4\beta_0^2}{16}\left(\frac{\partial^2[\partial_t U^n]}{\partial x \partial y}, \frac{\partial^2 \chi}{\partial x \partial y}\right)$$

$$= (f(t_{n-1/2}), \chi), \quad \chi \in S_h$$

ADI Crank–Nicolson FEG Method

With $v^n = U^n - U^{n-1}$, the method becomes:

$$(v^n, \chi) + \frac{k^2\beta_0}{4}\mathcal{A}(v^n, \chi) + \frac{k^4\beta_0^2}{16}\left(\frac{\partial^2[\partial_t v^n]}{\partial x \partial y}, \frac{\partial^2 \chi}{\partial x \partial y}\right)$$

$$= k(f(t_{n-1/2}), \chi) - k^2 \sum_{j=1}^{n-1} \beta_{n-j}\mathcal{A}(U^{j-1/2}, \chi) - \frac{k^2\beta_0}{2}\mathcal{A}(U^{n-1}, \chi)$$

$$\equiv F^n$$

Algebraic Problem

Let $\{\phi_i\}_{i=1}^{N'}$ be a basis for S_h and set

$$U^n = \sum_{i=1}^{N'} \sum_{j=1}^{N'} U_{i,j}^n \phi_i(x)\phi_j(y),$$

Then, with $\chi = \phi_l \phi_m$,

$$(B \otimes B) + \frac{k^2\beta_0}{4}(A \otimes B + B \otimes A) + \frac{k^4\beta_0^2}{16}(A \otimes A)\boldsymbol{\nu}^{(n)} = \mathbf{F}^{(n)}$$

where

$$\boldsymbol{\nu}^{(n)} = \mathbf{U}^{(n)} - \mathbf{U}^{(n-1)}$$

$$\mathbf{U}^{(n)} = \left[U_{1,1}^{(n)}, \dots U_{1,N'}^{(n)}, U_{2,1}^{(n)}, \dots U_{2,N'}^{(n)}, \dots, U_{N',1}^{(n)}, \dots U_{N',N'}^{(n)}\right]^T$$

$$A = [(\phi_i', \phi_j')]_{i,j=1}^{N'}, \quad B = [(\phi_i, \phi_j)]_{i,j=1}^{N'}$$

Error Estimate

Theorem

Assume that $\beta \in C^2[0,T]$ *and* $U^0 = W^0$. *Then*

$$\max_{1\le n\le M} \|u(t_n) - U^n\| \le Ch^{r+1}\left(\|u_0\|_{H^{r+1}} + \int_0^T \|u_t\|_{H^{r+1}}\,ds\right)$$
$$+Ck^2\left(\|u_0\|_{H^2} + \int_0^T [\|u\|_{H^2} + \|u_t\|_{H^2} + \|u_{tt}\|_{H^2} + \|u_{ttt}\|]\,ds\right)$$

Outline

Second Order BDF FEG Method

Using the trapezoidal rule:

$$q_n(\varphi) = \sum_{j=0}^{n} \omega_{nj}\varphi^j \approx \int_0^{t_n} \beta(t-s)\varphi(s)\,ds, \quad \varphi \in C^0[0,T]$$

where

$$\omega_{nn} = k\beta_0/2, \qquad \omega_{n0} = k\beta_n/2, \qquad \omega_{nj} = k\beta_{n-j}, \quad 1 \le j \le n-1$$

If $\beta \in C[0,T]$, then $q_n(\varphi)$ is ω_0-positive; that is,

$$k\sum_{n=1}^{m} q_n(\phi)\varphi^n \ge -\omega_0(\varphi^0)^2, \quad 1 \le m \le M$$

with $\omega_0 = k^2\beta_0/4$

Second Order BDF FEG Method

Definition

Given U^0, determine $\{U^n\}_{n=1}^M \subset S_h$ such that, for $\chi \in S_h$,

$$(\partial_t U^1, \chi) + Q_1(U)(\chi) = (f(t_1), \chi)$$

$$(\partial_t U^n, \chi) + \frac{2}{3} Q_n(U)(\chi) = \frac{1}{3}(\partial_t U^{n-1}, \chi) + \frac{2}{3}(f(t_n), \chi),$$

$$2 \leq n \leq M$$

where

$$Q_n(U)(\chi) = \frac{k}{2}\beta_n \mathcal{A}(U^0, \chi) + k \sum_{j=1}^{n-1} \beta_{n-j} \mathcal{A}(U^j, \chi) + \frac{k}{2}\beta_0 \mathcal{A}(U^n, \chi)$$

