Satellite Conference ICM Bangalore
13-17 August 2010

Some results about singular and quasilinear parabolic and elliptic equations

(Joint works with [Mehdi Badra, Kaushik Bal], [Habib Maagli, Paul Sauvy].)

Jacques Giacomoni
LMAP (UMR 5142), Université de Pau et des pays de l'Adour
Avenue de l'université, 64013 Pau Cedex.
(jgiacomo@univ-pau.fr)

We will mainly be concerned by the following problem :

$$(\mathrm{P}_t)\left\{\begin{array}{c} u_t - \Delta_p u = \frac{1}{u^\delta} + f(t,x) \;\text{ in }\; Q_T \stackrel{\text{def}}{=} (0,T]\times\Omega \\ u = 0 \;\text{ on }\; \Gamma = [0,T]\times\partial\Omega,\; u > 0 \text{ in } Q_T \\ u(0,x) = u_0(x) \;\text{ in }\; \Omega \end{array}\right.$$

where

i) Ω is an open bounded domain in $\mathbb{R}^N$ with smooth boundary,

ii) $1 < p < \infty$, $0 < \delta$, $T > 0$,

iii) $f \in L^\infty(Q_T)$.

We assume in addition that $u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$.

Such problems arise in different models:

1) non Newtonian flows,
2) chemical heterogeneous catalyst kinetics.

Bibliography:

J. Hernández and F. J. Mancebo, *Singular Elliptic and Parabolic Equations*, Handbook of Differential Equations Vol. 3 (2006), 317-400.

M. Ghergu and V. Radulescu, *Singular elliptic problems. Bifurcation and asymptotic analysis.* Oxford University Press, 2008.

Previous works mostly concern the stationnary equation with ($p = 2$): existence, uniqueness, multiplicity, asymptotic behaviour, regularity.

M. G. Crandall, P. H. Rabinowitz, and L. Tartar, *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations, **2** (1977), 193-222.

Concerning the case $p \neq 2$ and the stationnary equation:

[C. Aranda and T. Godoy] Electron. J. Diff. Equations (2004), [K. Perera and E. Silva] Diff. Integral Equations (2007), [G., I. Schindler and Peter Takáč] Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) (2007), [El Manouni, K. Perera and R. Shivaji] to appear, [G., J. Hernandez and A. Moussaoui] to appear.

Concerning the singular parabolic equation: existence, uniqueness/nonuniqueness, asymptotic behaviour, stability $\to p = 2$

. J. Hernández, F. J. Mancebo and L. M. Vega Ann. Inst. H. Poincaré Anal. Non Linéaire (2002),

. J. Davila and M. Montenegro Trans. Amer. Math. Soc. (2005),

. M. Winkler Math. Ann. (2007), Adv. Diff. Equations (2008).

Questions relative to Problem (P_t)**:**

. Existence of solutions (strong solutions? mild solutions? weak solutions?) $\rightarrow$ regularity,

. Uniqueness of solutions.

Difficulties : Equation in (P_t) involves a nonlinear degenerate operator and a singular term $\rightarrow$
Let $A(u) \stackrel{\text{def}}{=} -\Delta_p u - \frac{1}{u^\delta}$, $\mathcal{X} = L^q(\Omega)$, $1 \leq q \leq \infty$.

$$\mathcal{D}(A) \stackrel{\text{def}}{=} \{u \in W_0^{1,p}(\Omega) \mid A(u) \in \mathcal{X}\}?$$

Then, Semi-group Theory for nonlinear operators (V. Barbu, Non-Linear Differential Equations of Monotone Type On Banach Space, Springer Monographs in Mathematics) can not directly applied.

Netherveless, we use a similar method to get existence of *weak* solutions to (P_t): semi-discretization in time method. Precisely,
→ construction of a discrete approximated solution by an iterative scheme via the study of a quasilinear elliptic and singular equation posed in Ω,
→ Energy estimates and the weak comparison principle to pass to the limit and get one solution to (P_t).

