Existence and uniqueness for some two-dimensional problems

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- Differential geometry (Prescribing curvature)
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1. Prescribing curvature

Let (S, g_0) be a compact Riemannian surface

Let K_0 be the Gauss curvature relative to g_0

After conformal change of the metric $g := e^{2u} g_0$, the Gauss curvature K_g becomes

$$\frac{-\Delta_0 u + K_0(x)}{e^{2u}} = K_g(x)\,.$$

Equivalent to:

$$-\Delta_0 u = K_g(x) e^{2u} - K_0(x)$$

Kazdan-Warner (1970), main paper (1975).

Set $\bar{u} := \frac{1}{|S|}\int_S u$, and let us write equation for $U := 2(u - \bar{u})$

$$-\frac{1}{2}\Delta U = K_g(x) e^{\bar{u}} e^{U} - K_0(x)$$

Integrate on (S, g_0):

$$e^{\bar{u}} = \frac{\int_S K_0}{\int_S K_g e^U},$$

$$-\Delta U = 2(\int_S K_0)\left(\frac{K_g e^U}{\int_S K_g e^U} - \frac{K_0}{\int_S K_0}\right)$$

We are led to the PDE

$$-\Delta U = \lambda \left(\frac{K_g e^U}{\int_S K_g e^U} - \frac{K_0}{\int_S K_0} \right), \quad \int_S U = 0\,.$$

On the sphere, problem of prescribing curvature leads

$$-\Delta U = 8\pi \left(\frac{K_g e^U}{\int_{S^2} K_g e^U} - \frac{1}{|S^2|} \right)$$

2. Minimal surface

K. Uhlenbeck, *Closed Minimal surface in hyperbolic* 3*-manifolds* (1983).

Hyperbolic manifold: Riemannian manifold with sectional curvature -1.

Consider a closed surface S of genus ≥ 2, a 3-hyperbolic manifold and an immersion $S \to M^3$.

Let (S, g) be the surface S with the induced metric, and set g_0 to be a metric such that:

- g conformally equivalent to g_0;
- Gauss curvature of g_0 is -1.

Using isothermal coordinates, we have

$$g_0 = e^{2v_0} id \qquad g = e^{2v} g_0$$

Furthermore, the second fundamental form is the real part of a holomorphic quadratic differential α on (S, g_0).

Fix α, and consider $t\alpha(x)$.

The function $u := v + v_0$ satisfies the Gauss equation:

$$\frac{-\Delta_0 u - 1}{e^{2u}} = -1 - t^2 |\alpha(x)|^2 e^{-4u} \,.$$

Equivalently

$$-\Delta_0 u = 1 - e^{2u} - t^2|\alpha(x)|^2 e^{-2u}\,.$$

Realization theorem:
Any solution of the Gauss equation, can be realized locally as the data for some embedding in a hyperbolic 3 manifold.

Remark: The 3-hyperbolic manifold is not necessarily complete.

3. Chemotaxis model

Keller-Segel model (1970):
Macroscopic model to describe the positive chemotactic behavior of the slime molds in its aggregation phase:

$$
\begin{aligned}
u(t,x) &:= \text{density of the cellular slime mold,} \\
v(t,x) &:= \text{chemoattractant concentration.}
\end{aligned}
$$

$$
\left\{
\begin{array}{ccll}
\frac{\partial u}{\partial t} & = & \nabla\big(\nabla u - u\nabla v\big), & \text{in } \Omega, \\
\frac{\partial v}{\partial t} & = & \Delta v + \alpha u - \gamma v, & \text{in } \Omega, \\
\frac{\partial u}{\partial n} & = & \frac{\partial v}{\partial n} \; = \; 0, & \text{on } \partial\Omega, \\
u(0,\cdot) & = & u_0, \;\; v(0,\cdot) \; = \; v_0, & \text{in } \Omega.
\end{array}
\right.
$$

Following Jäger, Luckhaus (1992), we introduce the new functions

$$U(t,x) = \frac{u(t,x)}{\int\limits_{\Omega} u(t,s)ds},$$

$$V(t,x) = v(t,x) - \frac{1}{|\Omega|}\int\limits_{\Omega} v(t,s)ds,$$

and setting $\lambda := \alpha \int_{\Omega} u(0,\cdot)$ we get

$$\begin{cases} \frac{\partial U}{\partial t} & = & \nabla\big(\nabla U - U\nabla V\big), & \text{in } \Omega, \\ \frac{\partial V}{\partial t} & = & \Delta V + \lambda\big(U - \frac{1}{|\Omega|}\big) - \gamma V, & \text{in } \Omega, \\ \frac{\partial U}{\partial n} & = & \frac{\partial V}{\partial n} \ = \ 0, & \text{on } \partial\Omega, \\ U(0,\cdot) & = & U_0, \ V(0,\cdot) \ = \ V_0, & \text{in } \Omega. \end{cases}$$

