Semiclassical and spectral analysis of a model arising in oceanography

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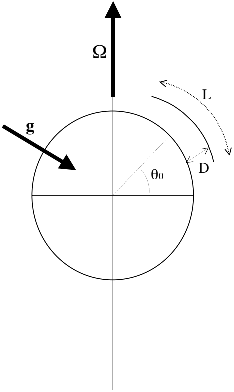
Large-scale oceanic motions

Ocean dynamics are obtained by the superposition of different motions :

- the **solid-body rotation** together with the Earth
- the **macroscopic flow** (associated with a stationary oceanic current such as the Gulf Stream)
- small scale **fluctuations** - due for instance to the wind - on smaller geographical zones

Our goal here is to study a simplified model which nevertheless allows to account for time-persistent structures of vortex-type spreading over dozens of kilometers in the ocean.



$$\left(\frac{du_I}{dt}\right)_I = \left(\frac{du_R}{dt}\right)_R + 2\Omega \wedge u_R - \nabla \frac{|\Omega \wedge r|^2}{2}$$

Neglect the third component of u_3 so that $\Omega \sim \Omega_3 = |\Omega| \sin x_2/R$ where x_2 is the latitude.

Related results

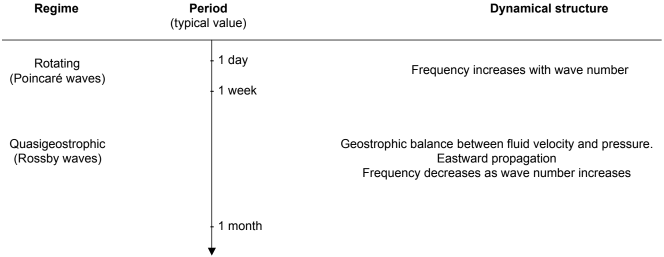
Geophysical flows have been under mathematical study since the mid nineties. One should mention among others the works of

- A. Babin, A. Mahalov, B. Nicolaenko
- D. Bresch, B. Desjardins, D. Gérard-Varet
- F. Charve
- J.-Y. Chemin
- A.-L. Dalibard
- A. Dutrifoy
- E. Grenier
- J.-L. Lions, R. Temam, X. Wang
- A. Majda
- N. Masmoudi
- M. Paicu
- F. Rousset
- S. Schochet
- M. Ziane

Classification of waves

Fluctuations can be decomposed into sums of different waves (which are elementary solutions of some linear equations).

Waves are then classified according to their **main qualitative features :**



Regime	**Period** (typical value)	**Dynamical structure**
Rotating (Poincaré waves)	1 day 1 week	Frequency increases with wave number
Quasigeostrophic (Rossby waves)		Geostrophic balance between fluid velocity and pressure. Eastward propagation Frequency decreases as wave number increases
	1 month	

Persistent structures

Typical scales for oceanic eddies are

- horizontal extent : 10 to 100 km
- persistence : 1 to 10 years

Physical observations show that the existence of these structures is related to

- wind forcing
- macroscopic flow (zonal current)
- propagation of Rossby waves

These waves, the speed of which is comparable to the fluid velocity ($\bar{u} \sim 10\,\mathrm{ms}^{-1}$), may indeed be trapped in some regions, called **ventilation zones**.

A simple mathematical model

We consider the ocean as an **incompressible inviscid fluid with free surface**.

We further assume

- the density of the fluid is homogeneous,
- the pressure law is given by the hydrostatic approximation,
- the motion is essentially horizontal and does not depend on the vertical coordinate,

leading to the so-called **shallow water approximation**.

We will not discuss the effects of the interaction with the lateral boundaries.

The shallow water equations with Coriolis force

The evolution of the water height h and velocity v is then governed by

$$\begin{aligned} \partial_t h + \nabla \cdot (hv) &= 0 \\ \partial_t (hv) + \nabla \cdot (hv \otimes v) + \omega (hv)^\perp + gh\nabla h &= h\tau \end{aligned}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and

- $\omega = |\Omega| \sin x_2 / R$ denotes the vertical component of the earth rotation vector Ω
- $v^\perp = (-v_2, v_1)$
- τ is the stationary forcing responsible for the macroscopic flow (It depends in particular on temperature gradients and topography)

In order to analyze the **influence of the macroscopic convection** on the trapping of Rossby waves, we will consider only small fluctuations and linearize the equations around a stationary solution $(\bar{h}, \bar{v})$ with

$$\bar{h} = constant, \quad \left\{ \begin{array}{c} \nabla \cdot (\bar{v} \otimes \bar{v}) + \omega (\bar{v})^\perp = \tau \\ \nabla \cdot \bar{v} = 0. \end{array} \right.$$