Definition.

$$\mathbf{V}(Q_T) \stackrel{\text{def}}{=} \{u \,:\, u \in L^\infty(Q_T),\, u_t \in L^2(Q_T),\, |\nabla u| \in L^\infty(0,T;L^p(\Omega))\}$$

We look for *weak* solutions to (P_t), that is for functions $u \in \mathbf{V}(Q_T)$ satisfying

1. for any compact $K \subset Q_T$, $\operatorname*{ess\,inf}_K u > 0$,
2. for every test function $\phi \in \mathbf{V}(Q_T)$,
$$\int_{Q_T} \left[\phi \frac{\partial u}{\partial t} - |\nabla u|^{p-2} \nabla u \nabla \phi - \phi (\frac{1}{u^\delta} + f(t,x)) \right] dz = 0, \; z = (t,x),$$
3. for every $\phi(x) \in W_0^{1,p}(\Omega)$
$$\int_\Omega \phi(x) u(x,t) \to \int_\Omega \phi(x) u_0(x) \text{ as } t \to 0^+.$$

We need to control the singular term $\frac{1}{u^\delta}$. Let the cone $\mathcal{C}$ be the set of functions $v \in L^\infty(\Omega)$ such that $\exists\, c_1, c_2 > 0$ with

$$\begin{cases} c_1 d(x) \leq v \leq c_2 d(x) \;\; \text{if} \;\; \delta < 1, \\ c_1 d(x) log^{\frac{1}{p}}(\frac{k}{d(x)}) \leq v \leq c_2 d(x) log^{\frac{1}{p}}(\frac{k}{d(x)}) \;\; \text{if} \;\; \delta = 1, \\ c_1 d(x)^{\frac{p}{\delta+p-1}} \leq v \leq c_2 d(x)^{\frac{p}{\delta+p-1}} \;\; \text{if} \;\; \delta > 1, \end{cases}$$

where $d(x) \stackrel{\text{def}}{=} \text{dist}(x, \partial\Omega)$.

Remark. $v \in \mathcal{C} \Rightarrow \frac{1}{v^\delta} \in W^{-1,p'}(\Omega)$ if and only if $\delta < 2 + \frac{1}{p-1}$ (by Hardy Inequality)!

One of the main result is the following theorem :

Theorem 1.[M. Badra-K. Bal-G] Let $1 < p < +\infty$, $p' \stackrel{\text{def}}{=} \frac{p}{p-1}$, $0 < \delta < 2 + \frac{1}{p-1}$, $f \in L^\infty(Q_T)$ and $u_0 \in W_0^{1,p}(\Omega) \cap \mathcal{C}$. Then, there exists a unique weak solution $u \in \mathbf{V}(Q_T)$ to (P_t) such that

(i) for any $t \in [0,T]$, $u(t) \in \mathcal{C}$ (uniformly),

(ii) $\frac{1}{u^\delta} \in L^\infty(0,T; W^{-1,p'}(\Omega))$, $\frac{\partial u}{\partial t} \in L^\infty(0,T; W^{-1,p'}(\Omega))$.

(iii) $u \in C_w(0,T;\, W_0^{1,p}(\Omega))$ and $\lim_{t \to 0^+} u(t) = u_0$ in $W_0^{1,p}(\Omega)$.

Remark. $u \in \mathbf{V}(Q_T) \Rightarrow u \in C([0,T];\, L^q(\Omega))$ for any $1 < q < \infty$.
$u \in C([0,T];\, W_0^{1,p}(\Omega))$?

