SOME SHAPE OPTIMIZATION PROBLEMS FOR THE p-LAPLACIAN [1]

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Dedicated to V.S. Sunder

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Plan

- Problem Statement.
- Existence.
- Optimality.
- Shape Derivative Calculus.
- Analysis.

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Optimization Problem.

- $0 < r < R$. $|a| < R - r$. $\Omega_a = B(0,R) \setminus \overline{B(a,r)} \subset \mathbb{R}^n$.
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$$\inf \left\{ J(\Omega_a) : a \in \overline{B(0, R-r)} \right\} .$$

$$J(\Omega_a) := \int_{\Omega_a} u_a \, dx .$$

$$\begin{aligned} -\Delta_p u_a &= 1 \text{ in } \Omega_a \\ u_a &= 0 \text{ on } \partial\Omega_a . \end{aligned}$$

Questions of Interest

- Existence - Does there exist an a which minimizes?
- Characterization - For what a?

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Known results

p=2

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- Minimum for $a = 0$; Maximum for $a = R - r$.

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Existence

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- $J(\Omega_a)$ is a continuous function of a.
- $\inf \left\{ J(\Omega_a) : a \in \overline{B(0, R-r)} \right\}$ is attained.

Many minimizers

- a is a minimizer. u_a corresponding state.
- R orthogonal transformation. $Ra = |a|\, e_1$.

$$-\Delta_p(u_a \circ R^{-1}) = (-\Delta_p u_a) \circ R^{-1} = -1 \text{ in } \Omega_{|a|e_1}$$

$$J(\Omega_a) = J(\Omega_{|a|\, e_1})$$

Equivalent Problem

$$\min \{ J(\Omega_{te_1}) : t \in [0, R-r] \}$$

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Criterion

- $j(t) := J(\Omega_{te_1})$.
- To show $j : [0, R-r] \to \mathbb{R}$ has a minimum at $t = 0$.
- Will show that j is differentiable; $j'(0) = 0$ and $j'(t) > 0$ for all $0 < t < R - r$.

Connection with Shape Derivative

$$j'(t) = \lim_{h \to 0} \frac{J(\Omega_{(t+h)\, e_1}) - J(\Omega_{t\, e_1})}{h}.$$

- $B = B(0, R)$ and $B_t = B(t\, e_1, r)$ where $t \in [0, R-r]$. $\Omega_t = B \setminus \overline{B_t}$.
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- $(\mathrm{id} + hV)\Omega_t = \Omega_{t+h}$.

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- Existence of the derivative $j'(t)$.
- Obtaining an expression for $j'(t)$.
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Shape Derivative

u_t the corresponding state for Ω_t. $\Phi_h = (\mathrm{id} + hV)$

$$
\begin{aligned}
j'(t) &= \lim_{h\to 0} \frac{\int_{\Omega_{t+h}} u_{t+h}\ dx - \int_{\Omega_t} u_t\ dx}{h} \\
&= \lim_{h\to 0} \int_{\Omega_t} \frac{u_{t+h}\circ\Phi_h |(D\Phi_h)| - u_t}{h}\ dx \\
&= \lim_{h\to 0} \int_{\Omega_t} \left(\frac{(u_{t+h}\circ\Phi_h - u_t)}{h} |(D\Phi_h)| + \frac{|(D\Phi_h)| - 1}{h} u_t \right)\ dx \\
&= \int_{\Omega_t} (\dot{u}_t + \mathrm{div}\, V\, u_t)\ dx = \int_{\Omega_t} (\dot{u}_t - \nabla u_t \cdot V)\ dx\,.
\end{aligned}
$$

material der. $\dot{u}_t := \displaystyle\lim_{h\to 0} \frac{(u_{t+h}\circ\Phi_h - u_t)}{h}$ shape der. $u_t' = \dot{u}_t - \nabla u_t \cdot V$.

An expression

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j'(t) = \int_{\Omega_t} u_t'\ dx\,.
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Existence of Shape Derivative

Linked to the existence of $\dot{u}_t$.

Preliminaries

$$\begin{aligned} -\Delta_p u &= f(u) \text{ in } \Omega \text{ (smooth bdd).} \\ u &> 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega . \end{aligned}$$

$f : [0, \infty] \to \mathbb{R}^+$ locally Lipschitz in $(0, \infty)$. $f(s) > 0$ for $s > 0$.
$\mathbf{p \geq 2}$. $m = |\nabla u|^{p-2}$.

