Existence and Non-Existence of TV Bounds for Scalar Conservation Laws with Discontinuous Flux

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Outline

- Conservation Laws with Continuous Flux.
- The Discontinuous Flux Problem.
- Explicit formulas for solutions
- Interface entropy conditions
- Existence of non-classical shocks and (A, B) entropy conditions.
- Total variation bound for the solutions.

Conservation Laws,Continuous Flux

$$u_t + f_x(x,u) = 0, \quad (x,t) \in \mathbb{R} \times (0,\infty).$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}.$$

$x \longrightarrow f(x,u)$ is Lip. continuous

f is smooth in u

Weak Formulation, Rankine Hugoniot Condition and Entropy condition

Let $u_0 \in L^\infty(\mathbb{R})$.
We say $u \in L^\infty(\mathbb{R} \times [0,\infty))$ is a **weak solution** if
$\forall \varphi \in C_0^1(\mathbb{R} \times [0,\infty))$,

$$\int_0^\infty \int_{\mathbb{R}} \{u\frac{\partial \varphi}{\partial t} + f(x,u)\frac{\partial \varphi}{\partial x}\}dxdt + \int_{\mathbb{R}} u_0\varphi(x,0)dx = 0.$$

$$\Leftrightarrow$$

Jump condition(**R-H** condition)

$$(f(x,u_+) - f(x,u_-)) = \frac{dx}{dt}(u_+ - u_-)$$

Where $x = x(t)$ is the curve of discontinuity of u.

Kruzkov Entropy Condition

- Infinitely many Weak solutions.

Kruzkov Entropy Condition(Uniqueness):

$$\int_0^T \int_{\mathbb{R}} |u(x,t) - k|\phi_t + \text{sign}(u(x,t) - k)[f(x, u(x,t)) - f(k)]\phi_x$$
$$- \int_0^T \int_{\mathbb{R}} sign(u - k)(f(x,k))_x \varphi)dxdt \quad \geq \quad 0$$

(0.1)

$\forall\ k \in \mathbb{R}$, and any $\phi \in C_0^1(\mathbb{R} \times [0, \infty))$, with $\phi(t,x) \geq 0$,

Different technique to prove existance

- If $f(x,u) = f(u)$ is Convex $\Rightarrow$ **Hopf-Lax** formula.
  For **general** f
- Vanishing-viscosity method by **Kruzkov**.
- Numerical Schemes.
- All the methods $\Longrightarrow$ u is in **BV**.
- To prove the convergence (of approximations) using BV $\hookrightarrow L^1_{\text{loc}}$.

Discontinuous Flux

We consider

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(F(x,u)) = 0, x \in \mathbb{R}, t > 0,$$

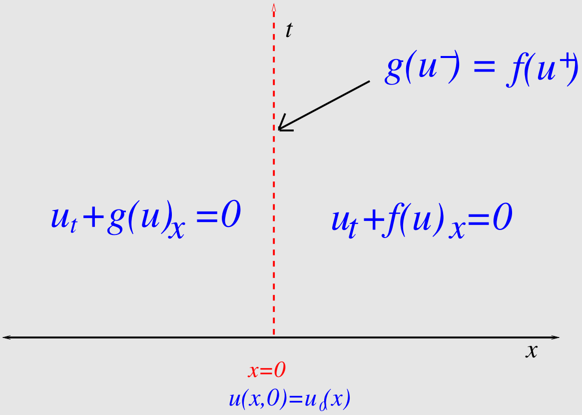
$$u(x,0) = u_0(x),\ \ x \in \mathbb{R}$$

- Where the flux function $F(x,u)$ is given by $F(x,u) = H(x)f(u) + (1 - H(x))g(u)$

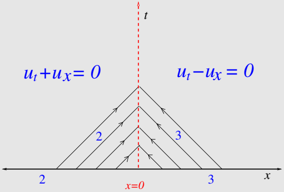
which is a **discontinuous function** of x, H is the Heaviside function, where both f and g are smooth and convex type.

- **Application**: Modeling two phase flow in porous media, sedimentation problem and in traffic flow.