We recall that (by the Minty-Browder Theorem, or by variational methods) for any $g \in W^{-1,p'}(\Omega)$, there exists a unique $u \in W_0^{1,p}(\Omega)$ satisfying

$$\begin{cases} -\Delta_p u = g \text{ in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases}$$

Similarly, we can prove

Theorem 2. Let $g \in L^\infty(\Omega)$, $\lambda > 0$ and $0 < \delta < 2 + \frac{1}{p-1}$. Then there exists a unique u_λ in $W_0^{1,p}(\Omega) \cap \mathcal{C}$ such that

$$(\mathrm{P}_\lambda) \begin{cases} u - \lambda(\Delta_p u + \frac{1}{u^\delta}) = g \text{ in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases}$$

Theorem 3. Let $g \in L^\infty(\Omega)$, $\lambda > 0$ and $\delta \geq 2 + \frac{1}{p-1}$. Then there exists u_λ in $W^{1,p}_{loc}(\Omega) \cap \mathcal{C}$ such that

$$\begin{cases} u - \lambda(\Delta_p u + \frac{1}{u^\delta}) = g \text{ in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases}$$

is satisfied in the sense of distributions. $u_\lambda \notin W^{1,p}_0(\Omega)$.

Remarks.

1. The restriction $\delta < 2 + \frac{1}{p-1}$ is sharp.
2. If $\delta < 1$, the solution u in Theorem 2 is the unique global minimizer of the strict convex energy functional E defined by
$$E(u) \stackrel{\text{def}}{=} \frac{1}{2}\int_\Omega u^2 + \lambda(\frac{1}{p}\int_\Omega |\nabla u|^p - \frac{1}{1-\delta}\int_\Omega (u^+)^{1-\delta}) - \int_\Omega gu.$$
3. If $0 < \delta < 1$ and $0 \leq g \in L^\infty(\Omega)$ then $u \in C^{1,\alpha}(\overline{\Omega})$ (from [G,Schindler, Takáč] Annali Sc. Sup. Pisa 2007).

Idea of the proof: sub and supersolution method, monotonicity of $-\Delta_p u - \frac{1}{u^\delta}$:

. if $0 < \delta < 1$: variational method and convexity arguments, existence of a supersolution;

. if $1 \leq \delta < 2 + \frac{1}{p-1}$: approximated problem, existence of a subsolution and a supersolution;

. if $\delta \geq 2 + \frac{1}{p-1}$: $\Omega_k \uparrow \Omega$, monotone iterative scheme in domains Ω_k.

Remark. If $\delta < 1$, the solution to (P_λ) $(\lambda > 0)$ is unique and belongs to $\mathcal{C}$.

Main tools used in the proof of Theorem 1: Let $N \geq 2$, $\Delta_t = \frac{T}{N}$, $t_n = n\Delta_t$ for $n \in [0, \cdot\cdot, N]$. From Theorem 2 (with $\lambda = \Delta_t$, $g = f^n + u^{n-1}\Delta_t \in L^\infty(\Omega)$), we define by iteration $u^n \in W_0^{1,p}(\Omega) \cap \mathcal{C}$ with the following scheme:

$$(\mathrm{P}_s) \begin{cases} \frac{u^n - u^{n-1}}{\Delta_t} - \Delta_p u^n - \frac{1}{(u^n)^\delta} = f^n \stackrel{\text{def}}{=} \frac{1}{\Delta_t} \int_{(n-1)\Delta_t}^{n\Delta_t} f(s,\cdot)\,\mathrm{d}s \text{ in } \Omega \\ u^n|_{\partial\Omega} = 0 \end{cases}$$

and $u^0 = u_0 \in W_0^{1,p}(\Omega) \cap \mathcal{C}$. From $u_0 \in \mathcal{C}$, we can show the existence of $\underline{u}$, $\bar{u} \in \mathcal{C}$ <u>independent</u> of Δ_t such that

$$\underline{u} \leq u^n \leq \bar{u},\ \forall n \in [0, \cdot\cdot, N]$$

(by the weak comparison principle).

