On Stability Estimates for Backward Heat Conduction Problem

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Plan of the talk

- BHCP - an introduction.
- Ill-posedness of BHCP - illustration.
- Form of the solution: $\Omega = \mathbb{R}^d$ and bounded $\Omega \subset \mathbb{R}^d$.
- BHCP as an operator equation.
- Stability estimates.
- Regularization
While dealing with a heat conducting body $\Omega \subseteq \mathbb{R}^d$, one may have to investigate the temperature profile

$$u(x, t), \quad x \in \Omega, \quad t \geq 0,$$

from the known data at a particular time, say $t = \tau$.

From this knowledge, one would like to know the temperature for the time $t < \tau$ as well as for $t > \tau$.

It is well known that the latter is a well-posed problem.

However, the former, the so called \textit{backward heat conduction problem} (BHCP) is an ill-posed problem.
BHCP: Introduction

While dealing with a heat conducting body \( \Omega \subseteq \mathbb{R}^d \), one may have to investigate the temperature profile

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    u(x, t), \quad x \in \Omega, \quad t \geq 0,
\]

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It is well known that the latter is a well-posed problem.

However, the former, the so called \textit{backward heat conduction problem} (BHCP) is an ill-posed problem.
In this talk, we shall discuss the case of the ill-posedness of the backward heat conduction problem (BHCP).

Recall that the heat equation associated with $\Omega$ is given by

$$\frac{\partial u}{\partial t} = c^2 \Delta u, \quad x \in \Omega, \quad t > 0,$$

(1)

where $u(x, t)$ represents the temperature at the point $x \in \Omega$ at time $t$.

We have the following two situations:

**Direct Problem**: From the knowledge of the temperature at time $t = t_0$, that is, $u(x, t_0)$, determine the temperature at a later time $t = \tau$, that is, $u(x, \tau)$ for $\tau > t_0$.

**Inverse Problem**: From the knowledge of the temperature at a particular time $\tau > 0$, that is, $u(x, \tau)$, determine the temperature at an earlier time $t = t_0$, that is, $u(x, t_0)$ for $t_0 < \tau$. 
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**Inverse Problem**: From the knowledge of the temperature at a particular time $\tau > 0$, that is, $u(x, \tau)$, determine the temperature at an earlier time $t = t_0$, that is, $u(x, t_0)$ for $t_0 < \tau$. 
We shall see that the direct problem is well-posed in the setting of $L^2(\Omega)$:

Given $u(\cdot, t_0)$ in $L^2(\Omega)$ for some $t_0 \geq 0$ and $t > t_0$, there exists a unique solution $u(\cdot, t)$ which depends continuously on the data $u(\cdot, t_0)$,

whereas the inverse problem, the BHCP, is ill-posed:

A solution need not exist unless the data $u(x, \tau)$ is too smooth, and even if a unique solution exists, it does not depend continuously on the data.

In fact, the BHCP belongs to a class of problems called severely ill-posed problems.
• In order to obtain stable approximate solutions for the BHCP, some *regularization methods* have to be used.

• For obtaining error estimates, it is necessary to assume some *a priori source conditions* on the unknown entities.

• The derived error estimates are usually compared with certain known *stability estimates* based on the source conditions.

• Standard result in this regard\(^1\) is for determining stability estimates for \(u(\cdot, t_0)\) for \(t_0 > 0\).

• Such results are not valid for \(t_0 = 0\).

• To deal with the case of \(t_0 = 0\), advanced analytic tools, developed recently\(^2\), have to be employed.

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\(^1\)See, e.g., Kirsch (1996)

\(^2\)Tautenhahn (1998), Nair, Schock and Tautenhahn (2003), Nair, Pereverzev and Tautenhahn (2005)
Ill-Posedness of the Problem: Illustration

Let us first look at the form of the solution.

We consider the cases of $\Omega = \mathbb{R}^d$ and $\Omega \subset \mathbb{R}^d$ a bounded domain separately.

