On Stability Estimates for Backward Heat Conduction Problem

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ICM Satellite Conference on "PDE and Related Topics"
TIFR-CAM, Bangalore
August 13-17,2010.

Plan of the talk

- BHCP - an introduction.
- Ill-posednes of BHCP - illustration.
- Form of the solution: $\Omega = \mathbb{R}^d$ and bounded $\Omega \subset \mathbb{R}^d$.
- BHCP as an operator equation.
- Stability estimates.
- Regularization

BHCP: Introduction

While dealing with a heat conducting body $\Omega \subseteq \mathbb{R}^d$, one may have to investigate the temperature profile

$$u(x,t), \qquad x \in \Omega, \quad t \geq 0,$$

from the known data at a particular time, say $t = \tau$.

From this knowledge, one would like to know the temperature for the time $t < \tau$ as well as for $t > \tau$.

It is well known that the latter is a well-posed problem.

However, the former, the so called *backward heat conduction problem* (BHCP) is an ill-posed problem.

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However, the former, the so called *backward heat conduction problem* (BHCP) is an ill-posed problem.

In this talk, we shall discuss the case of the ill-posedness of the backward heat conduction problem (BHCP).

Recall that the heat equation associated with Ω is given by

$$\frac{\partial u}{\partial t} = c^2 \triangle u, \quad x \in \Omega, \quad t > 0, \tag{1}$$

where $u(x, t)$ represents the temperature at the point $x \in \Omega$ at time t.

We have the following two situations:

Direct Problem: From the knowledge of the temperature at time $t = t_0$, that is, $u(x, t_0)$, determine the temperature at a later time $t = \tau$, that is, $u(x, \tau)$ for $\tau > t_0$.

Inverse Problem: From the knowledge of the temperature at a particular time $\tau > 0$, that is, $u(x, \tau)$, determine the temperature at an earlier time $t = t_0$, that is, $u(x, t_0)$ for $t_0 < \tau$.

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We shall see that the direct problem is well-posed in the setting of $L^2(\Omega)$:

> *Given* $u(\cdot, t_0)$ *in* $L^2(\Omega)$ *for some* $t_0 \geq 0$ *and* $t > t_0$*, there exists a unique solution* $u(\cdot, t)$ *which depends continuously on the data* $u(\cdot, t_0)$*,*

whereas the inverse problem, the BHCP, is ill-posed:

> *A solution need not exist unless the data* $u(x, \tau)$ *is too smooth, and even if a unique solution exists, it does not depend continuously on the data.*

In fact, the BHCP belongs to a class of problems called *severely ill-posed problems*.

- In order to obtain stable approximate solutions for the BHCP, some *regularization methods* have to be used.
- For obtaining error estimates, it is necessary to assume some *a priori source conditions* on the unknown entities.
- The derived error estimates are usually compared with certain known *stability estimates* based on the source conditions.
- Standard result in this regard[1] is for determining stability estimates for $u(\cdot, t_0)$ for $t_0 > 0$.
- Such results are not valid for $t_0 = 0$.
- To deal with the case of $t_0 = 0$, advanced analytic tools, developed recently[2], have to be employed.

[1] See, e.g., Kirsch (1996)

[2] Tautenhahn (1998), Nair, Schock and Tautenhahn (2003), Nair, Pereverzev and Tautenhahn (2005)

Ill-Posedness of the Problem: Illustration

Let us first look at the form of the solution.

We consider the cases of $\Omega = \mathbb{R}^d$ and $\Omega \subset \mathbb{R}^d$ a bounded domain separately.

Case (i): $\Omega = \mathbb{R}^d$:

In this case, applying Fourier transform to the equation

$$\frac{\partial u}{\partial t} = c^2 \triangle u,$$

with $f_0 := u(\cdot, 0)$, we get

$$\frac{\partial \hat{u}}{\partial t}(\xi, t) + 4\pi^2 c^2 |\xi|^2 \hat{u}(\xi, t) = 0,$$

with $\hat{u}(\xi, 0) = \hat{f}_0(\xi)$.

The solution of the above ODE is given by

$$\hat{u}(\xi,t) = \hat{f}_0(\xi)e^{-4\pi^2c^2|\xi|^2t}.$$

Then, for $0 \le t \le \tau$, we have

$$\hat{u}(\xi,\tau) = \hat{u}(\xi,t)e^{-4\pi^2c^2|\xi|^2(\tau-t)}.$$

Thus, $0 \le t_0 \le \tau$,

$$\hat{u}(\xi,t_0) = \hat{u}(\xi,\tau)e^{4\pi^2c^2|\xi|^2(\tau-t_0)}.$$

From this it follows that

- small error in the data $u(\cdot,\tau)$ leads to large deviation in the solution $u(\cdot,t_0)$.