For $t \in [(n-1)\Delta_t, n\Delta_t)$, $\forall\, n \in [1, \cdots, N]$, let

$$\left[\begin{array}{c} u_{\Delta_t}(t) \stackrel{\text{def}}{=} u^n, \\ \tilde{u}_{\Delta_t}(t) \stackrel{\text{def}}{=} \frac{(t-(n-1)\Delta_t)}{\Delta_t}(u^n - u^{n-1}) + u^{n-1}, \\ f_{\Delta_t}(t) \stackrel{\text{def}}{=} f^n. \end{array}\right.$$

we have

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} - \Delta_p u_{\Delta_t} - \frac{1}{{u_{\Delta_t}}^{\delta}} = f_{\Delta_t} \in L^\infty(Q_T). \qquad (1)$$

in the sense of distributions. We pass to the limit $\Delta_t \to 0^+$ via energy estimates.

Multiplying (P_s) by u^n and summing from $n = 1$ to N:

$$\Delta_t[\sum_{n=1}^{N}\int_\Omega \frac{u^n - u^{n-1}}{\Delta_t}u^n - \sum_{n=1}^{N} < \Delta_p u^n, u^n > - \sum_{n=1}^{N}\int_\Omega (u^n)^{1-\delta}] =$$

$$\Delta_t \textstyle\sum_{n=1}^{N}\int_\Omega f^n u^n.$$

Then, for $\epsilon > 0$ small, by Young Inequality, Sobolev imbedding and Jensen inequality :

$$\sum_{n=1}^{N}\int_\Omega (u^n - u^{n-1})u^n + (\Delta_t - \epsilon)\sum_{n=1}^{N}\|u^n\|^p_{W_0^{1,p}(\Omega)} - \Delta_t\sum_{n=1}^{N}\int_\Omega (u^n)^{1-\delta}$$

$$\leq \Delta_t \sum_{n=1}^{N}[C(\epsilon)\|f^n\|^{\frac{p}{p-1}}_{W^{-1,\frac{p}{p-1}}(\Omega)} \leq C\|f\|^{\frac{p}{p-1}}_\infty.$$

From above estimates together with

$$\Delta_t \sum_{n=1}^{N} \int_\Omega (u^n)^{1-\delta} \leq \begin{cases} T\int_\Omega \bar{u}^{1-\delta} < +\infty & \text{if } \delta \leq 1 \\ T\int_\Omega \underline{u}^{1-\delta} < +\infty & \text{if } 1 < \delta < 2 + \frac{1}{p-1}. \end{cases}$$

we get that u_{Δ_t}, $\tilde{u}_{\Delta_t} \in \mathcal{C}$ uniformly and are bounded in $L^p(0,T;\, W_0^{1,p}(\Omega)) \cap L^\infty(0,T;\, L^2(\Omega))$ independently of Δ_t.
We use a second energy estimate. Multiplying (P_s) by $\frac{u^n - u^{n-1}}{\Delta_t}$ and summing from $n = 1$ to N,

$$\Delta_t \sum_{n=1}^{N} \int_\Omega \left(\frac{u^n - u^{n-1}}{\Delta_t}\right)^2 - \sum_{n=1}^{N} < \Delta_p u^n, u^n - u^{n-1} >$$
$$- \sum_{n=1}^{N} \int_\Omega \frac{u^n - u^{n-1}}{(u^n)^\delta} = \sum_{n=1}^{N} \int_\Omega f^n (u^n - u^{n-1}).$$

From above and convexity arguments, we get the following estimate:

$$\frac{\Delta_t}{2}\sum_{n=1}^{N}\int_\Omega(\frac{u^n-u^{n-1}}{\Delta_t})^2+\frac{1}{p}[\int_\Omega|\nabla u^N|^p-\int_\Omega|\nabla u_0|^p]+$$
$$\tfrac{1}{1-\delta}[\textstyle\int_\Omega(u_0)^{1-\delta}-\int_\Omega(u^N)^{1-\delta}]\leq|\Omega|\tfrac{T}{2}\|f\|^2_{L^\infty(Q_T)}.$$

