Γ-lower limit for energy convergence for weak data

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Some Notations

- $M(a, b, \Omega)$ is the set of all matrices $A = A(x) \in (L^\infty(\Omega))^{n\times n}$, such that
$$a|\xi|^2 \leq A(x)\xi.\xi \leq b|\xi|^2, \forall \xi \in \mathbb{R}^n \text{ and } a.e. \text{ on } \Omega$$
where $0 < a < b$. $\Omega \subset \mathbb{R}^n$, a bounded open set.

- The energy functional
$$E_\varepsilon^A(v) = \int_\Omega A_\varepsilon \nabla v . \nabla v \, dx$$
and
$$E_0^A(v) = \int_\Omega A_0 \nabla v . \nabla v \, dx$$
.

Energy Convergence

For any $A_\varepsilon \subset M(a, b, \Omega)$ which H-converges to A_0, the energies converge, i.e.,

$$\int_\Omega A_\varepsilon \nabla v_\varepsilon . \nabla v_\varepsilon \, dx \stackrel{\varepsilon \to 0}{\longrightarrow} \int_\Omega A_0 \nabla v_0 . \nabla v_0 \, dx$$

where $v_0 \in H_0^1(\Omega)$ and v_ε is the solution of

$$\begin{cases} -\text{div}(A_\varepsilon(x)\nabla v_\varepsilon) & = -\text{div}(A_0(x)\nabla v_0) \quad \text{in } \Omega \\ v_\varepsilon & = 0 \quad \text{on } \partial\Omega \end{cases}$$

The energy convergence is true as long as the data converge strongly in $H^{-1}(\Omega)$.

A consequence of Energy convergence

For any sequence of symmetric matrices $\{B_\varepsilon\} \subset M(c,d,\Omega)$ G-converging to B_0 and any sequence w_ε weakly converging to w_0 in $H_0^1(\Omega)$, we have

$$\liminf_{\varepsilon\to 0} \int_\Omega B_\varepsilon \nabla w_\varepsilon \cdot \nabla w_\varepsilon \, dx \geq \int_\Omega B_0 \nabla w_0 \cdot \nabla w_0 \, dx$$

In particular,

$$\liminf_{\varepsilon\to 0} \int_\Omega B_\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon \, dx \geq \int_\Omega B_0 \nabla v_0 \cdot \nabla v_0 \, dx$$

where v_ε and v_0 are as given in previous slide. Thus, when $B_\varepsilon = I$, identity matrix, we have the usual weak lower semicontinuity of norm.

Is there a better estimate?

A question of interest is: Is there a $B^\sharp \geq B_0$ such that

$$\liminf_{\varepsilon \to 0} \int_\Omega B_\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon \, dx \geq \int_\Omega B^\sharp \nabla v_0 \cdot \nabla v_0 \, dx \geq \int_\Omega B_0 \nabla v_0 \cdot \nabla v_0 \, dx.$$

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We are motivated by some problems to answer the question in two parts:

- When data converge strongly in $H^{-1}(\Omega)$.

- When data converge weakly in $H^{-1}(\Omega)$.

Strong Data

(S. Kesavan, M. Rajesh, J. Saint Jean Paulin, M. Vanninathan)

Let $g_\varepsilon \to g$ strongly in $H^{-1}(\Omega)$ and let $v_\varepsilon \in H_0^1(\Omega)$ be the weak solution of

$$\begin{cases} -\text{div}(A_\varepsilon \nabla v_\varepsilon) & = g_\varepsilon \text{ in } \Omega \\ v_\varepsilon & = 0 \text{ on } \partial\Omega, \end{cases} \qquad (A_\varepsilon v_\varepsilon = g_\varepsilon)$$

then $E_\varepsilon^B(v_\varepsilon) \overset{\varepsilon \to 0}{\longrightarrow} E_\sharp^B(v_0)$ where $v_0 \in H_0^1(\Omega)$ is the unique solution of

$$\begin{cases} -\text{div}(A_0 \nabla v_0) & = g \text{ in } \Omega \\ v_0 & = 0 \text{ on } \partial\Omega. \end{cases} \qquad (A_0 v_0 = g)$$

and $B^\sharp$ be the weak-* limit of $\{P_\varepsilon^t B_\varepsilon P_\varepsilon\}$ in $(L^\infty(\Omega))^{n\times n}$ with $\{P_\varepsilon\}$ being the corrector matrix associated with $\{A_\varepsilon\}$.