An Illustration

For instance, let $\xi_0 \in \mathbb{R}^d$ and $g \in L^2(\mathbb{R}^d)$ be such that

$$\hat{g}(\xi) = \begin{cases} \hat{u}(\xi,\tau), & |\xi - \xi_0| > 1 \\ \hat{u}(\xi,\tau) + \delta\sqrt{\eta_d}, & |\xi - \xi_0| \le 1. \end{cases}$$

where η_d is the volume[3] of the sphere in $\mathbb{R}^d$. Then we have

$$\|g - u(\cdot,\tau)\|_2^2 = \|\hat{g} - \hat{u}(\cdot,\tau)\|_2^2 = \int_{|\xi-\xi_0|\le 1} \eta_d \delta^2 d\xi = \delta^2.$$

If f is the solution corresponding to the noisy data g, then we have

$$\hat{f}(\xi) = \hat{g}(\xi) e^{4\pi^2 c^2 |\xi|^2 (\tau - t_0)}.$$

[3] Jason D.M. Rennie, Nov. 22 (2005): For $d \ge 2$, $\eta_d = \frac{2^{(d+1)/2}\pi^{(d-1)/2}}{d(d-2)!}$ for d odd, and $\eta_d = \frac{2\pi^{d/2}}{d(d/2-1)!}$ for d even.

Note that

$$\|f - u(\cdot, t_0)\|_2^2 = \|\hat{f} - \hat{u}(\cdot, t_0)\|_2^2 = \eta_d \delta^2 \int_{|\xi - \xi_0| \le 1} e^{8\pi^2 c^2 |\xi|^2 (\tau - t_0)} d\xi.$$

Since $|\xi| \ge |\xi_0| - |\xi - \xi_0|$, it follows that

$$\begin{aligned} \|f - u(\cdot, t_0)\|_2^2 &\ge \eta_d \delta^2 \int_{|\xi - \xi_0| \le 1} e^{8\pi^2 c^2 (|\xi_0| - 1)^2 (\tau - t_0)} d\xi \\ &= \delta^2 e^{8\pi^2 c^2 (|\xi_0| - 1)^2 (\tau - t_0)}. \end{aligned}$$

Thus,

$$\|g - u(\cdot, \tau)\|_2 \le \delta \quad \text{but} \quad \|f - u(\cdot, t_0)\|_2 \ge \delta e^{4\pi^2 c^2 (|\xi_0| - 1)^2 (\tau - t_0)}.$$

- The error gets amplified by a factor of $e^{4\pi^2 c^2 (|\xi_0| - 1)^2 (\tau - t_0)}$.

Form of the solution

Case (i): $\Omega = \mathbb{R}^d$.

Note that the right hand side of

$$\hat{u}(\xi, \tau) = \hat{u}(\xi, t) e^{-4\pi^2 c^2 |\xi|^2 (\tau - t)}.$$

is a product of convolution of the functions $u(x, t)$ and

$$v(x, t) := \frac{1}{[4\pi c^2 (\tau - t)]^{d/2}} e^{-|x|^2 / 4\pi c^2 (\tau - t)},$$

the so called *heat kernel*. Therefore,

$$u(x, \tau) = \frac{1}{[4\pi c^2 (\tau - t)]^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2 / 4\pi c^2 (\tau - t)} u(y, t)\, dy.$$

To obtain the above, we used the following result:

$$f(x) = e^{-a\pi |x|^2} \iff \hat{f}(\xi) = \frac{1}{a^{d/2}} e^{-\pi |\xi|^2 / a}.$$

Thus, for $0 < t_0 < \tau$, we have

$$f_\tau := u(x,\tau) = \frac{1}{[4\pi c^2(\tau - t_0)]^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/4\pi c^2(\tau - t_0)} f_{t_0}(y)\, dy.$$

The above equation also shows that *small error in the data $u(\cdot,\tau)$ leads to large deviation in the solution $u(\cdot, t_0)$*.