As in Jäger, Luckhaus, we consider the case γ small.
We are led to the system:

$$\begin{cases} \frac{\partial U}{\partial t} = \nabla\big(\nabla U - U\nabla V\big), & \text{in } \Omega \\ \frac{\partial V}{\partial t} = \Delta V + \lambda\big(U - \frac{1}{|\Omega|}\big), & \text{in } \Omega \\ \frac{\partial U}{\partial n} = \frac{\partial V}{\partial n} = 0, & \text{on } \partial\Omega, \\ U(0,\cdot) = U_0,\ V(0,\cdot) = V_0, & \text{in } \Omega. \end{cases}$$

What are the steady states ?

Steady states

$$\begin{cases} -\Delta V = \lambda \left(\frac{e^V}{\int\limits_\Omega e^V dx} - \frac{1}{|\Omega|} \right), & \text{in } \Omega, \\ \frac{\partial V}{\partial n} = 0, & \text{on } \partial\Omega, \\ \int_\Omega V = 0. \end{cases}$$

Remark: $V \equiv 0$ is a trivial solution

Question: When is the solution unique ?

II.1 Minimal surface in hyperbolic space

Gauss equation

$$-\Delta u = 1 - e^{2u} - t^2|\alpha(x)|^2 e^{-2u}$$

Uhlenbeck: Applying implicit function theorem, existence of a branch of solution

Applying maximum principle, we deduce $0 \leq 1 - e^{2u}$, hence

Solutions are *negative*.

With Z. Huang:

(1) no solution for t large.
(2) Problem admits a variational structure for which Palais-Smale is satisfied.
(3) Functional admits a mountain pass geometry. Hence existence of a second solution.

II.2 A meanfield equation with constant boundary data

Given $\Omega \subset\subset \mathbb{R}^2$, we consider

$$-\Delta u = \lambda \left(\frac{e^u}{\int_\Omega e^u} - \frac{1}{|\Omega|} \right), \quad \mathrm{osc}_{\partial\Omega}(u) = 0.$$

A general result: Given

$$g \in C^1(\mathbb{R}), \qquad g \geq g' > 0, \qquad A > 0, \tag{1}$$

Theorem

Assume (1) *and let* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *satisfying*

$$-\Delta u = g(u) - A, \quad osc_{\partial\Omega}(u) \equiv 0.$$

If

$$\int_\Omega g(u) \; = \; A|\Omega| \; \leq \; 8\pi.$$

then u *is necessarily constant.*

CONSEQUENCE:

Theorem (with L. Zhang, 2005)

Consider $\Omega \subset\subset \mathbb{R}^2$. Then for $\lambda \leq 8\pi$ the constant functions are the unique solution of

$$\begin{cases} -\Delta u = \lambda \left(\dfrac{e^u}{\int\limits_\Omega e^u dx} - \dfrac{1}{|\Omega|} \right), \\ u \in C^2(\Omega) \cap C^0(\overline{\Omega}), \quad osc_{\partial\Omega}(u) \equiv 0. \end{cases} \tag{2}$$

Particular case: In a disk Ω, for $\lambda \leq 8\pi$ the constant functions are the unique *radial* solutions of (2).

- If $\lambda < 8\pi$ and Ω simply connected, result follows from Suzuki (1992).
  Main point: NO topological assumption on the domain, and result applies at $\lambda = 8\pi$.
- Generalized with D. Horstmann (2010) to

$$-\Delta u = \lambda \left(\frac{e^{\Phi(u)}}{\int_\Omega e^{\Phi(u)}} - \frac{1}{|\Omega|} \right)$$

  under suitable assumptions on Φ.

Result is optimal ! Beyond 8π we always have solution (M.L. 2007).

Variational formulation

$$J(\lambda, u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \lambda \log\left(\int_\Omega e^u\right) - \int_\Omega u, \qquad u \in H_0^1(\Omega).$$

Main difficulty: Palais-Smale is not satified. Can be overcome:

- either using the monotonicity trick of Struwe (general formulation given by Tintarev (1994), and jeanjean Toland (1998))
- or using an appropriate pseudogradient flow (M.L., 2006).

II.3 Neumann case, or manifold

Sphere:

- Onofri (1982): For $\lambda < 8\pi$, $u \equiv 0$ is the unique minimizer of the functional

$$J(u) = \frac{1}{2}\int_{S^2} |\nabla u|^2 - \lambda \log\left(\int_{S^2} e^u\right), \qquad \int_{S^2} u = 0$$

- Kiessling, Chanillo (1994): For $\lambda < 8\pi$, $u \equiv 0$ is actually the unique critical point, i.e. unique solution of

$$-\Delta u = \lambda \left(\frac{e^u}{\int_{S^2} e^u} - \frac{1}{|S^2|}\right), \qquad \int_{S^2} u = 0.$$

What happens on other Riemannian surfaces ?