Together with $\int_\Omega(u^n)^{1-\delta}\leq\max\{\int_\Omega(\bar{u})^{1-\delta},\ \int_\Omega(\underline{u})^{1-\delta}\}$, it follows that $\frac{\partial\tilde{u}_{\Delta_t}}{\partial t}$ is bounded in $L^2(Q_T)$, u_{Δ_t}, $\tilde{u}_{\Delta_t}$ are bounded in $L^\infty(0,T;\,W_0^{1,p}(\Omega))$ uniformly in Δ_t. In addition, $\exists\, C>0$ independent of Δ_t,

$$\|u_{\Delta_t}-\tilde{u}_{\Delta_t}\|_{L^\infty(0,T;\,L^2(\Omega))}\leq\max_{n\in[1,\cdot\cdot,N]}\|u^n-u^{n-1}\|_{L^2(\Omega)}\leq C(\Delta_t)^{\frac{1}{2}}. \quad (2)$$

Taking $N \to \infty$ ($\Rightarrow\ \Delta_t \to 0^+$) and up to a subsequence, we get that

$$\tilde{u}_{\Delta_t} \overset{*}{\rightharpoonup} u,\ u_{\Delta_t} \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(0,T;\, W_0^{1,p}(\Omega)) \text{ and } \ L^\infty(Q_T),$$
$$\tfrac{\partial \tilde{u}_{\Delta_t}}{\partial t} \rightharpoonup \tfrac{\partial u}{\partial t} \quad \text{in } L^2(Q_T),$$
$$\text{from (2)},\ u \equiv v.$$

By compactness imbedding, and interpolation inequality as
$\Delta_t \to 0^+$

$$u_{\Delta_t}, \tilde{u}_{\Delta_t} \to u \quad \text{in } L^\infty(0,T;\, L^q(\Omega)),\, \forall\, 1 < q < \infty.$$

and by Lebesgue Theorem

$$\frac{1}{{u_{\Delta_t}}^\delta} \to \frac{1}{u^\delta} \quad \text{in } L^\infty(0,T;\, W^{-1,\frac{p}{p-1}}(\Omega)).$$

Using (P_s), $u_{\Delta_t} \to u$ in $L^p(0,T;\, W_0^{1,p}(\Omega))$ and u satisfies equation (P_t) weakly. $\limsup\limits_{t \to 0^+} \|u(t)\|_{W_0^{1,p}(\Omega)} \le \|u_0\|_{W_0^{1,p}(\Omega)}$. Then, u is a weak solution to (P_t).

Remarks:

1. $f(t,x) = f(x,u) : f : \Omega \times I\!R^+ \to I\!R^+$ Caratheodory function such that $f(x,0) = 0$, $f(x,\cdot)$ is locally Lipschitz uniformly in $x \in \bar{\Omega}$, $\lim_{s\to\infty} \dfrac{f(x,s)}{s^{p-1}} = 0$ (uniformly in $x \in \bar{\Omega}$) : Theorem 1 holds.

2. The nonlinear operator $Au \stackrel{\text{def}}{=} -\Delta_p u - \frac{1}{u^\delta}$ is m-accretive in $L^\infty(\Omega)$,

$$\mathcal{D}(A) = \{u \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega) | \Delta_p u + \frac{1}{u^\delta} \in L^\infty(\Omega)\}.$$

   Then from the theory of semi-group in convex sets (see [Barbu] "Nonlinear Differential equations of monotone type on Banach space, Springer Monographs in Mathematics", we get that if $u_0 \in \overline{\mathcal{D}(A)}$ then $u \in C([0,T], L^\infty(\Omega))$.