Case (ii): $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary $\partial\Omega$ and the solution $u(x,t)$ is required to satisfy the boundary condition

$$u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0. \tag{2}$$

In this case, by method of separation of variables, for every $f_0 := u(\cdot, 0) \in L^2(\Omega)$, the solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \langle f_0, \varphi_n \rangle \varphi_n(x), \quad x \in \Omega,\ t > 0. \tag{3}$$

Here, (λ_n) is a sequence of positive real numbers such that $\lambda_n \to \infty$ as $n \to \infty$ and (φ_n) in $L^2(\Omega)$ is a complete orthonormal sequence in $L^2(\Omega)$. In fact,

$$\triangle \varphi_n + \lambda_n^2 \varphi_n = 0 \qquad \forall\, n \in \mathbb{N}.$$

Remark: It can be seen that if $\Omega = [0, \ell]$ for some $\ell > 0$, then

$$\lambda_n := \frac{cn\pi}{\ell} \quad \text{and} \quad \varphi_n(x) := \sqrt{\frac{2}{\ell}} \sin(n\pi x/\ell), \quad x \in [0, \ell], \quad n \in \mathbb{N}.$$

Now, let $0 \leq t_0 < \tau$ and let us denote

$$f_t := u(\cdot, t), \qquad t \geq 0.$$

Then, from (3), it follows that

$$f_\tau := u(\cdot, \tau) = \sum_{n=1}^{\infty} e^{-\lambda_n^2(\tau - t_0)} \langle f_{t_0}, \varphi_n \rangle \varphi_n \tag{4}$$

so that

$$f_{t_0} := u(\cdot, t_0) = \sum_{n=1}^{\infty} e^{\lambda_n^2(\tau - t_0)} \langle f_\tau, \varphi_n \rangle \varphi_n. \tag{5}$$

From expressions (4) and (5), we can infer the following.

- The problem of finding $f_\tau := u(\cdot, \tau)$ from the knowledge of $f_{t_0} := u(\cdot, t_0)$ for $t_0 < \tau$ is a well posed problem.
- The BHCP of determining $f_{t_0} := u(\cdot, t_0)$ from the knowledge of $f_\tau := u(\cdot, \tau)$ for $\tau > t_0$ is an ill-posed problem.

More precisely, we have the following:
(i) The problem has *no solution unless* $f_\tau := u(\cdot, \tau)$ satisfies the *Piccard condition*

$$\sum_{n=1}^{\infty} e^{2\lambda_n^2(\tau - t_0)} |\langle f_\tau, \varphi_n \rangle|^2 < \infty. \tag{6}$$

(ii) Also, (5) shows that closeness of $\tilde{f}_\tau$ to f_τ *does not imply* closeness of $\tilde{f}_{t_0}$ to f_{t_0}, as $\lambda_n \to \infty$ as $n \to \infty$.

Illustration

For example, if

$$f_{\tau,k} := f_\tau + e^{-\lambda_k(\tau-t_0)}\varphi_k$$

then the solution at t_0 is given by

$$f_{t_0,k} = \sum_{n=1}^{\infty} e^{\lambda_n^2(\tau-t_0)}\langle f_{\tau,k}, \varphi_n\rangle\varphi_n.$$

Then we have

$$\|f_{\tau,k} - f_\tau\|_2 = e^{-\lambda_k(\tau-t_0)} \to 0 \quad \text{as} \quad k \to \infty$$

but

$$\|f_{t_0,k} - f_{t_0}\|_2 \to \infty \quad \text{as} \quad k \to \infty.$$

BHCP as an Operator Equation

Case (i): $\Omega = \mathbb{R}^d$.

Recall that

$$u(x,\tau) = \frac{1}{[4\pi c^2(\tau - t_0)]^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/4\pi c^2(\tau - t)} u(y, t_0)\, dy.$$

Thus, the problem is to solve the operator equation

$$Af = f_\tau,$$

where $A : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is defined by

$$(Af)(x) = \frac{1}{[4\pi c^2(\tau - t_0)]^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/4\pi c^2(\tau - t_0)} f(y)\, dy.$$

We observe that

$$A = \mathcal{F}^{-1}\widehat{A}\mathcal{F},$$

where $\mathcal{F}$ is the Fourier transform operator,

$$(\mathcal{F}f)(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-2\pi i x.\xi} dx,$$

and $\widehat{A}$ is the multiplication operator

$$(\widehat{A}f)(\xi) = e^{-4\pi^2 c^2 |\xi|^2 (\tau - t_0)} f(\xi), \qquad f \in L^2(\mathbb{R}^d).$$

- A is a non-compact operator with non-closed range.

Case (ii): $\Omega \subset \mathbb{R}^d$ is a bounded domain in $\mathbb{R}^d$.