ADI Second Order BDF FEG Method

Definition

$$(\partial_t U^1,\chi)+\frac{\beta_0}{2}\mathcal{A}(U^1,\chi) \quad + \quad \frac{k^4\beta_0^2}{4}\left(\frac{\partial^2[\partial_t U^1]}{\partial x\partial y},\frac{\partial^2\chi}{\partial x\partial y}\right) = (f^1,\chi)-\frac{k\beta_1}{2}\mathcal{A}(U^0,\chi)$$

$$(\partial_t U^n,\chi)+\frac{1}{3}\beta_0\mathcal{A}(U^n,\chi)+\frac{k^4\beta_0^2}{9}\left(\frac{\partial^2[\partial_t U^n]}{\partial x\partial y},\frac{\partial^2\chi}{\partial x\partial y}\right)$$

$$= \frac{1}{3}\left(\partial_t U^{n-1},\chi\right)+\frac{2}{3}\left(f^n,\chi\right)-\frac{2}{3}\sum_{j=0}^{n-1}\beta_{n-j}\mathcal{A}(U^j,\chi)$$

$$-\frac{1}{3}\beta_n\mathcal{A}(U^0,\chi), \qquad 2\le n\le M$$

Error Estimate

Theorem

Assume that $\beta \in C^2([0, T])$. *Then with* $U^0 = W^0$,

$$\max_{1\le n\le M} \|u(t_n) - U^n\|_{L^2} \le Ch^{r+1}\left(\|u\|_{L^\infty(H^{r+1})} + \int_0^T \|u_t\|_{H^{r+1}}\, ds\right)$$

$$+Ck^2 \int_0^T \left[\|u\|_{H^2} + \|u_t\|_{H^2} + \|u_{tt}\|_{H^2} + \|u_{ttt}\|\right] ds$$

Outline

Preliminaries

Example:

$$\beta(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad 0 < \alpha < 1$$

- Γ is the gamma function

Use a graded mesh near $t = 0$:

$$t_n = (nk)^\gamma \quad n \geq 0, \quad \gamma \geq 1, \quad k = \frac{T^{1/\gamma}}{J}.$$

- $\gamma = 1$ – uniform stepsize $k_n = k$
- $\gamma > 1$ – graded mesh near $t = 0$

Set $I_n = (t_{n-1}, t_n), \quad k_n = t_n - t_{n-1}, \quad 1 \leq n \leq M$

Quadrature Rule

$$Q_{n-1/2}(\varphi) = \omega_{n1} k_1 \varphi^1 + \sum_{j=2}^{n} \omega_{nj} k_j \varphi^{j-1/2}$$

where

$$\omega_{nj} = \frac{1}{k_n k_j} \int_{t_{n-1}}^{t_n} \int_{t_{j-1}}^{\min(t_n,t)} \beta(t-s)\, ds\, dt$$

It can be shown that

$$\sum_{n=1}^{m} k_n Q^{n-1/2}(\varphi) \varphi^{n-1/2} \geq 0$$

ADI Crank–Nicolson FEG Method

Definition

Given U^0, determine $\{U_h^n\}_{n=1}^J \subset S_h$ such that

$$(\partial_t U_h^n, \chi) + Q_{n-1/2}(U_h)(\chi) + \frac{k_n^4 w_{nn}^2}{4}\left(\frac{\partial^2(\partial_t U_h^n)}{\partial x \partial y}, \frac{\partial^2 \chi}{\partial x \partial y}\right)$$

$$= (f(t_{n-1/2}), \chi), \quad \chi \in S_h, \quad 1 \le n \le M$$

where

$$Q_{n-1/2}(U_h)(\chi) = \sum_{j=1}^{n} \omega_{nj} k_j \mathcal{A}(U_h^{j-1/2} \chi)$$

and

$$\omega_{nn} = \frac{k_n^{\alpha-1}}{\Gamma(\alpha+2)}$$

Error Estimate

Theorem

Under certain smoothness assumptions,

$$\max_{1\le n\le M} \|u(t_n) - U^n\| \le C(h^2 + \psi(k))$$

where

$$\psi(k) = \begin{cases} k^\gamma, & 1 \le \gamma < 2 \\ k^2|\log k|, & \gamma = 2 \\ k^2, & \gamma > 2 \end{cases}$$

Thank you for your attention