Concerning the case $p = 2$, using maximal regularity results for the heat equation, we get additional properties:

Theorem 4 [M. Badra, K. Bal, G.] $\forall\, 0 < \eta$ small enough,

1. If $\delta < \frac{1}{2}$ and $u_0 \in \mathcal{C} \cap H^{2-\eta}(\Omega)$ then $u \in C(0, T; H^{2-\eta}(\Omega))$,
2. If $\frac{1}{2} \leq \delta < 1$ and $u_0 \in \mathcal{C} \cap H^{2-\delta+\frac{1}{2}-\eta}(\Omega)$, then $u \in C([0, T]; H^{2-\delta+\frac{1}{2}-\eta}(\Omega))$.
3. If $1 \leq \delta$ and $u_0 \in \mathcal{C} \cap H^{\frac{1}{2}+\frac{2}{1+\delta}-\eta}(\Omega)$, then $u \in C([0, T], H^{\frac{1}{2}+\frac{2}{\delta+1}-\eta}(\Omega))$.

Idea of the proof: Interpolation Sobolev spaces Theory + Hardy Sobolev Inequalities+ $L^p - L^q$ maximal regularity results for the heat equation.

Remarks.

1. We don't require any restriction on $\delta > 0$.
2. For $\delta < 3$, $u \in C([0, T], H_0^1(\Omega))$.

$-\Delta$ + homogeneous Dirichlet B.C.($\mathcal{D}(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$)
obeys:

1. $\mathcal{D}((-\Delta)^\theta) = H^{2\theta}(\Omega)$ if $0 \le \theta < \frac{1}{4}$,
2. $\mathcal{D}((-\Delta)^{\frac{1}{4}}) = \{v \in H^{\frac{1}{2}}(\Omega) | \frac{v}{d^{\frac{1}{2}}(x)} \in L^2(\Omega)\} = \tilde{H}^{\frac{1}{2}}(\Omega)$.
3. $\mathcal{D}((-\Delta)^\theta) = \{v \in H^{2\theta}(\Omega) | v = 0 \,\text{on}\, \partial\Omega\}$ if $\frac{1}{4} < \theta \le 1$,

and for $\theta \in [0, 1]$,

$-\Delta$ is an isomorphism from $\mathcal{D}((-\Delta)^{1-\theta})$ onto $[\mathcal{D}((-\Delta)^\theta)]'$.

Definition. Let $1 < q < \infty$, $X_0 \subset X_1$ (densely), $\alpha \in \mathbb{R}$, $\ell \in \mathbb{R}^+$.

$$W_q(\alpha, \ell; X_0, X_1) \stackrel{\text{def}}{=} \{t^\alpha v \in L^q(0, \ell; X_0) | t^\alpha v_t \in L^q(0, \ell; X_1)\}$$

equipped with norm:

$$\|v\|_{W_q(\alpha,\ell;X_0,X_1)} \stackrel{\text{def}}{=} \left(\int_0^\ell (\|t^\alpha v(t)\|_{X_0}^q + \|t^\alpha v_t\|_{X_1}^q) dt \right)^{\frac{1}{q}}$$

Lemma.
Let $\theta \in [0, 1)$ and $q > \frac{2}{1-\theta}$.

(i) The linear operator L defined by $L(f) = z$ where z satisfies

$$z_t - \Delta z = f \text{ in } (0, T] \times \Omega,\ z = 0 \text{ on } \partial\Omega \text{ for } t \geq 0,\ z(t = 0) = 0,$$

is bounded from $L^q(0, T; [\mathcal{D}((-\Delta)^\theta)]')$ onto $W_q(0, T; \mathcal{D}((-\Delta)^{1-\theta}), [\mathcal{D}((-\Delta)^\theta)]')$.

(ii) The space $W_q(0, T; \mathcal{D}((-\Delta)^{1-\theta}), [\mathcal{D}((-\Delta)^\theta)]')$ is continuously imbedded in $C([0, T); \mathcal{D}((-\Delta)^{1-\theta-\frac{2}{q}}))$.

(In our situation, q can be taken as large as necessary).