Case (i): $\Omega = \mathbb{R}^d$:

In this case, applying Fourier transform to the equation

$$\frac{\partial u}{\partial t} = c^2 \triangle u,$$

with $f_0 := u(\cdot, 0)$, we get

$$\frac{\partial \hat{u}}{\partial t}(\xi, t) + 4\pi^2 c^2 |\xi|^2 \hat{u}(\xi, t) = 0,$$

with $\hat{u}(\xi, 0) = \hat{f}_0(\xi)$.
The solution of the above ODE is given by

\[ \hat{u}(\xi, t) = \hat{f}_0(\xi) e^{-4\pi^2 c^2 |\xi|^2 t}. \]

Then, for \( 0 \leq t \leq \tau \), we have

\[ \hat{u}(\xi, \tau) = \hat{u}(\xi, t) e^{-4\pi^2 c^2 |\xi|^2 (\tau - t)}. \]

Thus, \( 0 \leq t_0 \leq \tau \),

\[ \hat{u}(\xi, t_0) = \hat{u}(\xi, \tau) e^{4\pi^2 c^2 |\xi|^2 (\tau - t_0)}. \]

From this it follows that

- small error in the data \( u(\cdot, \tau) \) leads to large deviation in the solution \( u(\cdot, t_0) \).
An Illustration

For instance, let $\xi_0 \in \mathbb{R}^d$ and $g \in L^2(\mathbb{R}^d)$ be such that

$$
\hat{g}(\xi) = \begin{cases} 
\hat{u}(\xi, \tau), & |\xi - \xi_0| > 1 \\
\hat{u}(\xi, \tau) + \delta \sqrt{\eta_d}, & |\xi - \xi_0| \leq 1.
\end{cases}
$$

where $\eta_d$ is the volume\(^3\) of the sphere in $\mathbb{R}^d$. Then we have

$$
\|g - u(\cdot, \tau)\|_2^2 = \|\hat{g} - \hat{u}(\cdot, \tau)\|_2^2 = \int_{|\xi - \xi_0| \leq 1} \eta_d \delta^2 d\xi = \delta^2.
$$

If $f$ is the solution corresponding to the noisy data $g$, then we have

$$
\hat{f}(\xi) = \hat{g}(\xi) e^{4\pi^2 \sigma^2 |\xi|^2 (\tau - t_0)}.
$$

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\(^3\)Jason D.M. Rennie, Nov. 22 (2005): For $d \geq 2$, $\eta_d = \frac{2^{(d+1)/2} \pi^{(d-1)/2}}{d(d-2)!}$ for $d$ odd, and $\eta_d = \frac{2\pi^{d/2}}{d(d/2-1)!}$ for $d$ even.
Note that
\[ \| f - u(\cdot, t_0) \|_2^2 = \| \hat{f} - \hat{u}(\cdot, t_0) \|_2^2 = \eta_d \delta^2 \int_{|\xi - \xi_0| \leq 1} e^{8\pi^2 c^2 |\xi|^2 (\tau - t_0)} d\xi. \]

Since \(|\xi| \geq |\xi_0| - |\xi - \xi_0|\), it follows that
\[ \| f - u(\cdot, t_0) \|_2^2 \geq \eta_d \delta^2 \int_{|\xi - \xi_0| \leq 1} e^{8\pi^2 c^2 (|\xi_0| - 1)^2 (\tau - t_0)} d\xi \]
\[ = \delta^2 e^{8\pi^2 c^2 (|\xi_0| - 1)^2 (\tau - t_0)}. \]

Thus,
\[ \| g - u(\cdot, \tau) \|_2 \leq \delta \quad \text{but} \quad \| f - u(\cdot, t_0) \|_2 \geq \delta e^{4\pi^2 c^2 (|\xi_0| - 1)^2 (\tau - t_0)}. \]

- The error gets amplified by a factor of \( e^{4\pi^2 c^2 (|\xi_0| - 1)^2 (\tau - t_0)}. \)
Form of the solution

Case (i): $\Omega = \mathbb{R}^d$.

Note that the right hand side of

$$\hat{u}(\xi, \tau) = \hat{u}(\xi, t)e^{-4\pi^2 c^2 |\xi|^2 (\tau - t)}.$$ 

is a product of convolution of the functions $u(x, t)$ and

$$v(x, t) := \frac{1}{[4\pi c^2 (\tau - t)]^{d/2}} e^{-|x|^2/4\pi c^2 (\tau - t)},$$

the so called heat kernel. Therefore,

$$u(x, \tau) = \frac{1}{[4\pi c^2 (\tau - t)]^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/4\pi c^2 (\tau - t)} u(y, t) \, dy.$$

To obtain the above, we used the following result:

$$f(x) = e^{-a\pi |x|^2} \iff \hat{f}(\xi) = \frac{1}{a^{d/2}} e^{-\pi |\xi|^2/a}.$$
Thus, for $0 < t_0 < \tau$, we have

$$f_\tau := u(x, \tau) = \frac{1}{[4\pi c^2(\tau - t_0)]^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/4\pi c^2(\tau-t_0)} f_{t_0}(y) \, dy.$$ 