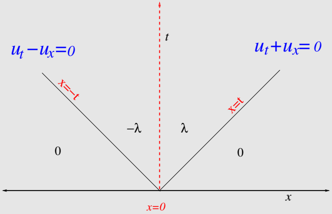
Weak Formulation and R-H condition



Non-Existence and Non-Uniqueness

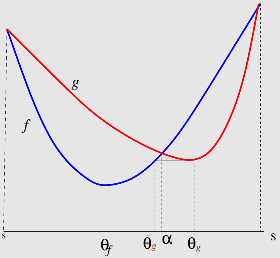


$u^+ \neq -u^-$ (No R–H condition)



$u(x,0) \equiv 0$

Choise of f and g



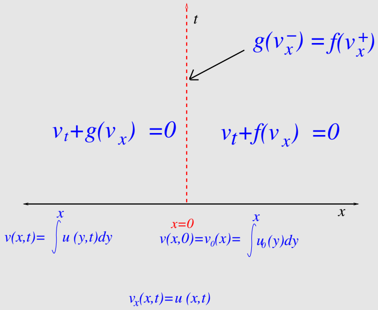
Gimse and Risebro $\longrightarrow$ *RiemannProblem*(1990) :

Uniqueness:$min\{|u^- - u^+|\}$

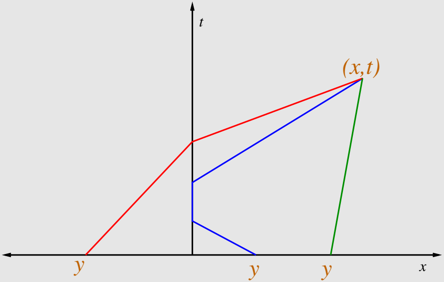
Diel:Γ Condition (small total variation)

$$u_t + F^\delta(x,u)_x = \epsilon u_{xx} \qquad x \in \mathbb{R},\ \ t > 0$$
$$u(x,0) = u_0(x) \qquad x \in \mathbb{R}.$$

Explicit Formula for Discontinuous H-J Eqn.



Explicit Formula for Discontinuous H-J Eqn.



Hopf-Lax formula for discontinuous Hamiltonian

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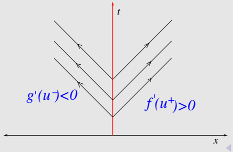
Adimurthi and Gowda, *J. Math. Kyoto Univ., 2003*

- Viscosity solution.
- set $u = v_x$ is a solution to conservation law
- Away from interface satisfies **Lax entropy condition**.i.e $u^- > u^+$
- **Interface Entropy Condition**: At $x = 0$,

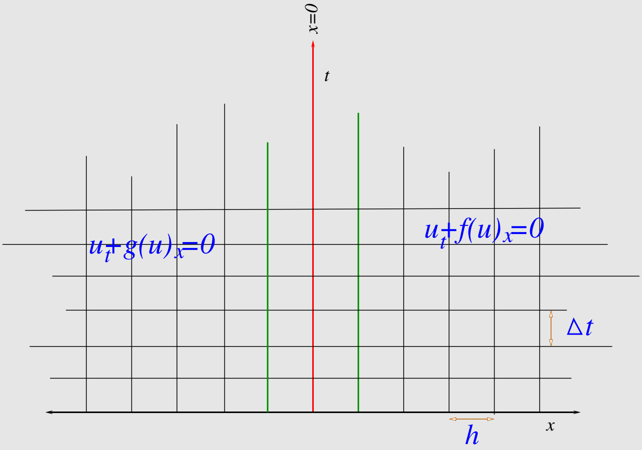
$$\text{meas}\{t : f^{'}(u^+(t)) > 0,\ g^{'}(u^-(t)) < 0\} = 0.$$

Undercompressive waves are not allowed.



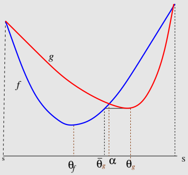
- Under the **interface entropy condition at** $x = 0$ **and Lax entropy condition away from interface** uniqueness results are proved.

Godunov Scheme



- Compute the solution of **Riemann Problem** by using explicit formula.

Explicit formula for Godunov Flux



- At the interface for convex fluxes f and g:
  $\overline{F}(a,b) = \max\{g(\max(a,\theta_g)), f(\min(\theta_f, b))\}$
- Scheme is monotone under CFL$=\frac{\Delta t}{h} \sup \max\{f', g'\} \leq 1$
- **Failure of the consistency property**