(ii) follows from the interpolation theory : We recall that for A_0, A_1 two Banach spaces, we can define (by K-method, J. PEETRE) for $0 < \theta < 1$, $1 \leq q < \infty$ and $q = \infty$:

$$(A_0, A_1)_{\theta,q} = \{a \in A_0 + A_1 | \|a\|_{(A_0,A_1)_{\theta,q}} = (\int_0^\infty [t^{-\theta} K(t,a)]^q)^{\frac{1}{q}} < \infty\},$$

$$(A_0, A_1)_{\theta,\infty} = \{a \in A_0 + A_1 | \|a\|_{(A_0,A_1)_{\theta,\infty}} = \sup_{0<t<\infty} t^{-\theta} K(t,a) < \infty\},$$

with

$$K(t,a) \stackrel{\text{def}}{=} \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \quad a \in A_0 + A_1.$$

Examples : Lorentz space $L^{p,q}(\Omega) = (L^1(\Omega), L^\infty(\Omega))_{\frac{p-1}{p},q}$, $L^{p,p}(\Omega) = L^p(\Omega)$, $L^{p,\infty}(\Omega) =$ the weak L^p space of Marcinkiewicz.

Properties of the interpolation space $(A_0, A_1)_{\theta,q}$ (for $0 < \theta < 1$, $1 \leq q \leq \infty$) :

1. $A_0 \cap A_1 \subset (A_0, A_1)_{\theta,q} \subset A_0 + A_1$, $(A_0, A_1)_{\theta,q} = (A_1, A_0)_{1-\theta,q}$,
2. if $q \leq \tilde{q} \leq \infty$, then
   $(A_0, A_1)_{\theta,1} \subset (A_0, A_1)_{\theta,q} \subset (A_0, A_1)_{\theta,\tilde{q}} \subset (A_0, A_1)_{\theta,\infty}$,
3. if $A_0 \subset A_1$, then for $\theta < \tilde{\theta} < 1$, $1 \leq \tilde{q} \leq \infty$,
   $(A_0, A_1)_{\theta,q} \subset (A_0, A_1)_{\tilde{\theta},\tilde{q}}$ holds.
4. $\exists\, c_{\theta,q}$ such that $\forall\, a \in A_0 \cap A_1$, $\|a\|_{(A_0,A_1)_{\theta,q}} \leq c_{\theta,q} \|a\|_{A_0}^{1-\theta} \|a\|_{A_1}^{\theta}$.

References : H. Triebel, L. Tartar, R. A. Adams

Definition (*Space of traces*, J. L. LIONS) [$\ell = \infty$]
$T(q, \alpha, X_0, X_1) \stackrel{\text{def}}{=} \{x \in X_1 \,:\, \exists\, u \in W_q(\alpha, X_0, X_1),\, u(0) = x\}$,
endowed by the norm
$\|x\|_{T(q,\alpha,X_0,X_1)} \stackrel{\text{def}}{=} \inf\{\|u\|_{W_q(\alpha,X_0,X_1)} \,:\, u(0) = x\}$, is a Banach space.

Some properties of $T(q, \alpha, X_0, X_1)$:

1. $X_0 \subset T(q, \alpha, X_0, X_1) \subset X_1$,
2. $u \in C([0, \infty); T(q, \alpha, X_0, X_1))$ for any $u \in W_q(\alpha, X_0, X_1)$.
3. (*Equivalence Theorem,* P. GRISVARD) Let $0 < \alpha + \frac{1}{q} < 1$, $\theta = \alpha + \frac{1}{q}$. Then, $T(q, \alpha, X_0, X_1) \cong (X_0, X_1)_{\theta,q}$.

