Nonexistence of k-peak solutions to elliptic problems in convex domains

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The Theorem of Gidas, Ni and Nirenberg (1979)

Let us consider a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain which is convex with respect to $x_1, .., x_N$ and is symmetric with respect to the planes $x_1 = 0, .., x_N = 0$ and f is a locally Lipschitz function.
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Corollary

A consequence of the previous theorem is that 0 is the unique critical point of u.

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$$\sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x)(x_i - x_{0,i}) < 0 \quad \text{for any } x \in \Omega \setminus \{x_0\}.$$

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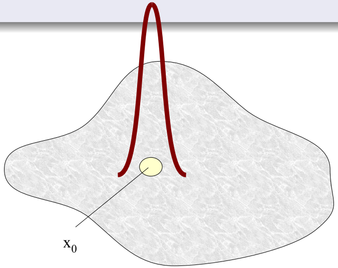

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Conjecture

Let u be a solution of

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

with $1 < p < \frac{N+2}{N-2}$. Then, if Ω is (strictly) convex, the monotonicity property holds.

A brief history on the problem

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Cabré and Chanillo (1998)

Let Ω be a bounded, convex domain of $\mathbb{R}^2$ where the boundary has positive curvature. Suppose that $f \in C^\infty(\mathbb{R})$, $f \geq 0$ and let u be a semi-stabile solution of (i.e. the first eigenvalue of the linearized operator is nonnegative)

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Applications

The previous result applies to the minimal solutions to $f(s) = \lambda e^s$, $\lambda > 0$, and $f(s) = \lambda(1+s)^p$, $\lambda > 0$ and $p > 1$.

Other results

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Grossi and Molle (2003)

Let Ω be a bounded, convex domain of $\mathbb{R}^N$, $N \geq 3$. Let us suppose that u_ϵ is a positive solution of

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}-\epsilon} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which satisfies (S is the best constant in Sobolev embedding)

$$\frac{\int_\Omega |\nabla u_\epsilon|^2}{\left(\int_\Omega |u_\epsilon|^{\frac{2N}{N-2}-\epsilon}\right)^{\frac{2(N+2)}{2N+\epsilon(N+2)}}} \to S$$

Then, denoting by x_ϵ the point where the maximum of u_ϵ is achieved, we have that $(x - x_\epsilon) \cdot \nabla u_\epsilon(x) < 0 \quad$ for any $x \in \Omega \setminus \{x_\epsilon\}$, provided ϵ is small enough.

A problem in the plane: the case of one peak

Let us consider the following "model problem",

The Gelfand problem

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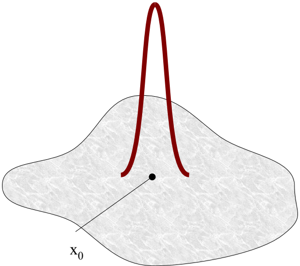
Theorem

For any $\Omega \subset \mathbb{R}^2$, there exists a solution u_λ with one peak, i.e.

$$u_\lambda(x) \to G(x, x_0) \quad \text{as } \lambda \to 0.$$

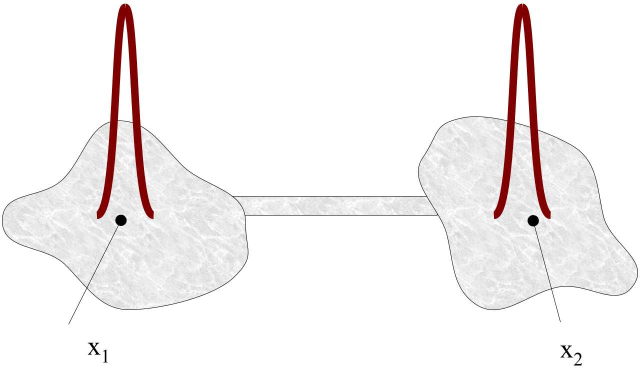


A problem in the plane: the case of k peaks.