On Bounds

If $A_\varepsilon \in M(a, b, \Omega)$, $B_\varepsilon \in M(c, d, \Omega)$, then

- $A_0 \in M(a, \frac{b^2}{a}, \Omega)$.
- $B_0 \in M(c, \frac{d^2}{c}, \Omega)$.
- $B^\sharp \in M(c, d\left(\frac{b}{a}\right)^2, \Omega)$.

Conjecture

Let $g_\varepsilon \rightharpoonup g$ weakly in $H^{-1}(\Omega)$ and let $v_\varepsilon \in H_0^1(\Omega)$, the weak solution of $A_\varepsilon v_\varepsilon = g_\varepsilon$, be such that $v_\varepsilon \rightharpoonup v_0$ weakly in $H_0^1(\Omega)$, where $v_0 \in H_0^1(\Omega)$ is the unique solution of $A_0 v_0 = g$. Then

$$\liminf_{\varepsilon \to 0} \int_\Omega B_\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon \, dx \geq \int_\Omega B^\sharp \nabla v_0 \cdot \nabla v_0 \, dx.$$

Conjecture

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The conjecture is open, in general. In this talk we shall prove it for the periodic case with additional hypothesis on g_ε.

Lemma 1 (M. Rajesh, TMK). *Let* $\nabla v_\varepsilon \overset{2s}{\rightharpoonup} \nabla v_0 + \nabla_y v_1(x,y)$ *where* $v_0 \in H_0^1(\Omega)$ *and* $v_1 \in L^2[\Omega; H^1_{\text{per}}(Y)]$ *and let* $B(x,y) \in M(c,d,\Omega \times Y)$, Y*-periodic in* y*, be a symmetric matrix, then*

$$\liminf_{\varepsilon \to 0} E_\varepsilon^B(v_\varepsilon) \geq \int_{\Omega \times Y} B(x,y)\,[\nabla v_0 + \nabla_y v_1] \cdot [\nabla v_0 + \nabla_y v_1]\; dy\, dx,$$

where $B_\varepsilon = B\left(x, \frac{x}{\varepsilon}\right)$.

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where $B_\varepsilon = B\left(x, \frac{x}{\varepsilon}\right)$.

Let $\Phi \in \left(\mathcal{D}[\Omega; C^\infty_{\text{per}}(Y)]\right)^n$. Then, by Fenchel's inequality,

$$\begin{aligned} I_\varepsilon &:= \frac{1}{2}\int_\Omega B\left(x, \frac{x}{\varepsilon}\right) \nabla v_\varepsilon \cdot \nabla v_\varepsilon \, dx \\ &\geq \int_\Omega \nabla v_\varepsilon \cdot \Phi\left(x, \frac{x}{\varepsilon}\right)\, dx \\ &\quad - \frac{1}{2}\int_\Omega B^{-1}\left(x, \frac{x}{\varepsilon}\right) \Phi\left(x, \frac{x}{\varepsilon}\right) \cdot \Phi\left(x, \frac{x}{\varepsilon}\right)\, dx \end{aligned}$$

Step 2

We take $\liminf$ on both sides. By two-scale convergence of v_ε, the first term on RHS becomes,

$$\lim_{\varepsilon\to 0}\int_\Omega \nabla v_\varepsilon \cdot \Phi\left(x, \frac{x}{\varepsilon}\right)\, dx = \int_{\Omega\times Y} [\nabla v_0 + \nabla_y v_1(x,y)] \cdot \Phi(x,y)\, dy\, dx.$$

By the product of weak and strong two-scale convergence, we have

$$\lim_{\varepsilon\to 0}\int_\Omega B^{-1}\left(x, \frac{x}{\varepsilon}\right)\Phi\left(x, \frac{x}{\varepsilon}\right)\cdot\Phi\left(x, \frac{x}{\varepsilon}\right)\, dx = \int_{\Omega\times Y} B^{-1}(x,y)\Phi(x,y)\cdot\Phi(x,y)\, dy$$