Flat Torus:
Consider the flat torus T^2 with fundamental cell $(-1,1)^2$.

Struwe, tarantello (1998): Existence of a range $(-\infty, \lambda^*)$ for which $u \equiv 0$ is the unique solution.

Further investigations: With C. Cabré, M. Sanchón and C.S. Lin.

Theorem (with C.S. Lin)

For $\lambda \leq 8\pi$, $u \equiv 0$ is the unique solution to

$$-\Delta u = \lambda \left(\frac{e^u}{\int_{S^2} e^u} - \frac{1}{|S^2|} \right), \qquad \int_{S^2} u = 0.$$

Main ingredient: An inequality involving the isoperimetric function of the torus.

II.4 Solutions invariant under a group of isometry (with D. Horstmann)

B disc of $\mathbb{R}^2$ and G group of isometry of B,

$$-\Delta u = \lambda \left(\frac{e^u}{\int_B e^u} - \frac{1}{|B|} \right), \qquad \frac{\partial u}{\partial n} = 0 \text{ on } \partial B$$

In a disc, can we find non-trivial solutions which are invariant under a group of isometries G ?

EXAMPLES:

- the group $SO(2)$;
- $\langle R_\theta \rangle$, the group generated by a rotation R_θ of angle θ;
- Dihedral group

Define

$$H^G := \left\{ V \in H^1(B) \,:\, V = V \circ g \;\forall g \in G \right\},$$

$$\overset{\circ}{H}{}^G := \left\{ V \in H^G \,:\, \int_B V = 0 \right\}.$$

If $G = SO(2)$ we laready know the result (by section II.2).

Theorem

Let $G = \langle R_{\frac{2\pi}{n}} \rangle$, and assume there exists a non-constant solution $V \in H^G$. Then

$$\lambda > \Lambda_n := \begin{cases} \frac{64}{\pi} & \text{if } n = 2, \\ 8\pi & \text{if } n \geq 3. \end{cases}$$

Theorem

Let $G = \langle R_{\frac{2\pi}{n}} \rangle$*. Then* $V \equiv 0$ *is the unique solution to*

$$-\Delta u = \lambda \left(\frac{e^u}{\int_B e^u} - \frac{1}{|B|} \right), \qquad u \in \overset{\circ}{H}{}^G,$$

whenever

$$\lambda \leq \Lambda_n := \begin{cases} \frac{64}{\pi} & \text{if } n = 2, \\ 8\pi & \text{if } n \geq 3. \end{cases}$$

Result is optimal for $n \geq 3$, since there are radial solutions for $\lambda > 8\pi$.

II.5 Optimal inequalities

In dimension $N \geq 3$, best constants in the critical Sobolev embedding is known (Lieb, Talenti).

Optimal inequality in dimension two for moser-trudinger functional

Consider the functional:

$$\frac{1}{2}\int_S |\nabla u|^2 - \lambda \log\left(\frac{1}{|S|}\int_S e^u\right), \qquad \int_S u = 0.$$

On Riemannian surface: Functional is bounded from below (and minimizer achieved) whenever $\lambda \leq 8\pi$.

On a bounded domain: Bounded from below $\lambda < \lambda_\Omega$.

Sphere

$$\frac{1}{|S^2|}\int_{S^2} e^u \le e^{\frac{1}{16\pi}\|\nabla u\|_2^2}.$$

Flat torus $\mathbb{R}^2/\mathbb{Z}^2$

$$\frac{1}{|T^2|}\int_{T^2} e^u \le e^{\frac{1}{16\pi}\|\nabla u\|_2^2},$$

and equality holds iff $u \equiv 0$.

Disk

Let $G = \langle R_{2\pi/n}\rangle$ $(n \ge 3)$. Then

$$\frac{1}{|B|}\int_{B} e^u \le e^{\frac{1}{16\pi}\|\nabla u\|_2^2}, \quad \forall u \in \overset{\circ}{H}{}^{G},$$

and equality holds iff $u \equiv 0$.

III.1 Isoperimetric profile

Definition

Let M be a two dimensional manifold without boundary. Consider the class $\mathcal{O}$ of open subsets $\omega \subset M$ satisfying

$$\overline{\partial\omega \cap M^o} \text{ is a } 1-\text{submanifold of class } C^1.$$

The "isoperimetric profile" of M is the function $I_M : [0, |\Omega|] \to (0, \infty)$ defined as

$$I_M(s) := \inf \left\{ \mathcal{H}^1(\partial\omega \cap \Omega) \,:\, \omega \in \mathcal{O}, \quad \mathcal{H}^2(\omega) = s \right\},$$

for all $s \in (0, |M|]$, and we set $I_M(0) = 0$.