In this case, the equation to be solved is

$$Af = f_\tau, \tag{7}$$

where $A : L^2(\Omega) \to L^2(\Omega)$ is given by

$$Af := \sum_{n=1}^{\infty} e^{-\lambda_n^2(\tau - t_0)} \langle f, \varphi_n \rangle \varphi_n. \tag{8}$$

- A is a compact positive self-adjoint operator[4] on $L^2(\Omega)$.

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[4] Nair (2002), Functional Analysis: A First Course, PHI, New Delhi

- In both the cases, the problem is ill-posed.
- *Regularization methods* are to be used[5].

By a regularization method we mean a family of well-posed problems (depending on certain parameter) whose solutions approximate the solution of the ill-posed problem.

Before considering regularization methods, we shall discuss stability estimates based on certain source conditions which are used to measure the quality of a regularization method.

[5] Nair (2009), Linear Operator Equations: Approximation and Regularization, World Scientific

Stability Estimates

We would like to have estimates of the form

$$\|u(\cdot, t_0)\|_2 \leq \Phi(\|u(\cdot, \tau)\|_2) \tag{9}$$

for some function $\Phi(\cdot)$ which satisfies the condition $\Phi(\lambda) \to 0$ as $\lambda \to 0$.

Since the problem of determining $u(\cdot, t_0)$ from the knowledge of $u(\cdot, \tau)$ is ill-posed, an estimate such as the above will not be possible unless we restrict the solution $u(\cdot, t_0)$ to certain source set in $L^2(\Omega)$.

Thus, it is necessary to identify a source set $\mathcal{M} \subseteq L^2(\Omega)$ and obtain a function $\Phi_{\mathcal{M}}(\cdot)$ such that

$$\Phi_{\mathcal{M}}(\lambda) \to 0 \quad \text{as} \quad \lambda \to 0$$

and

$$\|u(\cdot, t_0)\|_2 \leq \Phi_{\mathcal{M}}(\|u(\cdot, \tau)\|_2) \tag{10}$$

whenever $u(\cdot, t_0) \in \mathcal{M}$.

Now, let us see how estimate of the form (10) is important when we deal regularization methods.

Any continuous function $R : L^2(\Omega) \to L^2(\Omega)$ can be called a regularization method for solving an operator equation

$$Af = g,$$

where $A : L^2(\Omega) \to L^2(\Omega)$ is a bounded operator with non-closed range.

However, if the data is noisy, say $\tilde{g}$ in place of g, with

$$\|g - \tilde{g}\| \leq \delta$$

for some noise level $\delta > 0$, then, in order that $R\tilde{g}$ approximate the solution f, it is necessary that the R has to have some additional properties with respect to certain appropriate source set $\mathcal{M}$.

So, given a function $R : L^2(\Omega) \to L^2(\Omega)$, a source set $\mathcal{M} \subseteq L^2(\Omega)$ and $\delta > 0$, consider the quantity

$$\mathcal{E}_\delta(\mathcal{M}, R) := \sup\{\|f - R\tilde{g}\|_2 : f \in \mathcal{M},\ \|Af - \tilde{g}\|_2 \leq \delta\}.$$

Then the requirement is:

$$\lim_{\delta \to 0} \mathcal{E}_\delta(\mathcal{M}, R) = 0.$$

A regularization method R_0 is said to be order optimal for the source set $\mathcal{M}$, if there exists a constant $\kappa > 0$ such that

$$\|f - R_0\tilde{g}\|_2 \leq \kappa \inf_R \mathcal{E}_\delta(\mathcal{M}, R)$$

whenever $f \in \mathcal{M}$ and $\tilde{f} \in L^2(\Omega)$ is such that $\|Af - \tilde{g}|_2 \leq \delta$.

The quantity

$$\widetilde{\mathcal{E}}_\delta(\mathcal{M}) := \inf_R \mathcal{E}_\delta(\mathcal{M}, R)$$

is called the *worst case error estimate* corresponding to the source set $\mathcal{M}$ and error level δ.

It is known that[6] if $\mathcal{M}$ is a convex and balanced set, then

$$\omega_\delta(\mathcal{M}) \le \widetilde{\mathcal{E}}_\delta(\mathcal{M}) \le 2\omega_\delta(\mathcal{M}),$$

where

$$\omega_\delta(\mathcal{M}) := \sup\{\|f\|_2 : \ f \in \mathcal{M}, \ \|Af\|_2 \le \delta\}.$$

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[6] Michelli and Rivlin (1977)

Thus:

- A regularization method R_0 is order optimal for the source set $\mathcal{M}$, if there exists a constant $\kappa > 0$ such that

$$\|f - R_0\tilde{g}\|_2 \leq \kappa\, \omega_\delta(\mathcal{M})$$

  whenever $f \in \mathcal{M}$ and $\tilde{g} \in L^2(\Omega)$ is such that $\|Af - \tilde{g}\|_2 \leq \delta$.