Theorem (Damascelli & Sciunzi J. Differential Equations 2004)

Poincaré Ineq. $\displaystyle\int_\Omega |v|^2 \, dx \leq C(|\Omega|)^2 \int_\Omega |\nabla v|^2 m(x) dx \qquad \forall v \in C_0^\infty(\Omega)$

Weighted Sobolev space

$$\|v\|^2_{H^1_{0,m}} = \int_\Omega |\nabla v|^2 m(x) dx , \qquad H^1_{0,m} = \overline{C_0^\infty(\Omega)} .$$

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Existence of Shape Derivative...continued

Theorem (Chorwadwala, Kesavan, M.)

- *The material derivative $\dot{u}_t$ exists and belongs to $H^1_{0,m}$.*
- *The shape derivative u'_t satisfies the equation:*

$$\begin{aligned} -\mathrm{div}\,(A\nabla v) &= 0 && \text{in } \Omega_t = B \setminus B_t \\ v &= 0 && \text{on } \partial B \\ v &= -\frac{\partial u_t}{\partial n} V \cdot n && \text{on } \partial B_t\,. \end{aligned}$$

where $A = \left(|\nabla u_t|^{p-2} I + (p-2)\,|\nabla u_t|^{p-4} \nabla u_t \otimes \nabla u_t\right)$.
(Note: $A \geq \min\{1, p-1\}\, m\, I$*)*

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Preprocessing

- $\partial B_t^+ = \{x \in \partial B_t : x_1 > 0\}$.
- For $x \in \partial B_t^+$, let $x' = (2t - x_1, x_2, \ldots, x_n)$.

An expression

$$j'(t) = \int_{\partial B_t^+} \left(\left| \frac{\partial u_t}{\partial n} \right|^p (x) - \left| \frac{\partial u_t}{\partial n} \right|^p (x') \right) n_1(x)\, dx .$$

- n_1 is negative.
- Enough to show $\frac{\partial u_t}{\partial n}(x') < \frac{\partial u_t}{\partial n}(x) < 0$ for all $x \in \partial B_t^+$.
- Can be obtained from a comparison result-uses the moving plane method.

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$$j'(t) = \int_{\partial B_t^+} \left(\left|\frac{\partial u_t}{\partial n}\right|^p (x) - \left|\frac{\partial u_t}{\partial n}\right|^p (x') \right) n_1(x)\, dx\,.$$

- n_1 is negative.
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- Can be obtained from a comparison result-uses the moving plane method.

Preprocessing

- $\partial B_t^+ = \{x \in \partial B_t : x_1 > 0\}$.
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- $O := \{x \in \partial\Omega_t : x_1 > 0\}$.
- For $x \in O$, $w_t(x) := u_t(x')$.

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\begin{array}{rclll}
-\Delta_p u_t = 1 & ; & -\Delta_p w_t = 1 & \text{in } O \\
u_t = 0 & ; & w_t = 0 & \text{on } \partial B_t^+ \\
u_t = g > 0 & ; & w_t = g > 0 & \text{on } \Gamma \\
u_t = 0 & ; & w_t > 0 & \text{on } \partial O^{\mathrm{ext}}
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Theorem (Chorwadwala,Kesavan,M.)

$$WCP \qquad w_t \geq u_t \ in\ O\,.$$

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More Comparison

Consequently,

Theorem (Chorwadwala,Kesavan,M.)

$$SCP \qquad w_t > u_t \text{ in } O \text{ and } \frac{\partial w_t}{\partial n} < \frac{\partial u_t}{\partial n} \text{ on } \partial O\,.$$

Eigenvalue Optimization

Optimization Problem.

$0 < r < R$. $|a| < R - r$. $\Omega_a = B(0,R) \setminus \overline{B(a,r)} \subset \mathbb{R}^n$.

$$\inf \left\{ \lambda_1(\Omega_a) : a \in \overline{B(0, R-r)} \right\}.$$

$$\begin{aligned} -\Delta_p u_a &= \lambda_1(\Omega_a) u_a \text{ in } \Omega_a \\ u_a &= 0 \text{ on } \partial\Omega_a . \end{aligned}$$

Results

- Chorwadwala, Kesavan, M. - Max. for $a = 0$; "Min." for $a = R - r$.
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