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Orders of magnitude and scaling

In order to exhibit structures like eddies, we have to choose **appropriate observation time and length scales**

$$T \sim 1 \text{ month}, \quad L \sim R \sim 10^4 \text{ km}, \quad \bar{h} \sim 1 \text{ km}, \quad U \sim \bar{v} \sim 10 \text{ ms}^{-1}.$$

Define the Froude and Rossby numbers

$$\mathrm{Fr}^2 := \frac{UL}{Tg\delta h} \sim 10^{-1} \sim \varepsilon, \quad \mathrm{Ro} := \frac{1}{T|\Omega|} \sim 10^{-2}$$

with $\delta h \sim 1\mathrm{m}$ the mean hight fluctuation.

The nondimensional equations on the **height** and **velocity** fluctuations

$$u := \frac{v - \bar{v}}{\delta v} \qquad \eta := \frac{h - \bar{h}}{\delta h} \qquad (\text{with } \delta v \sim 10^{-1} \text{ ms}^{-1})$$

then state, writing $b := \omega/|\Omega|$ and $\bar{u} := \bar{v}/U$ (later $\bar{u} = (\bar{u}_1(x_2), 0)$),

$$\partial_t \eta + \frac{1}{\varepsilon}\nabla \cdot u + \bar{u} \cdot \nabla \eta + \varepsilon^2 \nabla \cdot (\eta u) = 0$$

$$\partial_t u + \frac{1}{\varepsilon}\nabla \eta + \frac{1}{\varepsilon^2} b u^\perp + \bar{u} \cdot \nabla u + u \cdot \nabla \bar{u} + \varepsilon^2 u \cdot \nabla u = 0.$$

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Choice of the initial data : semiclassical framework

One can compute the response to the wind assuming that it prescribes the initial data (**pulse at time 0**) in the form

$$(\eta, u)_{|t=0} = (\eta_k(x), u_k(x)) \exp\left(\frac{ik \cdot x}{\varepsilon}\right).$$

More generally one assumes the initial data is **microlocalized** in a compact set K of $T^*\mathbb{R}^2$: for any $(x_0, \xi_0) \in {}^cK$, there is $\chi \in \mathcal{D}(T^*\mathbb{R}^2)$ equal to one near (x_0, ξ_0), such that

$$\int e^{i(x-y)\cdot\xi/\varepsilon} \chi(y, \xi)(\eta, u)_{|t=0}(y)\, dy = O(\varepsilon^\infty) \quad \text{in } L^2.$$

The natural functional framework is that of **semiclassical Sobolev spaces** :

$$\|U\|^2_{H^m_\varepsilon} := \sum_{|k| \leq m} \|(\varepsilon\partial)^k U\|^2_{L^2}.$$

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System reduction

The system may be reduced to

$$\varepsilon^2 \partial_t U + A(x_2, \varepsilon D) U + \varepsilon^3 Q(U) = 0, \quad \text{with}$$

$$U = (\eta, u_1, u_2) \quad \text{and} \quad A(x_2, \varepsilon D) := \begin{pmatrix} \varepsilon \bar{u} \cdot \varepsilon \nabla & \varepsilon \partial_1 & \varepsilon \partial_2 \\ \varepsilon \partial_1 & \varepsilon \bar{u} \cdot \varepsilon \nabla & -b + \varepsilon^2 \bar{u}_1' \\ \varepsilon \partial_2 & b & \varepsilon \bar{u} \cdot \varepsilon \nabla \end{pmatrix}$$

while $Q(U)$ is a symmetric quadratic form of the type

$$Q(U) = S_1(U) \varepsilon \partial_1 U + S_2(U) \varepsilon \partial_2 U$$

with S_j symmetric.

Using the usual theory of quasilinear symmetric hyperbolic systems and the fact that

$$\|U\|_{L^\infty} \leq \frac{C}{\varepsilon} \|U\|_{H^s_\varepsilon} \quad (s > 1),$$

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Main questions

1. Prove that T_ε^* may be bounded from below
2. Exhibit Rossby and Poincaré type waves (**diagonalization** of the linearized system)
3. Prove that Poincaré waves **disperse** while Rossby waves may be **trapped**.

Additional assumptions : b is a symbol-like function

$$|b^{(\alpha)}(y)| \leq C_\alpha (1 + b^2(y))^{\frac{1}{2}}$$

satisfying $b^2 \to \infty$ at infinity, while b has a finite number of critical points.