- The source set $\mathcal{M}$ has to be identified in such a way that

$$\omega_\delta(\mathcal{M}) \to 0 \quad \text{as} \quad \delta \to 0.$$

In this regard we observe the following[7]:

- If A is not bounded below and $\mathcal{M} = \{f \in L^2(\Omega) : \|f\|_2 \leq \rho\}$, then $\omega_\delta(\mathcal{M}) = \rho$.
- If $f \in \mathcal{M} \cap N(A)$, then $\omega_\delta(\mathcal{M}) \geq \|f\|$.

[7] Nair (2009)

Once we have an estimate of the form (10), i.e.,

$$\|f\| \leq \Phi_{\mathcal{M}}(\|Af\|_2) \tag{11}$$

whenever $f \in \mathcal{M}$, it follows that

$$\omega_\delta(\mathcal{M}) \leq \Phi_{\mathcal{M}}(\delta). \tag{12}$$

If we can show the relation (12) is sharp, then the efforts would be to obtain a regularization method R which leads to an estimate of the form

$$\|f - R\tilde{g}\|_2 \leq \kappa\, \Phi_{\mathcal{M}}(\delta)$$

so that the method R is order optimal.

For the BHCP, we now derive estimates of the form (11) for certain source set $\mathcal{M}$ and remark that the derived estimate is sharp for the proposed source set $\mathcal{M}$.

Case (i): $\Omega = \mathbb{R}^d$

(a) Assume $t_0 > 0$.

For $0 \leq t < \tau$, define

$$(A_{t,\tau} f)(x) = \frac{1}{[4\pi c^2(\tau - t)]^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/4\pi c^2(\tau - t)} f(y)\, dy.$$

Then the equation to be solved is

$$Af = f_\tau := u(\cdot, \tau),$$

where $A := A_{t_0,\tau}$.

In this case, we consider the source set as

$$M_\rho := \{u(\cdot, t_0) \in L^2(\mathbb{R}^d) : \|u(\cdot, 0)\|_2 \leq \rho\},$$

i.e.,

$$M_\rho = \{A_{0,t_0} f : \|f\|_2 \leq \rho\}.$$

Note that with $p > 1$, $q > 1$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and using Hölder's inequality, we have

$$\begin{aligned}
\|u(\cdot, t_0)\|_2^2 &= \|\hat{u}(\cdot, t_0)\|_2^2 = \int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 e^{-8\pi^2 c^2 |\xi|^2 t_0} d\xi. \\
&= \int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^{2/p} e^{-8\pi^2 c^2 |\xi|^2 t_0} |\hat{f}_0(\xi)|^{2/q} d\xi \\
&\leq \left(\int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 e^{-8\pi^2 c^2 |\xi|^2 t_0 p} \right)^{1/p} \left(\int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 d\xi \right)^{1/q}
\end{aligned}$$

Now, taking $p = \tau / t_0$, we obtain,

$$\begin{aligned}
\|u(\cdot, t_0)\|_2^2 &\leq \left(\int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 e^{-8\pi^2 c^2 |\xi|^2 \tau} \right)^{t_0/\tau} \left(\int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 d\xi \right)^{1 - t_0/\tau} \\
&= \|u(\cdot, \tau)\|_2^{\frac{2t_0}{\tau}} \|f_0\|_2^{2(1 - \frac{t_0}{\tau})}.
\end{aligned}$$

Note that with $p > 1$, $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and using Hölder's inequality, we have

$$\begin{aligned}
\|u(\cdot, t_0)\|_2^2 &= \|\hat{u}(\cdot, t_0)\|_2^2 = \int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 e^{-8\pi^2 c^2 |\xi|^2 t_0} d\xi. \\
&= \int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^{2/p} e^{-8\pi^2 c^2 |\xi|^2 t_0} |\hat{f}_0(\xi)|^{2/q} d\xi \\
&\leq \left(\int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 e^{-8\pi^2 c^2 |\xi|^2 t_0 p} \right)^{1/p} \left(\int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 d\xi \right)^{1/q}
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&= \|u(\cdot, \tau)\|_2^{\frac{2t_0}{\tau}} \|f_0\|_2^{2(1 - \frac{t_0}{\tau})}.
\end{aligned}$$