Thus, we conclude

$$\begin{aligned}\liminf_{\varepsilon\to 0} I_\varepsilon \;\geq\; & \int_{\Omega\times Y} [\nabla v_0 + \nabla_y v_1(x,y)] \cdot \Phi(x,y)\, dy\, dx \\ & -\frac{1}{2}\int_{\Omega\times Y} B^{-1}(x,y)\Phi(x,y)\cdot\Phi(x,y)\, dy\, dx.\end{aligned}$$

Step 3

Taking supremeum over all $\Phi \in (\mathcal{D}(\Omega \times Y))^n$ on the right hand side and interchanging sup and integral, we obtain,

$$\liminf_{\varepsilon \to 0} I_\varepsilon \;\geq\; \int_{\Omega \times Y} \sup_{\xi \in \mathbb{R}^n} \left\{ [\nabla v_0 + \nabla_y v_1(x,y)] \cdot \xi - \frac{1}{2} B^{-1}(x,y)\xi \cdot \xi \right\} dy\, dx.$$

Thus,

$$\liminf_{\varepsilon \to 0} E_\varepsilon^B(v_\varepsilon) \geq \int_{\Omega \times Y} B(x,y)\,[\nabla v_0 + \nabla_y v_1] \cdot [\nabla v_0 + \nabla_y v_1]\; dy\, dx.$$

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Thus,

$$\liminf_{\varepsilon\to 0} E_\varepsilon^B(v_\varepsilon) \geq \int_{\Omega\times Y} B(x,y)\,[\nabla v_0 + \nabla_y v_1]\cdot[\nabla v_0 + \nabla_y v_1]\; dy\, dx.$$

If we can show $\nabla_y v_1(x,y) = P(x,y)\nabla v_0$, then our conjecture is answered positively. However, such an equality is known only when data converge strongly.

Weak Data

Theorem 1 (A. K. Nandakumaran, TMK)**.** *Let* $\gamma < 1$ *be a fixed real number. Let* $v_\varepsilon \in H_0^1(\Omega)$ *be the weak solution of*

$$\begin{cases} -\mathrm{div}(A\left(x, \frac{x}{\varepsilon}\right) \nabla v_\varepsilon) & = g_\varepsilon \ \text{ in } \Omega \\ v_\varepsilon & = 0 \ \text{ on } \partial\Omega, \end{cases}$$

where $g_\varepsilon \in L^2(\Omega)$ *is such that* $g_\varepsilon \rightharpoonup g$ *weakly in* $H^{-1}(\Omega)$ *and* $\varepsilon^\gamma g_\varepsilon$ *is bounded in* $L^2(\Omega)$*. Then,*

$$\left.\begin{array}{r} v_\varepsilon \rightharpoonup v_0 \text{ weakly in } H_0^1(\Omega) \\ A_\varepsilon \nabla v_\varepsilon \rightharpoonup A_0 \nabla v_0 \text{ weakly in } (L^2(\Omega))^n. \end{array}\right\}$$

is satisfied, where $v_0 \in H_0^1(\Omega)$ *is the unique solution of* $A_0 v_0 = g$ *and* A_0 *is given as*

$$(A_0)_{ij} = \int_Y A(x,y)(P(x,y)e_i + e_i) \cdot (P(x,y)e_j + e_j)\, dy.$$

An Optimal Control Problem

Cost functional $J_\varepsilon : U \to \mathbb{R}$ is:

$$J_\varepsilon(\theta) = \frac{1}{2}\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx + \frac{N}{2}\|\theta\|_2^2$$

where $u_\varepsilon = u_\varepsilon(\theta)$ solves $A_\varepsilon u_\varepsilon = f + \theta$.

- Does the minimizer θ_ε^* converge to some θ^* in some topology?
- Is the limit θ^* also a minimizer for some optimal control problem?