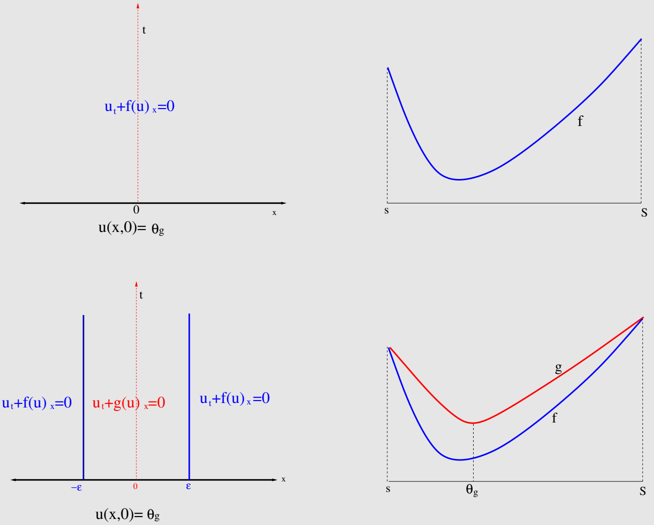
$$\overline{F}(a,a) \neq g(a) \text{ or } f(a)$$

Total variation Bound

- By using the **Singular Mapping technique** of Temple,convergence results are established. Ref:Adi+ Jaffre +Gowda,SINUM(2003)
- Lack of information about total variation bound even though numerical solution satisfies L^1 contraction property.
- solution computed by us agrees with numerical solution computed by petrolium engineers by using Upstream Mobility flux.

CL

PCL

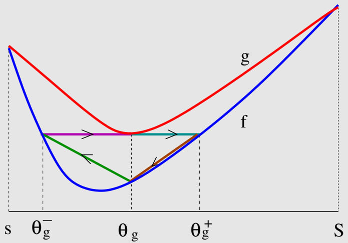


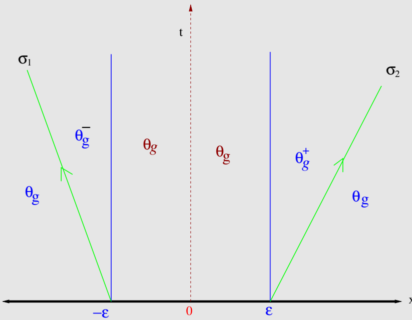
Winther et al. (Instability of Buckley-Leverett Flow in a Hetrogeneous Medium, Transport in Porous Media:1992)

and termed this phenomina as *instability* in the flow function

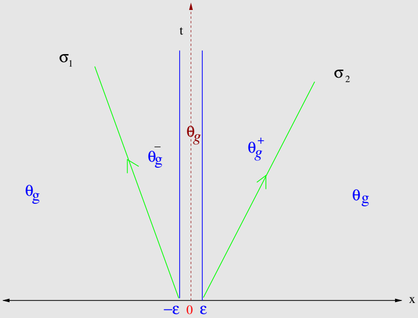
Existance of a Non-Classical Shocks

Solution for PCL





Existance of a Non-Classical Shocks



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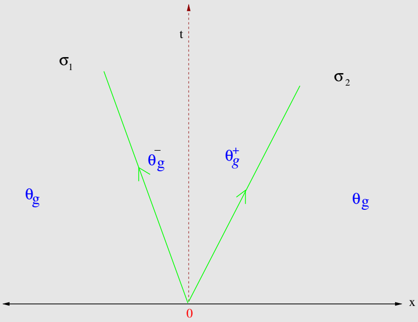
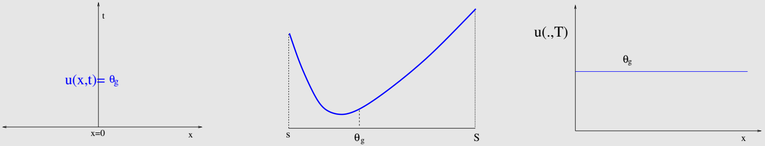
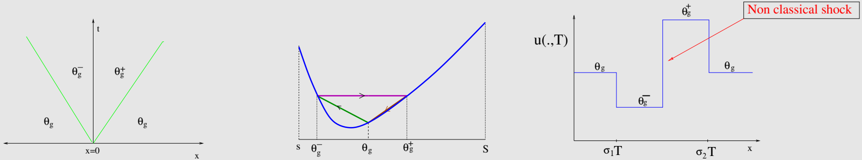