The above equation also shows that small error in the data $u(\cdot, \tau)$ leads to large deviation in the solution $u(\cdot, t_0)$. 
Case (ii): $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary $\partial\Omega$ and the solution $u(x, t)$ is required to satisfy the boundary condition

$$u(x, t) = 0, \quad x \in \partial\Omega, \ t > 0. \quad (2)$$

In this case, by method of separation of variables, for every $f_0 := u(\cdot, 0) \in L^2(\Omega)$, the solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \langle f_0, \varphi_n \rangle \varphi_n(x), \quad x \in \Omega, \ t > 0. \quad (3)$$

Here, $(\lambda_n)$ is a sequence of positive real numbers such that $\lambda_n \to \infty$ as $n \to \infty$ and $(\varphi_n)$ in $L^2(\Omega)$ is a complete orthonormal sequence in $L^2(\Omega)$. In fact,

$$\Delta \varphi_n + \lambda_n^2 \varphi_n = 0 \quad \forall \ n \in \mathbb{N}.$$
Remark: It can be seen that if \( \Omega = [0, \ell] \) for some \( \ell > 0 \), then

\[
\lambda_n := \frac{cn\pi}{\ell} \quad \text{and} \quad \varphi_n(x) := \sqrt{\frac{2}{\ell}} \sin(n\pi x/\ell), \quad x \in [0, \ell], \quad n \in \mathbb{N}.
\]

Now, let \( 0 \leq t_0 < \tau \) and let us denote

\[
f_t := u(\cdot, t), \quad t \geq 0.
\]

Then, from (3), it follows that

\[
f_\tau := u(\cdot, \tau) = \sum_{n=1}^{\infty} e^{-\lambda_n^2(\tau-t_0)} \langle f_{t_0}, \varphi_n \rangle \varphi_n \tag{4}
\]

so that

\[
f_{t_0} := u(\cdot, t_0) = \sum_{n=1}^{\infty} e^{\lambda_n^2(\tau-t_0)} \langle f_\tau, \varphi_n \rangle \varphi_n. \tag{5}
\]
From expressions (4) and (5), we can infer the following.

- The problem of finding $f_\tau := u(\cdot, \tau)$ from the knowledge of $f_{t_0} := u(\cdot, t_0)$ for $t_0 < \tau$ is a well posed problem.

- The BHCP of determining $f_{t_0} := u(\cdot, t_0)$ from the knowledge of $f_\tau := u(\cdot, \tau)$ for $\tau > t_0$ is an ill-posed problem.

More precisely, we have the following:
(i) The problem has no solution unless $f_\tau := u(\cdot, \tau)$ satisfies the Piccard condition

$$\sum_{n=1}^{\infty} e^{2\lambda_n^2(\tau-t_0)}|\langle f_\tau, \varphi_n \rangle|^2 < \infty.$$  \hspace{1cm} \text{(6)}

(ii) Also, (5) shows that closeness of $\tilde{f}_\tau$ to $f_\tau$ does not imply closeness of $\tilde{f}_{t_0}$ to $f_{t_0}$, as $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. 
For example, if
\[ f_{\tau,k} := f_\tau + e^{-\lambda_k(\tau-t_0)} \varphi_k \]
then the solution at \( t_0 \) is given by
\[ f_{t_0,k} = \sum_{n=1}^{\infty} e^{\lambda_n^2(\tau-t_0)} \langle f_{\tau,k}, \varphi_n \rangle \varphi_n. \]

Then we have
\[ \| f_{\tau,k} - f_\tau \|_2 = e^{-\lambda_k(\tau-t_0)} \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty \]
but
\[ \| f_{t_0,k} - f_{t_0} \|_2 \rightarrow \infty \quad \text{as} \quad k \rightarrow \infty. \]
BHCP as an Operator Equation

Case (i): $\Omega = \mathbb{R}^d$.