Thus,

$$\|u(\cdot,t_0)\|_2 \leq \|u(\cdot,\tau)\|_2^{\frac{t_0}{\tau}} \|f_0\|_2^{1-\frac{t_0}{\tau}}.$$

Hence, if $u(\cdot,t_0) \in M_\rho$, equivalently, if $\|u(\cdot,0)\|_2 \leq \rho$, then we have

$$\|u(\cdot,t_0)\|_2 \leq \rho^{1-\frac{t_0}{\tau}} \|u(\cdot,\tau)\|_2^{\frac{t_0}{\tau}}.$$

In particular,

$$\omega_\delta(M_\rho) \leq \rho^{1-\frac{t_0}{\tau}} \delta^{\frac{t_0}{\tau}}.$$

Note that the above stability estimate is not useful for the case of $t_0 = 0$.

In fact, if $t_0 = 0$, then

$$M_\rho = \{f \in L^2(\mathbb{R}^d) : \|f\|_2 \leq \rho\}.$$

In this case we have $\omega_\delta(M_\rho) = \rho$, since A is not bounded below.

(b): Let $t_0 = 0$ and $f_0 := u(\cdot, 0)$.

In this case, we consider the source set as

$$M_{\rho, s} := \{f \in H^s(\mathbb{R}^d) : \|f\|_{2,s} \leq \rho\},$$

for $s > 0$.

Here $H^s(\mathbb{R}^d)$ is the Sobolev space of order s, i.e., the space of all $f \in L^2(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi < \infty,$$

and $\|\cdot\|_{2,s}$ is the Sobolev norm on $H^s(\mathbb{R}^d)$ defined by

$$\|f\|_{2,s} := \left[\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi\right]^{1/2}.$$

Now, we write

$$\|f_0\|_2^2 = \int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 d\xi = \int_{\mathbb{R}^d} (1+|\xi|^2)^{-s} |\hat{f}_0(\xi)|^2 (1+|\xi|^2)^s d\xi$$

so that

$$\frac{\|f_0\|_2^2}{\|f_0\|_{2,s}^2} = \int_{\mathbb{R}^d} (1+|\xi|^2)^{-s} d\mu(\xi),$$

where

$$d\mu(\xi) = \frac{|\hat{f}_0(\xi)|^2 (1+|\xi|^2)^s}{\int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 (1+|\xi|^2)^s d\xi}$$

is a probability measure.

Therefore, by Jensen's inequality, we have

$$\psi\left(\frac{\|f_0\|_2^2}{\|f_0\|_{2,s}^2}\right) = \int_{\mathbb{R}^d} \psi\left((1+|\xi|^2)^{-s}\right) d\mu(\xi)$$
$$\leq \frac{\int_{\mathbb{R}^d} \psi\left((1+|\xi|^2)^{-s}\right) |\hat{f}_0(\xi)|^2 (1+|\xi|^2)^s d\xi}{\|f_0\|_{2,s}^2}$$

for any convex function ψ. We define ψ in such a way that

$$\psi\left((1+|\xi|^2)^{-s}\right) |\hat{f}_0(\xi)|^2 (1+|\xi|^2)^s = |\hat{u}(\xi,\tau)|^2.$$

Recall that

$$\hat{u}(\xi,\tau) = \hat{f}_0(\xi) e^{-4\pi^2 c^2 |\xi|^2 \tau}.$$

Thus, the relation ψ has to satisfy is

$$\psi\left((1+|\xi|^2)^{-s}\right) |\hat{f}_0(\xi)|^2 (1+|\xi|^2)^s = e^{-8\pi^2 c^2 \tau |\xi|^2} |\hat{f}_0(\xi)|^2.$$

This is accomplished by defining $\psi(\cdot)$ as

$$\psi(\lambda) := e^{8\pi^2 c^2 \tau} \lambda e^{-8\pi^2 c^2 \tau / \lambda^{1/s}}, \quad \lambda > 0.$$

It can be verified that

- ψ is convex, and
- $\lambda \mapsto \psi(\lambda)/\lambda$ is increasing.

Thus,

$$\psi\left(\frac{\|f_0\|_2^2}{\|f_0\|_{2,s}^2}\right) \leq \frac{\|u(\cdot,\tau)\|_2^2}{\|f_0\|_{2,s}^2}.$$

Now, let us assume that $f_0 \in M_{\rho,s}$ so that $\|f_0\|_{2,s} \leq \rho$.