Using above properties, we have that for $\theta \in [0,1)$

$$(\mathcal{D}((-\Delta)^{1-\theta}), [\mathcal{D}((-\Delta)^{\theta})]')_{\frac{1}{q},q} = (-\Delta)^{\theta}(\mathcal{D}(\Delta), L^2(\Omega))_{\frac{1}{q},q},$$

$$(\mathcal{D}(\Delta), L^2(\Omega))_{\frac{1}{q},q} \subset (\mathcal{D}(\Delta), L^2(\Omega))_{\frac{2}{q},1} \subset \mathcal{D}((-\Delta)^{1-\frac{2}{q}}),$$

and by Hardy Inequalities for $u \in \mathcal{C}$

$$\frac{1}{u^{\delta}} \in \begin{cases} (\mathcal{D}((-\Delta)^{\theta}))' \quad \text{with } \theta > \max(0, \dfrac{\delta}{2} - \dfrac{1}{4}) \text{ if } \delta < 1 \\ (\mathcal{D}((-\Delta)^{\theta}))' \quad \text{with } \theta > 1 + \frac{\delta-3}{2(\delta+1)} \text{ if } \delta \geq 1 \end{cases}$$

Theorem 4 follows.

Singular absorption case:

$$(S_t)\begin{cases} u_t - \Delta_p u + f(t)\frac{\chi_{\{u>0\}}}{u^\delta} = 0 \ \text{ in } \ Q_T \\ u = 0 \ \text{ on } \Gamma, \ u(0,x) = u_0(x) \ \text{ in } \Omega \end{cases}$$

with f a nonnegative continuous function on $I\!R^+$. We can prove :

1. For any $u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$, $0 < \delta < 1$, $u_0(x) \geq$ a.e. in Ω, Problem (S_t) admits at least one weak solution in $\mathbf{V}(Q_T)$ (method: Laydysenskaia type approximation +approximation of the singular term).
2. (quenching in a finite time) Under some additional conditions on f and $p \geq \frac{2N}{N+2}$, every solution to S_t satisfies $u(t) \equiv 0$ for $t > t^*$ (t^* depending on u).

Effect of singular absorption term:

$$(\mathrm{P}_\lambda)\begin{cases} -\Delta_p u = K(x)(-u^q + \lambda u^r) \text{ in } \Omega \\ u|_{\partial\Omega} = 0, \ u \geq 0 \text{ in } \Omega \end{cases}$$

where $K(x) \sim d(x)^{-k}$ with $0 < k < p$, $-1 < q < r < p-1$, $\lambda > 0$ large enough.

Theorem 5 [G., H. Maagli, P. Sauvy]

1. If $k < 1 + q$, there exists a positive solution $u_\lambda \in C^{1,\beta}(\bar{\Omega})$ $(0 < \beta < 1)$, to (P_λ).
2. If $k \geq 1 + q$ and $q > 0$, then there exists at least one non trivial bounded solution and every bounded solution has a compact support and belongs to $C^{1,\beta}(\bar{\Omega})$.

Existence of a supersolution to (P_λ): Let

$$(\mathrm{P}) \begin{cases} -\Delta_p u = K(x)u^r \text{ in } \Omega \\ u|_{\partial\Omega} = 0, \ u \geq 0 \text{ in } \Omega \end{cases}$$

Proposition There exists a unique weak solution u to (P) such that $\exists\, c_2 > c_1 > 0$

- . if $k < 1 + r$, $c_1 d(x) \leq u \leq c_2 d(x)$;
- . if $k = 1 + r$, $c_1 d(x) \ln(\frac{A}{d(x)})^{\frac{1}{p-1-r}} \leq u \leq c_2 d(x) \ln(\frac{A}{d(x)})^{\frac{1}{p-1-r}}$;
- . if $1 + r < k < p$, $c_1 d(x)^\alpha \leq u \leq c_2 d(x)^\alpha$ with $\alpha = \frac{p-k}{p-1-r}$.

Existence of a positive subsolution to P_λ (if $k < 1 + r$): ϕ_1^τ with $\tau \geq \frac{p-k}{p-1-q} > 1$.