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The particular case of the betaplane model

If $\bar{u} = 0$ and $b(x_2) = \beta x_2$ then the above system may be explicitly diagonalized : in particular the (generalized) eigenvalues are

$$\tau_{\pm}(\xi_1, n) = \pm\sqrt{\xi_1^2 + \varepsilon\beta(2n+1)}, \quad \tau_R(\xi_1, n) = \frac{\varepsilon\beta\xi_1}{\xi_1^2 + \varepsilon\beta(2n+1)}.$$

Note that the Rossby eigenvalue is a "subsymbol" compared to the Poincaré eigenvalues.

Theorem 1 : the linear equation

Consider the equation

$$\varepsilon^2 \partial_t V_\varepsilon + A(x_2, \varepsilon D) V_\varepsilon = 0.$$

There is a set $\Lambda \subset T^*\mathbb{R}^2$ such that the following holds. Suppose the initial data is microlocalized in a compact set K such that $K \cap \{\xi_1 = 0\} = \emptyset$. Then for all $t \geq 0$ one can write

$$V_\varepsilon(t) = V_\varepsilon^R(t) + V_\varepsilon^P(t), \quad \text{with}$$

1. If b^2 has only one, non degenerate critical value, then

$$\forall C \text{ compact in } \mathbb{R}^2, \forall t > 0, \quad \|V_\varepsilon^P(t)\|_{L^2(C)} = O(\varepsilon^\infty).$$

2. There is a compact set K' of $T^*\mathbb{R}^2$ such that

$$\forall t \geq 0, \quad \|V_\varepsilon^R(t)\|_{L^2(K')} \neq O(\varepsilon^\infty)$$

if and only if the frequency set of $V_\varepsilon^R(0)$ intersects Λ.

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Theorem 2 : the nonlinear equation

Assume that

$$(1-\varepsilon^2\partial_1^2)^{\frac{3}{2}}(1-\varepsilon^2\partial_2^2+b^2)U_{|t=0} \quad \text{is bounded.}$$

Then

1. There is $T^* > 0$ such that there is a unique solution U on $[0, T^*]$.
2. If $\varepsilon^3 Q(U)$ is replaced by $\varepsilon^{3+\delta} Q(U)$, $\delta > 0$, then for any $T > 0$ there is a unique solution U on $[0, T]$ and

$$\|U_\varepsilon(t) - V_\varepsilon(t)\|_{L^2} \to 0 \quad \text{with } \varepsilon.$$

Remark : A loss of a power of ε, due to the poor Sobolev embedding, is unavoidable. That accounts for the ε^3 factor in front of $Q(U)$. But a priori another loss is due to the variable coefficients of the penalization operator. That is avoided by using variable-coefficient operators which almost commute with the linear propagator (in the spirit of Dutrifoy-Majda-Schochet).

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Polarization on Rossby and Poincaré waves

The strategy is the following : suppose that $i\tau V = A(x_2, \varepsilon D)V$. This allows to deduce a PDE on one component, say V_1 :

$$h(x_2, \varepsilon D; \varepsilon, \tau)V_1 = 0.$$

Then one computes the roots of the polynomial $h(x_2, \xi; \varepsilon, \tau)$. One finds at leading order

$$\tau_{\pm} = \pm\sqrt{\xi_1^2 + \xi_2^2 + b^2}, \quad \tau_R = \frac{\varepsilon b' \xi_1}{\xi_1^2 + \xi_2^2 + b^2} + \varepsilon \bar{u}_1(x_2)\xi_1.$$

By microlocal techniques one can then compute three linear operators $T_{\pm}$ and T_R of principal symbols $\tau_{\pm}$ and τ_R, as well as three operators $\Pi_{\pm}$ and Π_R such that if

$$i\varepsilon^2 \partial_t V_1 = T_{\pm} V_1, \quad i\varepsilon \partial_t \widetilde{V}_1 = T_R \widetilde{V}_1$$

then $h(x_2, \varepsilon D; \varepsilon, \tau)V_1 = O(\varepsilon^{\infty})$ and $h(x_2, \varepsilon D; \varepsilon, \tau)\widetilde{V}_1 = O(\varepsilon^{\infty})$, and the vector fields $\Pi_{\pm} V_1$ and $\Pi_R \widetilde{V}_1$ satisfy the original system up to $O(\varepsilon^{\infty})$.