Hence, using the property that $\lambda \mapsto \psi(\lambda)/\lambda$ is an increasing function, we have

$$\frac{\rho^2}{\|f_0\|_2^2}\psi\left(\frac{\|f_0\|_2^2}{\rho^2}\right) \leq \frac{\|f_0\|_{2,s}^2}{\|f_0\|_2^2}\psi\left(\frac{\|f_0\|_2^2}{\|f_0\|_{2,s}^2}\right) \leq \frac{\|u(\cdot,\tau)\|_2^2}{\|f_0\|_2^2}.$$

Thus,

$$\psi\left(\frac{\|f_0\|_2^2}{\rho^2}\right) \leq \frac{\|u(\cdot,\tau)\|_2^2}{\rho^2}$$

so that

$$\|f_0\|_2 \leq \rho\sqrt{\psi^{-1}\left(\frac{\|u(\cdot,\tau)\|_2^2}{\rho^2}\right)}$$

In particular,

$$\omega_\delta(M_{\rho,s}) \leq \rho\sqrt{\psi^{-1}\left(\frac{\delta^2}{\rho^2}\right)}.$$

Remarks

It can be seen that

$$\psi(\lambda) = \lambda\varphi^{-1}(\lambda), \quad \lambda > 0,$$

where

$$\varphi(\lambda) := \left[\frac{1}{8\pi^2 c^2 \tau} \log\left(\frac{e^{8\pi^2 c^2 \tau}}{\lambda}\right)\right]^{-s}.$$

Also, it can be verified that

$$M_{\rho,s} = \{f = [\varphi(A^*A)]^{1/2} h : \|h\|_2 \leq \rho\},$$

where $A := A_{0,\tau}$.

In regularization theory for the ill-posed operator equations $Af = g$, it is known[8] that if a source set is given as

$$M = \{f = [\varphi(A^*A)]^{1/2} h : \|h\| \le \rho\},$$

where $\varphi(\cdot)$ is increasing with $\lim_{\lambda\to 0} \varphi(\lambda) = 0$ and $\lambda \mapsto \psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ is convex, then

$$\omega_\delta(M) \le \rho\sqrt{\psi^{-1}\left(\frac{\delta^2}{\rho^2}\right)}$$

and this estimate is sharp.
Thus, for the BHCP: $A_{0,\tau} f = f_\tau$, the derived estimate for the source set

$$M_{\rho,s} := \{f \in H^s(\mathbb{R}^d) : \|f\|_{2,s} \le \rho\},$$

is sharp.

[8] Tautenhahn (1998), Nair, Schock & Tautenhahn (2003)

Case (ii): Ω is a bounded domain in $\mathbb{R}^d$.

Let $f_0 = u(\cdot, 0)$.

For $0 \le t < \tau$, define

$$(A_{t,\tau} f)(x) = \sum_{n=1}^{\infty} e^{-\lambda_n^2(\tau - t)} \langle u(\cdot, t_0), \varphi_n \rangle \varphi_n$$

Then equation to be solved is

$$Af = f_\tau := u(\cdot, \tau)$$

where $A := A_{t_0,\tau}$.

Note that the operator A is compact, positive and self adjoint, with singular values (which are in fact eigenvalues)

$$\sigma_n := e^{-\lambda_n^2(\tau - t_0)}, \qquad n \in \mathbb{N}.$$

Assume first that $t_0 > 0$. As in the previous case, we consider the source set as

$$M_\rho := \{u(\cdot, t_0) \in L^2(\Omega) : \|u(\cdot, 0)\|_2 \le \rho\},$$

i.e.,

$$M_\rho = \{A_{0,t_0} f : \|f\|_2 \le \rho\}.$$

Recall that

$$u(\cdot, t_0) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 t_0} \langle f_0, \varphi_n \rangle \varphi_n.$$

Using Hölder's inequality with $p \rangle 1$ and $q > 1$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$, we have

$$\begin{aligned} \|u(\cdot,t_0)\|_2^2 &= \sum_{n=1}^{\infty} e^{-2\lambda_n^2 t_0} |\langle f_0, \varphi_n \rangle|^2 \\ &\leq \left(\sum_{n=1}^{\infty} e^{-2p\lambda_n^2 t_0} |\langle f_0, \varphi_n \rangle|^2 \right)^{1/p} \left(\sum_{n=1}^{\infty} |\langle f_0, \varphi_n \rangle|^2 \right)^{1/q} . \end{aligned}$$