Existance of a Non-Classical Shocks

Solution of CL



Limit of PCL

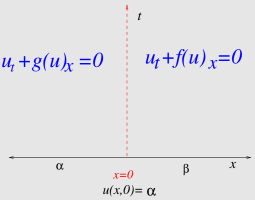


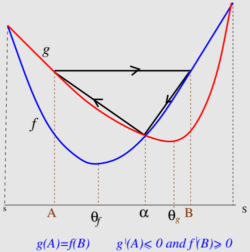
Adimurthi,Sudharshan Kumar and Gowda

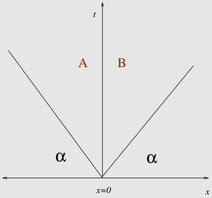
Entropy Condition Disagree With Other

- **Gimse and Risebro** $\Rightarrow$ Minimal of $|u^+ - u^-|$ Condition $\Rightarrow$ Non Uniqueness.
- **Karlsen, Risebro and Towers** $\Rightarrow$ Vanishing Viscosity $\Rightarrow$ Undercompressive Waves.
- **Diehl** $\Rightarrow$ Γ condition $\Rightarrow$ Undercompressive Waves.
- **Adimurthi,Gowda**'s entropy condition is compatible with the experimental results obtained from **oil industry**.
- On the other hand, the solution obtained by **Karlsen et. al.** has a physical evidence for sedimentation problem.
- It is not clear which solution is **physically relevant**
- **Depending on the physical situation?**

Generalised AB Entropy Condition







Interface AB Entropy condition

$$I_{AB}(t) = sign(u^{-}(t) - A)(g(u^{-}(t)) - g(A)) - sign(u^{+}(t) - B)(f(u^{+}(t)) - f(B)).$$

$$I_{AB}(t) \geq 0 \ \ a.e \ \ t.$$

$$\overline{F}(a, b) = \max\{g(\max(a, A)), f(\min(B, b))\}$$

Adimurthi, Mishra and Gowda, *JHDE, 2005,JDE,2007*

Existence of TV Bounds

- When the flux function is discontinuous is it possible to get total variation bound?
- Burger, Karlsen, and Towers proved $u \in \mathrm{BV}$ away from **Interface**.
- What happens around the **Interface**?
- We settle the complete answer. Adimurthi, Dutta,Shyam and Gowda,*to appear in Comm.Pure and Appl.Math*

Existence of TV Bounds

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(F(x,u)) = 0, x \in \mathbb{R}, t > 0,$$
$$u(x,0) = u_0(x), \ \ x \in \mathbb{R}$$

Where the flux function $F(x,u)$ is given by

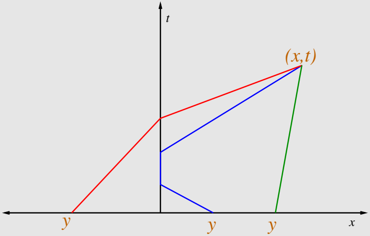
$$F(x,u) = H(x)f(u) + (1 - H(x))g(u),$$

H is the Heaviside function

Equivalently,

$$\begin{aligned} u_t + (vf(u) + (1-v)g(v))_x &= 0 \\ v_t + (v(1-v))_x &= 0 \end{aligned}$$

$$\mathrm{u(x,0)} = \mathrm{u}_0(x), \quad v(x,0) = \begin{cases} 0 & \text{if} \quad x < 0, \\ 1 & \text{if} \quad x > 0 \end{cases} .$$



By **R-H** condition

$$f(u(0^+,t)) = g(u(0^-,t))$$

$$f(u(x,t)) = f(u(0^+,t)) = g(u(0^-,t)) = g(u_0(z))$$

$$u(x,t) = f^{-1}g(u_0(z)) \notin \; BV.$$

Critical Points playing a role.

- If we do not allow critical points we have shown that $u \in \mathrm{BV}$.
- i.e Let θ_f and θ_g be the critical points of f and g,

$$A \neq \theta_g \text{ and } B \neq \theta_f$$

$$\Longrightarrow$$

$$u \in \mathrm{BV}.$$

Main Results: Existence of TV Bound

Theorem

1) Let $u_0 \in BV$, $T > 0$ *and* $A \neq \theta_g$ *and* $B \neq \theta_f$. *Then there exists* $C > 0$ *such that for all* $0 < t \leq T$,

$$TV\big(u(.,t)\big) \leq C \ TV(u_0) + 6||u||_\infty.$$

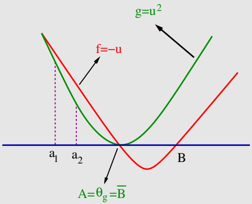
2) Let $u_0, \ f_+^{-1}g(u_0), \ g_+^{-1}f(u_0) \in \mathrm{BV}, \ T > 0$ and $A = \theta_g$. Then for all $0 < t \leq T$,

$$\begin{aligned} &\mathrm{TV}\big(u(.,t)\big) \\ &\leq \mathrm{TV}(u_0) + \max\big(TV(f_+^{-1}g(u_0)), \mathrm{TV}(g_+^{-1}f(u_0))\big) \\ &+6||u||_\infty. \end{aligned}$$