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Dispersion of Poincaré waves

Goal : prove that Poincaré waves propagating according to $T_\pm$ with principal symbols

$$\tau_\pm = \pm\sqrt{\xi_1^2 + \xi_2^2 + b^2(x_2)}$$

do not affect the dynamics on the time scale considered.
The energy carried by these waves is indeed expected to exit from any fixed compact set for all positive times.

Difficulty : no simple dispersion estimate on times of order $O(\varepsilon^{-2})$.
The Wigner transform does not converge in general to the solution of the semiclassical transport equation for such large times (possible spreading in the phase space).
Two methods : a direct spectral method, or the use of Mourre estimates.
Here we present the first method.

Bohr-Sommerfeld quantization

Consider the equation

$$i\varepsilon^2 \partial_t \varphi = T_\pm \varphi.$$

The **Bohr-Sommerfeld quantization** method allows to prove that the eigenvalues of $T_\pm$ are of the form

$$\lambda_\pm^n = \pm i \sqrt{\xi_1^2 + f(2n+1) + \varepsilon \mu_\pm^n(\xi_1; \varepsilon)}$$

where $\mu_\pm^n(\xi_1; \varepsilon)$ and all its derivatives in ξ_1 are bounded. Similarly the eigenvectors are smooth and bounded.

Decomposition of the initial data

First **Fourier transform** the initial data in x_1,

$$\varphi_0(x) = c_\varepsilon \int e^{ix_1\xi_1/\varepsilon} \widehat{\varphi}_0(\xi_1, x_2)\, d\xi_1$$

and then decompose the resulting x_2-dependent function onto the above **basis** of eigenvectors of $T_\pm$:

$$\varphi_0(x) = c_\varepsilon \sum_n \int e^{ix_1\xi_1/\varepsilon} \underbrace{\varphi_{0,n}(\xi_1)}_{\text{smooth coefficient}} \underbrace{\psi_n(\xi_1, x_2; \varepsilon)}_{\text{smooth bounded eigenvector}} d\xi_1.$$

To recover localization in x_1, decompose $\varphi_{0,n}(\xi_1)$ into **coherent states** :

$$\varphi_0(x) = c_\varepsilon \sum_n \int e^{ix_1\xi_1/\varepsilon} e^{-i\xi_1 q/\varepsilon} e^{-(\xi_1-p)^2/2\varepsilon} \varphi_{qpn}\, \psi_n(\xi_1, x_2; \varepsilon)\, d\xi_1 dpdq.$$

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A non stationary phase argument

The resulting wave on the considered time scale is given by a formula of the type :

$$c_\varepsilon \sum_n \int e^{a(p,\xi_1;\varepsilon)+ib(t,n,q,\xi_1;\varepsilon)} \varphi_{qpn}\, \psi_n(\xi_1, x_2; \varepsilon)\, d\xi_1 dp dq$$

with

$$a(p,\xi_1;\varepsilon) := -(\xi_1 - p)^2/2\varepsilon \quad \text{and}$$

$$b(t,n,q,\xi_1;\varepsilon) := (x_1 - q)\xi_1/\varepsilon \pm \sqrt{\xi_1^2 + f(2n+1) + \varepsilon\mu_\pm^n(\xi_1;\varepsilon)}\, t/\varepsilon^2.$$

Such integrals are $O(\varepsilon^\infty)$ except if there exists a **stationary point for the phase**, given by the conditions :

$$\xi_1 = p \quad \text{and} \quad \varepsilon(x_1 - q) \pm \frac{(2\xi_1 + \varepsilon\partial_{\xi_1}\mu_\pm^n)}{2\sqrt{\xi_1^2 + f(2n+1) + \varepsilon\mu_\pm^n}}\, t = 0.$$

Therefore there is **no critical point** for x_1 in a compact set, and the energy carried by Poincaré waves **exits from any compact set**.

A non stationary phase argument

The resulting wave on the considered time scale is given by a formula of the type :

$$c_\varepsilon \sum_n \int e^{a(p,\xi_1;\varepsilon)+ib(t,n,q,\xi_1;\varepsilon)} \varphi_{qpn}\, \psi_n(\xi_1, x_2; \varepsilon)\, d\xi_1 dpdq$$

with

$$a(p, \xi_1; \varepsilon) := -(\xi_1 - p)^2/2\varepsilon \quad \text{and}$$

$$b(t, n, q, \xi_1; \varepsilon) := (x_1 - q)\xi_1/\varepsilon \pm \sqrt{\xi_1^2 + f(2n+1) + \varepsilon\mu_\pm^n(\xi_1; \varepsilon)}\, t/\varepsilon^2.$$

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Trapping of Rossby waves

Goal : analyze the transport equation for Rossby modes obtained by a standard semiclassical expansion (Wigner transform)

$$\partial_t f + \{\tau_R, f\} = 0$$

with

$$\tau_R(\xi_1, x_2, \xi_2) = \frac{b'(x_2)\xi_1}{\xi_1^2 + \xi_2^2 + b^2(x_2)} + \bar{u}_1(x_2)\xi_1 \,.$$

Difficulty : no global qualitative behaviour for a general convection field $\bar{u}$.