Then, taking $p = \tau / t_0$, it follows that

$$\|u(\cdot,t_0)\|_2 \leq \|g\|_2^{t_0/\tau} \|f_0\|_2^{1-t_0/\tau}.$$

Hence, we obtain

$$\omega_\delta(M_\rho) \leq \rho^{1-\frac{t_0}{\tau}} \delta^{\frac{t_0}{\tau}}.$$

Since

$$f_0 := u(\cdot, 0) = \sum_{n=1}^{\infty} e^{\lambda_n^2 t_0} \langle u(\cdot, t_0), \varphi_n \rangle \varphi_n$$

we have

$$\begin{aligned} u(\cdot, t_0) \in M_\rho \quad &\Longleftrightarrow \quad \|f_0\|_2 \leq \rho \\ &\Longleftrightarrow \quad \sum_{n=1}^{\infty} e^{2\lambda_n^2 t_0} |\langle u(\cdot, t_0), \varphi_n \rangle|^2 \leq \rho^2. \end{aligned}$$

Thus,

$$M_\rho = \{ f \in L^2(\Omega) : \sum_{n=1}^{\infty} e^{2\lambda_n^2 t_0} |\langle f, \varphi_n \rangle|^2 \leq \rho^2 \}.$$

In view of this, one look for estimates under a less restrictive assumption.

Thus, consider the source set

$$\widetilde{M}_\rho := \{f \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^2 |\langle f, \varphi_n \rangle|^2 \le \rho^2\}.$$

We note the following:

- The requirement $u(\cdot, t_0) \in \widetilde{M}_\rho$ is less restrictive than the requirement $u(\cdot, t_0) \in \widetilde{M}_\rho$.
- $\widetilde{M}_\rho$ is independent of t_0.

Hence, the estimates associated with $\widetilde{M}$ would be applicable to the case of $t_0 = 0$ as well.

It can be seen that

$$\sigma_n := e^{-\lambda_n^2(\tau - t_0)} \iff \lambda_n = \left[\frac{1}{\tau - t_0} \ln\left(\frac{1}{\sigma_n}\right)\right]^{1/2}.$$

Hence, we have

$$M_\rho = \left\{ f : \sum_{n=1}^{\infty} \frac{|\langle f, \varphi_n \rangle|^2}{\sigma_n^{2\mu}} \le \rho^2 \right\}, \quad \mu := \frac{t_0}{\tau - t_0}.$$

and

$$\widetilde{M}_\rho = \left\{ f : \sum_{n=1}^{\infty} \frac{|\langle f, \varphi_n \rangle|^2}{\varphi(\sigma_n)^2} \le \rho^2 \right\}$$

where

$$\varphi(\sigma_n) := \left[\frac{1}{\tau - t_0} \ln\left(\frac{1}{\sigma_n}\right)\right]^{-1/2}, \quad n \in \mathbb{N}.$$

Note that $\{\varphi(\sigma_n)\}$ converges to 0 more slowly than $\{\sigma_n^\mu\}$.

Associated with the source set $\widetilde{M}_\rho$ we have the following stability estimate[9].

$$\omega_\delta(\widetilde{M}_\rho) \leq \rho\psi^{-1}\left(\frac{\delta}{\rho}\right),$$

where $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$.

It can be seen that

$$\rho\,\psi^{-1}(\delta/\rho) = \rho\,\sqrt{\tau - t_0}\left[\ln\left(\frac{\rho}{\delta}\right)\right]^{-1/2}[1 + o(1)].$$

[9] Nair, Schock and Tautenhahn (2003)

Regularization

Suppose the available data is noisy, i.e., we have $\tilde{g}$ in place of $g := u(\cdot, \tau)$ with

$$\|g - \tilde{g}\|_2 \leq \delta$$

for some known noise level $\delta > 0$. Then one would like to have a regularized solution $\tilde{f}$ in place of $f := u(\cdot, t_0)$ such that

$$\|f - \tilde{f}\|_2 \leq c_0 \rho \psi^{-1}\left(\frac{\delta}{\rho}\right)$$

whenever $u(\cdot, t_0) \in \widetilde{M}_\rho$.

The above result has been proved in the context of Tikhonov regularization[10] and Lavrentiev regularization[11] by appropriate choices of the regularization parameter.

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[10] Nair, Schock and Tautenhahn (2003)

[11] Nair and Tautenhahn (2004)

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