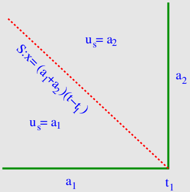
Non Existence of TV Bound

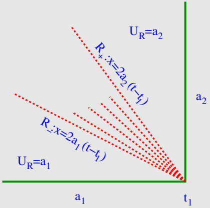
Theorem

There exist $u_0 \in BV \cap L^\infty$ such that $TV(u(.,t)) = \infty$ if $A = \theta_g$ or $B = \theta_f$.



Non Existence of TV Bound





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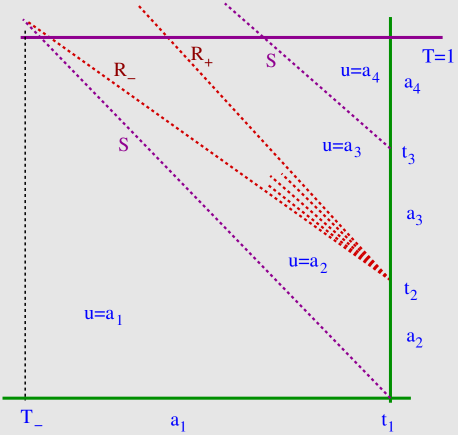
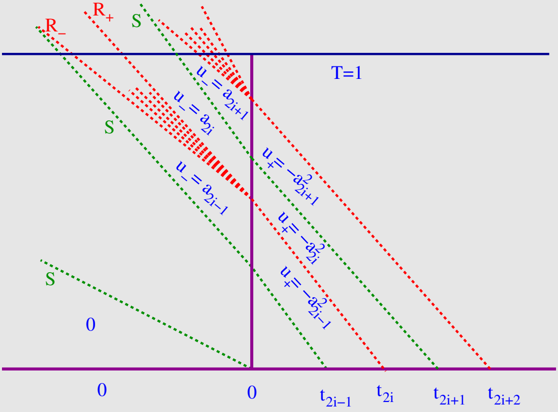


Non Existence of TV Bound



Non Existence of TV Bound

Let $0 = t_0 < t_1 < t_2 < \ldots\ldots$ be a sequence to be determined. Let $a_0 = 0$ and $\{a_i\}_{i \geq 1}$ be defined by

$$a_{2i-1} = -\frac{1}{(i+1)},\ a_{2i} = -\frac{1}{(i+1)^2}$$

$$u_0(x) = \begin{cases} 0 & \text{if } x < 0, \\ -a_{2i+1}^2 & \text{if } t_{2i} < x < t_{2i+1}, \\ -a_{2i+2}^2 & \text{if } t_{2i+1} < x < t_{2i+2}, \\ -1 & \text{if } x \geq 1. \end{cases}$$

$$\begin{aligned} \mathrm{TV}\big(u(.,1)\big) &\geq \sum_{n=1}^{\infty} |u(x_n,1) - u(x_{n+1},1)| \\ &= \sum_{n=1}^{\infty} |a_n - a_{n+1}| \\ &\geq \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n^2} = \infty. \end{aligned}$$

Remark

Let $\epsilon > 0$ and $v_{0,\epsilon} = \epsilon^2 u_0$ where u_0 is as in Step 4. Then $\mathrm{TV}(v_{0,\epsilon}) = \epsilon^2 \mathrm{TV}(u_0)$. Let v_ϵ be the corresponding solution. Observe that time steps in previous figure are independent of ϵ. Hence $\mathrm{TV}(v_\epsilon(.,1)) = \infty$. This shows that smallness of the total variation of initial data does not guarantee the boundedness of the total variation at the interface.

Applications of DFLUX to solve Systems

$$
\begin{aligned}
u_t + f(u,v)_x &= 0 \\
v_t + g(u,v)_x &= 0
\end{aligned}
$$

with initial conditions:

$$
\begin{aligned}
u(x,0) &= u_0(x) \\
v(x,0) &= v_0(x)
\end{aligned}
$$