According to the shape of the domain, we can have solution with more peaks

Theorem, Esposito, Grossi e Pistoia (2005)

Under some suitable assumptions on $\Omega \subset \mathbf{R}^2$, there exist solution with k peaks, for any $k \geq 2$.



Location of the peaks

The location of the peaks of the solution has been characterized.
Let us denote by $P_{1,\lambda} \to P_1$,.., $P_{k,\lambda} \to P_k$ the peaks of the solution u_λ.
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If we have that

$$\lambda \int_\Omega e^{u_\lambda} \to 8k\pi \quad \text{as } \lambda \to 0,$$

then

$$\frac{1}{2} \nabla R(P_i) - \sum_{j=1, j \neq i}^{k} \nabla_x G(P_i, P_j) = \mathbf{0}$$

Corollary (the case of 2 peaks)

Let us suppose to have a 2-peak solution to

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Then, we have that

$$\lambda \int_\Omega e^{u_\lambda} \to 16\pi \quad \text{per } \lambda \to 0,$$

$$\begin{cases} \frac{1}{2}\nabla R(P_1) = \nabla_x G(P_1, P_2) \\ \frac{1}{2}\nabla R(P_2) = \nabla_x G(P_2, P_1). \end{cases}$$

Theorem, Grossi and Takahashi (2010)

Let Ω be convex and u_λ be a solution of

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega \subset \mathbf{R}^2, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $||u_\lambda|| \to +\infty$ as $\lambda \to 0$. Then we have that,

$$\lambda \int_\Omega e^{u_\lambda} \to 8\pi \quad \text{per } \lambda \to 0.$$

Hence, for λ small enough, all the solutions have just one peak.

Lemma

Let $G(x, y)$ be the Green function of $-\Delta$ with zero Dirichlet boundary condition.

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$$\int_{\partial\Omega}(x-Q)\cdot\nu(x)\left(\frac{\partial G(x,P_1)}{\partial\nu_x}\right)\left(\frac{\partial G(x,P_2)}{\partial\nu_x}\right)ds_x$$

$$=(2-N)G(P_1,P_2)+(Q-P_1)\cdot\nabla_x G(P_1,P_2)+(Q-P_2)\cdot\nabla_x G(P_2,P_1)$$

where $Q\in\mathbf{R}^2$ and $\nu(x)$ is the unit outer normal to $x\in\partial\Omega$.

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$$(1) \qquad w(x) = (x-Q)\cdot\nabla_x G(x,P_1),$$

which satisfies $-\Delta w = (x-Q)\cdot\nabla\delta_{P_1} + 2\delta_{P_1}$.

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$$-\int_\Omega \Delta w G(x,P_2)=-\int_\Omega \Delta G(x,P_2)w+\int_{\partial\Omega}\frac{\partial G(x,P_2)}{\partial\nu_x}w$$

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$$=(P_2-Q)\cdot\nabla_x G(P_2,P_1)+\int_{\partial\Omega}\frac{\partial G(x,P_2)}{\partial\nu_x}(x-Q)\cdot\nu_x\frac{\partial G(x,P_1)}{\partial\nu_x}$$

Proof of the main result

Sketch of the proof

Let Ω be convex and u_λ be a solution of

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Sketch of the proof

- Step1: boundedness of $\lambda \int e^u$
- Step2: nonexistence of k-peak solution with $k \geq 2$ via the integral identity.

Theorem

Let Ω be convex and u_ϵ be a solution of

$$(P_\epsilon) \qquad \begin{cases} -\Delta u = u^{\frac{N+2}{N-2}-\epsilon} & \text{in } \Omega \subset \mathbb{R}^N, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Then, for $\epsilon > 0$ small enough, all solutions have just one peak.

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- Location of the peaks. It is (almost) the same of the two dimensional problem, (Bahri,Li e Rey (1995)).
- Convexity of the Robin function. In dimensions $N \geq 3$ the convexity of the Robin function was proved by Cardaliaguet and Tahraoui (2002).

Thank you

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