Limit Control Problem

The limit cost functional is given as

$$J(\theta) = \frac{1}{2}\int_{\Omega} B^{\sharp}\nabla u.\nabla u\, dx + \frac{N}{2}\|\theta\|_2^2$$

where $u = u(\theta)$ solves $A_0 u = f + \theta$. In addition, $\theta^*_\varepsilon \to \theta^*$ strongly in $L^2(\Omega)$.

"Low-Cost" Control Problems

Consider the optimal control problem whose cost of control is of the same order as that of oscillations. Let the cost functional J_ε be defined over $U \subseteq L^2(\Omega)$, closed and convex subset, as

$$J_\varepsilon(\theta) = \frac{1}{2}\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx + \frac{\varepsilon}{2}\|\theta\|_2^2$$

where $u_\varepsilon = u_\varepsilon(\theta)$ solves $A_\varepsilon u_\varepsilon = f + \theta$.

- No *a priori* bounds for minimizers θ_ε^* on $L^2(\Omega)$.

$$\frac{\varepsilon}{2}\|\theta_\varepsilon^*\|_2^2 \leq J_\varepsilon(\theta_\varepsilon^*) \leq J_\varepsilon(\theta)$$

- However, the minimizers $\theta_\varepsilon^* \rightharpoonup \theta^*$ weakly in $H^{-1}(\Omega)$.

Equality for Optimal States

Lemma 2. *For the optimal states u_ε^* and u^*, we have*

$$\lim_{\varepsilon\to 0}\int_\Omega B\left(x,\frac{x}{\varepsilon}\right)\nabla u_\varepsilon^*\cdot\nabla u_\varepsilon^*\,dx=\int_\Omega B^\sharp\nabla u^*\cdot\nabla u^*\,dx$$

A consequence of obtaining Γ-limit is that θ^* is a minimizer of the limit problem and

$$\frac{1}{2}\int_\Omega B\left(x,\frac{x}{\varepsilon}\right)\nabla u_\varepsilon^*\cdot\nabla u_\varepsilon^*\,dx+\frac{\varepsilon}{2}\|\theta\|_2^2\to\frac{1}{2}\int_\Omega B^\sharp\nabla u^*\cdot\nabla u^*\,dx.$$

Observe that,

$$\frac{1}{2}\int_\Omega B\left(x,\frac{x}{\varepsilon}\right)\nabla u_\varepsilon^*\cdot\nabla u_\varepsilon^*\,dx\le\frac{1}{2}\int_\Omega B\left(x,\frac{x}{\varepsilon}\right)\nabla u_\varepsilon^*\cdot\nabla u_\varepsilon^*\,dx+\frac{\varepsilon}{2}\|\theta\|_2^2.$$

Now, taking lim sup both sides we have,

$$\limsup_{\varepsilon\to 0}\int_\Omega B\left(x,\frac{x}{\varepsilon}\right)\nabla u_\varepsilon^*\cdot\nabla u_\varepsilon^*\,dx\le\int_\Omega B^\sharp\nabla u^*\cdot\nabla u^*\,dx$$

Corrector result for Optimal States

Theorem 2. *Let* $P(x,y)e_i = \nabla_y \chi_i(x,y)$ *be the corrector matrix where* $\chi_i(x,y)$ *is defined as*

$$\begin{cases} -\mathrm{div}_y\left(A(x,y)[\nabla_y \chi_i(x,y) + e_i]\right) & = 0 \quad \text{in } Y \\ y \mapsto \chi_i(x,y) & \quad\quad \text{is } Y\text{-periodic,} \end{cases}$$

where $\{e_1, \cdots, e_n\}$ *is the standard basis of* $\mathbb{R}^n$*. We also assume that both* A *and* B *are in* $C[\Omega; L^\infty_{\mathrm{per}}(Y)]^{n\times n}$*. Then,*

$$\nabla u^*_\varepsilon - \left[P\left(x, \frac{x}{\varepsilon}\right) + I\right] \nabla u^* \to 0 \text{ strongly in } L^2(\Omega).$$