However the system is integrable, so the motion in the reduced phase space (x_2, ξ_2) can be **decoupled**.

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Energy surfaces

Trajectories are **submanifolds of the energy surfaces**

$$\tau_R(\xi_1, x_2, \xi_2) = \tau$$

or equivalently

$$\xi_2^2 = \frac{b'(x_2)\xi_1}{\tau - \bar{u}(x_2)\xi_1} - \xi_1^2 - b^2(x_2) \equiv V_\tau(x_2).$$

Remarks :

- The projection of the energy surfaces onto the x_2 axis is always **bounded**.
- If $V_\tau \to -\infty$ at infinity, then "particle in a well" type behaviour.
- τ/ξ_1 may belong to the image of $\bar{u}$, so **singularities** are also possible.

Motion along x_2

The trajectory (x_2^t, ξ_2^t) starting from (x_2^0, ξ_2^0)

$$\dot{x}_2^t = \frac{\partial \tau_R}{\partial \xi_2}, \quad \dot{\xi}_2^t = -\frac{\partial \tau_R}{\partial x_2}$$

can be of two types, depending on the **possible existence of a singularity** of the function V_τ between two roots of this same function.

For any (x_2^0, ξ_2^0) define $\tau = \tau_0(\xi_1, x_2^0, \xi_2^0)$ and

$$x_{min} = \max\{x < x_2^0 \,/\, x \text{ zero or singularity of } V_\tau\}$$
$$x_{max} = \min\{x > x_2^0 \,/\, x \text{ zero or singularity of } V_\tau\}$$

If x_{min} and x_{max} are turning points, the trajectory is **periodic**.
If x_{min} or x_{max} is a singular point, the trajectory is called **asymptotic**.

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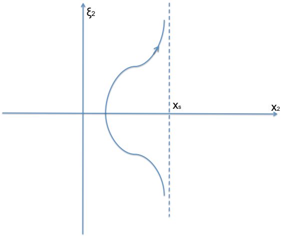
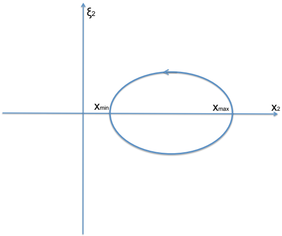


Asymptotic trajectory



Periodic trajectory

Propagation along x_1

We obtain qualitative information on the energy propagation by integrating the equation on $\dot{x}_1^t$.

- If the motion along x_2 is periodic, the asymptotic behaviour of x_1^t depends on the **average of $\dot{x}_1^t$ over one period**.
  The trajectory in the physical space (x_1, x_2) exits from any fixed compact if and only if

$$\frac{1}{T}\int_0^T \dot{x}_1^t \, dt \neq 0$$

- If the motion along x_2 converges to a singular point x_s, the asymptotic behaviour of x_1^t depends only on the **limiting convection velocity**.

Trapped waves and eddies

Performing a change of variable, we can express the **trapping condition** in terms of the initial parameters (ξ_1, x_2^0, ξ_2^0)

$$F(\xi_1, x_2^0, \xi_2^0) = 0 \text{ for some piecewise continuous function.}$$

Proposition. Let $u \in \mathcal{D}(\mathbb{R})$ be a smooth function with a discrete set of critical points, and not identically positive. Then

$$\Lambda = \{(\xi_1, x_2^0, \xi_2^0) \in \mathbb{R}^* \times \mathbb{R} \times \mathbb{R} \,/\, F(\xi_1, x_2^0, \xi_2^0) = 0\}$$

contains a submanifold of $\mathbb{R}^* \times \mathbb{R} \times \mathbb{R}$ of codimension 1.

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Such trapped Rossby waves are, asymptotically as $t \to \infty$,
- **spatially concentrated** on lines (invariant by translation w.r.t. x_1 because of the simplified geometry)

$$x_2^t \to x_s;$$

- **strongly oscillating** with respect to x_2

$$|\xi_2^t| \to \infty.$$

In particular, the associated vorticity concentrates on zero measure sets. The trapping seems therefore to correspond to vortices, which are typical from the **oceanic eddies**.

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