Recall that

$$u(x, \tau) = \frac{1}{[4\pi c^2(\tau - t_0)]^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/4\pi c^2(\tau-t)} u(y, t_0) \, dy.$$  

Thus, the problem is to solve the operator equation

$$Af = f_\tau,$$

where $A : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ is defined by

$$(Af)(x) = \frac{1}{[4\pi c^2(\tau - t_0)]^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/4\pi c^2(\tau-t_0)} f(y) \, dy.$$
We observe that

\[ A = F^{-1} \hat{A} F, \]

where \( F \) is the Fourier transform operator,

\[ (Ff)(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x)e^{-2\pi i x \cdot \xi} dx, \]

and \( \hat{A} \) is the multiplication operator

\[ (\hat{A}f)(\xi) = e^{-4\pi^2 ct^2 |\xi|^2 (\tau-t_0)} f(\xi), \quad f \in L^2(\mathbb{R}^d). \]

- \( A \) is a non-compact operator with non-closed range.
Case (ii): $\Omega \subset \mathbb{R}^d$ is a bounded domain in $\mathbb{R}^d$.

In this case, the equation to be solved is

$$Af = f_\tau,$$

(7)

where $A : L^2(\Omega) \to L^2(\Omega)$ is given by

$$Af := \sum_{n=1}^{\infty} e^{-\lambda_n^2(\tau-t_0)} \langle f, \varphi_n \rangle \varphi_n.$$

(8)

- $A$ is a compact positive self-adjoint operator\(^4\) on $L^2(\Omega)$.

In both the cases, the problem is ill-posed.

Regularization methods are to be used\(^5\).

By a regularization method we mean a family of well-posed problems (depending on certain parameter) whose solutions approximate the solution of the ill-posed problem.

Before considering regularization methods, we shall discuss stability estimates based on certain source conditions which are used to measure the quality of a regularization method.

\(^5\)Nair (2009), Linear Operator Equations: Approximation and Regularization, World Scientific
Stability Estimates

We would like to have estimates of the form

$$\|u(\cdot, t_0)\|_2 \leq \Phi(\|u(\cdot, \tau)\|_2)$$

(9)

for some function $\Phi(\cdot)$ which satisfies the condition $\Phi(\lambda) \to 0$ as $\lambda \to 0$.

Since the problem of determining $u(\cdot, t_0)$ from the knowledge of $u(\cdot, \tau)$ is ill-posed, an estimate such as the above will not be possible unless we restrict the solution $u(\cdot, t_0)$ to certain source set in $L^2(\Omega)$.

Thus, it is necessary to identify a source set $\mathcal{M} \subseteq L^2(\Omega)$ and obtain a function $\Phi_{\mathcal{M}}(\cdot)$ such that

$$\Phi_{\mathcal{M}}(\lambda) \to 0 \quad \text{as} \quad \lambda \to 0$$

and

$$\|u(\cdot, t_0)\|_2 \leq \Phi_{\mathcal{M}}(\|u(\cdot, \tau)\|_2)$$

(10)

whenever $u(\cdot, t_0) \in \mathcal{M}$. 
Now, let us see how estimate of the form (10) is important when we deal regularization methods.

Any continuous function $R : L^2(\Omega) \to L^2(\Omega)$ can be called a \textit{regularization method} for solving an operator equation

$$Af = g,$$

where $A : L^2(\Omega) \to L^2(\Omega)$ is a bounded operator with non-closed range.

However, if the data is noisy, say $\tilde{g}$ in place of $g$, with

$$\|g - \tilde{g}\| \leq \delta$$

for some noise level $\delta > 0$, then, in order that $R\tilde{g}$ approximate the solution $f$, it is necessary that the $R$ has to have some additional properties with respect to certain appropriate source set $\mathcal{M}$. 
So, given a function $R : L^2(\Omega) \to L^2(\Omega)$, a source set $\mathcal{M} \subseteq L^2(\Omega)$ and $\delta > 0$, consider the quantity

$$E_\delta(\mathcal{M}, R) := \sup\{\|f - R\tilde{g}\|_2 : f \in \mathcal{M}, \|Af - \tilde{g}\|_2 \leq \delta\}.$$  

Then the requirement is:

$$\lim_{\delta \to 0} E_\delta(\mathcal{M}, R) = 0.$$  

A regularization method $R_0$ is said to be order optimal for the source set $\mathcal{M}$, if there exists a constant $\kappa > 0$ such that

$$\|f - R_0\tilde{g}\|_2 \leq \kappa \inf_R E_\delta(\mathcal{M}, R)$$

whenever $f \in \mathcal{M}$ and $\tilde{f} \in L^2(\Omega)$ is such that $\|Af - \tilde{g}\|_2 \leq \delta$. 
The quantity
\[ \tilde{E}_\delta(\mathcal{M}) := \inf_R E_\delta(\mathcal{M}, R) \]
is called the \textit{worst case error estimate} corresponding to the source set \( \mathcal{M} \) and error level \( \delta \).