Isoperimetric profile for a domain

Definition

Given $\Omega \subset\subset \mathbb{R}^2$, let $\mathcal{O}_\Omega$ be the class of open subsets $\omega \subset \Omega$ satisfying

$$\omega = \omega' \cap \Omega, \quad \omega' \subset\subset \mathbb{R}^2 \text{ of class } C^1.$$

The "isoperimetric profile" of Ω is the function $I_\Omega : (0, |\Omega|] \to (0, \infty)$ defined as

$$I_\Omega(s) := \inf \left\{ \mathcal{H}^1(\partial\omega \cap \Omega) \, : \, \omega \in \mathcal{O}_\Omega, \quad \mathcal{H}^2(\omega) = s \right\}.$$

Definition

Given a group G of isometries of Ω, we consider the class of open subsets:

$$\mathcal{O}_\Omega^G := \{\omega \in \mathcal{O}_\Omega : \ g(\omega) = \omega, \forall g \in G\}.$$

The "G-isoperimetric profile" of Ω is defined as

$$I_\Omega^G(s) := \inf \left\{ \mathcal{H}^1(\partial\omega \cap \Omega) \ : \ \omega \in \mathcal{O}_\Omega^G, \quad \mathcal{H}^2(\omega) = s \right\}.$$

We set $I_\Omega(0) = I_\Omega^G(0) = 0$.

III.2 An integral inequality

Theorem

Let $\Omega \subset\subset \mathbb{R}^2$ be a piecewise C^1-domain, g a function satisfying

$$g \in C^1(\mathbb{R}), \qquad g \geq g' > 0,$$

and $V \in C^2(\overline{\Omega})$ be a non-constant solution of the Neumann Problem:

$$\left.\begin{aligned} -\Delta V &= g(V) - A \quad \text{in } \Omega, \\ \frac{\partial V}{\partial n} &= 0 \qquad\qquad\quad \text{on } \partial\Omega. \end{aligned}\right\}$$

Setting $\mu(t) := \mathcal{H}^2\big(\{V > t\}\big)$, then the following inequality holds:

$$\frac{A}{2}\int_{-\infty}^{\infty} g'(t)\,\mu(t)\big(|\Omega| - \mu(t)\big)\,dt \geq \int_{-\infty}^{\infty} g(t)\, I_\Omega^2(\mu(t))\,dt.$$

"G-equivariant formulation"

Theorem

SAME ASSUMPTIONS, and let G be a group of isometries of Ω. If $V \in C^2(\overline{\Omega}) \cap H^G$ is a non-constant solution

$$\begin{cases} -\Delta V &= g(V) - A \quad \text{in } \Omega, \\ \frac{\partial V}{\partial n} &= 0 \qquad\qquad\quad \text{on } \partial\Omega, \end{cases}$$

then

$$\frac{A}{2} \int_{-\infty}^{\infty} g'(t)\, \mu(t)\big(|B| - \mu(t)\big)\, dt \geq \int_{-\infty}^{\infty} g(t)\, [I_B^G(\mu(t))]^2\, dt.$$

Application when Ω is a disk and $G = \langle R_{2\pi/n} \rangle$. Let u be a non-constant solution to

$$-\Delta u = \lambda \left(\frac{e^u}{\int_B e^u} - \frac{1}{|B|} \right), \quad u \in \overset{\circ}{H}{}^G.$$

Define

$$g(s) := \lambda \frac{e^s}{\int_B e^u} \qquad A = \frac{\lambda}{|B|}$$

Inequality previous Theorem yields:

$$\frac{\lambda}{|B|} \geq 2 \frac{\int_{-\infty}^{\infty} e^t \, [I_B^G(\mu(t))]^2 \, dt}{\int_{-\infty}^{\infty} e^t \, \mu(t) \big(|B| - \mu(t) \big) \, dt}.$$

Lemma

For a disc $B \subset \mathbb{R}^2$ it holds

$$\frac{[I_B(s)]^2}{s(|B|-s)} > \frac{16}{\pi|B|}, \quad \forall s \in (0, \frac{|B|}{2}).$$

If $G = \langle R_{\frac{2\pi}{n}} \rangle$ we have

$$\frac{[I_B^G(s)]^2}{s(|B|-s)} > \min\left\{\frac{4\pi}{|B|}, \frac{16n}{\pi|B|}\right\}.$$

THANK YOU !