Let

$$I_\varepsilon = \int_\Omega B\left(x, \frac{x}{\varepsilon}\right)\left[\nabla u^*_\varepsilon - \left[P\left(x, \frac{x}{\varepsilon}\right) + I\right]\nabla u^*\right] \cdot \left[\nabla u^*_\varepsilon - \left[P\left(x, \frac{x}{\varepsilon}\right) + I\right]\nabla u^*\right] dx$$

Then (also using the symmetry of B),

$$\begin{aligned} I_\varepsilon \;=\; & \int_\Omega B\left(x,\frac{x}{\varepsilon}\right)\nabla u_\varepsilon^*\cdot\nabla u_\varepsilon^*\,dx - 2\int_\Omega B\left(x,\frac{x}{\varepsilon}\right)\left[P\left(x,\frac{x}{\varepsilon}\right)+I\right]\nabla u^*\cdot\nabla u_\varepsilon^*\,dx \\ & + \int_\Omega B\left(x,\frac{x}{\varepsilon}\right)\left[P\left(x,\frac{x}{\varepsilon}\right)+I\right]\nabla u^*\cdot\left[P\left(x,\frac{x}{\varepsilon}\right)+I\right]\nabla u^*\,dx. \end{aligned}$$

To simplify notation, we rewrite above equation as,

$$I_\varepsilon = I_\varepsilon^1 - 2I_\varepsilon^2 + I_\varepsilon^3.$$

Using equality for optimal states, we see that

$$\lim_{\varepsilon\to 0} I_\varepsilon^1 = \int_\Omega B^\sharp\nabla u^*\cdot\nabla u^*\,dx.$$

Since $A \in C[\Omega; L^\infty_{\text{per}}(Y)]^{n\times n}$ by the continuous dependence on data for elliptic equation, we have $P \in C[\Omega; L^2_{\text{per}}(Y)]^{n\times n}$ and hence BP in the class of admissible functions and thus, strongly two-scale converges.

By product of weak and strong two scale

$$\lim_{\varepsilon\to 0} I_\varepsilon^2 = \int_{\Omega\times Y} B(x,y)(P(x,y)+I)\nabla u^* \cdot (P(x,y)+I)\nabla u^* \, dy \, dx$$
$$= \int_\Omega B^\sharp \nabla u^* \cdot \nabla u^* \, dx$$

Similarly, we compute the last term,

$$\lim_{\varepsilon\to 0} I_\varepsilon^3 = \int_{\Omega\times Y} B(x,y)(P(x,y)+I)\nabla u^* \cdot (P(x,y)+I)\nabla u^* \, dy \, dx$$
$$= \int_\Omega B^\sharp \nabla u^* \cdot \nabla u^* \, dx$$

Therefore, by the coercivity of B,

$$c\left\| \nabla u_\varepsilon^* - \left[P\left(x, \frac{x}{\varepsilon}\right) + I \right] \nabla u^* \right\|_2^2 \le I_\varepsilon.$$

Now taking $\limsup$ both sides, we have our desired result.

Energy Convergence for Optimal States

Corollary 1. *For the optimal states* u_ε^* *and* u^*, *we have*

$$\lim_{\varepsilon \to 0} \int_\Omega A_\varepsilon \nabla u_\varepsilon^* \cdot \nabla u_\varepsilon^* \, dx = \int_\Omega A_0 \nabla u^* \cdot \nabla u^* \, dx$$

The sequence of corrector matrices P_ε is taken as $P_\varepsilon = P\left(x, \frac{x}{\varepsilon}\right) + I$. Therefore, by corrector convergence, we have

$$\begin{aligned}
\int_\Omega A_\varepsilon \nabla u_\varepsilon^* \cdot \nabla u_\varepsilon^* \, dx &= \int_\Omega A_\varepsilon P_\varepsilon \nabla u^* \cdot P_\varepsilon \nabla u^* \, dx + o(1) \\
&= \int_\Omega P_\varepsilon^t A_\varepsilon P_\varepsilon \nabla u^* \cdot \nabla u^* \, dx + o(1) \\
&= \int_\Omega A_0 \nabla u^* \cdot \nabla u^* \, dx + o(1)
\end{aligned}$$

where the last equality is due to the properties of corrector matrices.

Thank you for your attention

....and that ends our conference too!