It is known that\(^6\) if \( \mathcal{M} \) is a convex and balanced set, then
\[ \omega_\delta(\mathcal{M}) \leq \tilde{E}_\delta(\mathcal{M}) \leq 2\omega_\delta(\mathcal{M}), \]
where
\[ \omega_\delta(\mathcal{M}) := \sup\{ \|f\|_2 : f \in \mathcal{M}, \|Af\|_2 \leq \delta \}. \]

\(^6\)Michelli and Rivlin (1977)
Thus:

- A regularization method $R_0$ is order optimal for the source set $\mathcal{M}$, if there exists a constant $\kappa > 0$ such that

$$\|f - R_0 \tilde{g}\|_2 \leq \kappa \omega_\delta(\mathcal{M})$$

whenever $f \in \mathcal{M}$ and $\tilde{g} \in L^2(\Omega)$ is such that $\|Af - \tilde{g}\|_2 \leq \delta$.

- The source set $\mathcal{M}$ has to be identified in such a way that

$$\omega_\delta(\mathcal{M}) \to 0 \quad \text{as} \quad \delta \to 0.$$

In this regard we observe the following\(^7\):

- If $A$ is not bounded below and $\mathcal{M} = \{f \in L^2(\Omega) : \|f\|_2 \leq \rho\}$, then $\omega_\delta(\mathcal{M}) = \rho$.

- If $f \in \mathcal{M} \cap N(A)$, then $\omega_\delta(\mathcal{M}) \geq \|f\|$.

\(^7\)Nair (2009)
Once we have an estimate of the form (10), i.e.,

$$
\|f\| \leq \Phi_{\mathcal{M}}(\|Af\|_2)
$$

(11)

whenever $f \in \mathcal{M}$, it follows that

$$
\omega_\delta(\mathcal{M}) \leq \Phi_{\mathcal{M}}(\delta).
$$

(12)

If we can show the relation (12) is sharp, then the efforts would be to obtain a regularization method $R$ which leads to an estimate of the form

$$
\|f - R\tilde{g}\|_2 \leq \kappa \Phi_{\mathcal{M}}(\delta)
$$

so that the method $R$ is order optimal.

For the BHCP, we now derive estimates of the form (11) for certain source set $\mathcal{M}$ and remark that the derived estimate is sharp for the proposed source set $\mathcal{M}$.
Case (i): $\Omega = \mathbb{R}^d$

(a) Assume $t_0 > 0$.

For $0 \leq t < \tau$, define

$$(A_{t,\tau} f)(x) = \frac{1}{[4\pi c^2(\tau - t)]^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/4\pi c^2(\tau - t)} f(y) \, dy.$$ 

Then the equation to be solved is

$$Af = f_\tau := u(\cdot, \tau),$$

where $A := A_{t_0, \tau}$.

In this case, we consider the source set as

$$M_\rho := \{u(\cdot, t_0) \in L^2(\mathbb{R}^d) : \|u(\cdot, 0)\|_2 \leq \rho\},$$

i.e.,

$$M_\rho = \{A_{0, t_0} f : \|f\|_2 \leq \rho\}.$$
Note that with $p > 1$, $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and using Hölder’s inequality, we have

$$\| u(\cdot, t_0) \|^2_2 = \| \hat{u}(\cdot, t_0) \|^2_2 = \int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 e^{-8\pi^2 c^2 |\xi|^2} t_0 d\xi.$$ 

$$= \int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^{2/p} e^{-8\pi^2 c^2 |\xi|^2} t_0 |\hat{f}_0(\xi)|^{2/q} d\xi$$

$$\leq \left( \int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 e^{-8\pi^2 c^2 |\xi|^2} t_0 p \right)^{1/p} \left( \int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 d\xi \right)^{1/q}$$

Now, taking $p = \tau / t_0$, we obtain,

$$\| u(\cdot, t_0) \|^2_2 \leq \left( \int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 e^{-8\pi^2 c^2 |\xi|^2} \right)^{t_0 / \tau} \left( \int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 d\xi \right)^{1 - t_0 / \tau}$$

$$= \| u(\cdot, \tau) \|_{2}^{2t_0 / \tau} \| f_0 \|_{2}^{2(1 - t_0 / \tau)}.$$
Note that with \( p > 1, \ q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \) and using Hölder’s inequality, we have

\[
\| u(\cdot, t_0) \|_2^2 = \| \hat{u}(\cdot, t_0) \|_2^2 = \int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 e^{-8\pi^2 c^2 |\xi|^2 t_0} d\xi.
\]

\[
= \int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^{2/p} e^{-8\pi^2 c^2 |\xi|^2 t_0} |\hat{f}_0(\xi)|^{2/q} d\xi.
\]

\[
\leq \left( \int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 e^{-8\pi^2 c^2 |\xi|^2 t_0} d\xi \right)^{1/p} \left( \int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 d\xi \right)^{1/q}
\]

Now, taking \( p = \tau / t_0 \), we obtain,

\[
\| u(\cdot, t_0) \|_2^2 \leq \left( \int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 e^{-8\pi^2 c^2 |\xi|^2 \tau} d\xi \right)^{t_0/\tau} \left( \int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 d\xi \right)^{1-t_0/\tau}
\]

\[
= \| u(\cdot, \tau) \|_2^{2t_0/\tau} \| f_0 \|_2^{2(1-t_0/\tau)}.
\]
Thus,
\[ \| u(\cdot, t_0) \|_2 \leq \| u(\cdot, \tau) \|_2^{t_0/\tau} \| f_0 \|_2^{1-t_0/\tau}. \]

Hence, if \( u(\cdot, t_0) \in M_\rho \), equivalently, if \( \| u(\cdot, 0) \|_2 \leq \rho \), then we have
\[ \| u(\cdot, t_0) \|_2 \leq \rho^{1-t_0/\tau} \| u(\cdot, \tau) \|_2^{t_0/\tau}. \]

In particular,
\[ \omega_\delta(M_\rho) \leq \rho^{1-t_0/\tau} \delta^{t_0/\tau}. \]

Note that the above stability estimate is not useful for the case of \( t_0 = 0 \).

In fact, if \( t_0 = 0 \), then
\[ M_\rho = \{ f \in L^2(\mathbb{R}^d) : \| f \|_2 \leq \rho \}. \]

In this case we have \( \omega_\delta(M_\rho) = \rho \), since \( A \) is not bounded below.
(b): Let $t_0 = 0$ and $f_0 := u(\cdot, 0)$.

In this case, we consider the source set as

$$M_{\rho, s} := \{ f \in H^s(\mathbb{R}^d) : \| f \|_{2,s} \leq \rho \},$$

for $s > 0$.

Here $H^s(\mathbb{R}^d)$ is the Sobolev space of order $s$, i.e., the space of all $f \in L^2(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi < \infty,$$

and $\| \cdot \|_{2,s}$ is the Sobolev norm on $H^s(\mathbb{R}^d)$ defined by

$$\| f \|_{2,s} := \left[ \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi \right]^{1/2}.$$
Now, we write
\[ \|f_0\|_2^2 = \int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 d\xi = \int_{\mathbb{R}^d} (1 + |\xi|^2)^{-s} |\hat{f}_0(\xi)|^2 (1 + |\xi|^2)^s d\xi \]
so that
\[ \frac{\|f_0\|_2^2}{\|f_0\|_{2,s}^2} = \int_{\mathbb{R}^d} (1 + |\xi|^2)^{-s} d\mu(\xi), \]
where
\[ d\mu(\xi) = \frac{|\hat{f}_0(\xi)|^2 (1 + |\xi|^2)^s}{\int_{\mathbb{R}^d} |\hat{f}_0(\xi)|^2 (1 + |\xi|^2)^s d\xi} d\xi \]
is a probability measure.
Therefore, by Jensen’s inequality, we have

\[
\psi \left( \frac{\|f_0\|_2^2}{\|f_0\|_{2,s}^2} \right) = \int_{\mathbb{R}^d} \psi \left( (1 + |\xi|^2)^{-s} \right) d\mu(\xi)
\]

\[
\leq \frac{\int_{\mathbb{R}^d} \psi \left( (1 + |\xi|^2)^{-s} \right) |\hat{f}_0(\xi)|^2(1 + |\xi|^2)^s d\xi}{\|f_0\|_{2,s}^2}
\]

for any convex function \(\psi\). We define \(\psi\) in such a way that

\[
\psi \left( (1 + |\xi|^2)^{-s} \right) |\hat{f}_0(\xi)|^2(1 + |\xi|^2)^s = |\hat{u}(\xi, \tau)|^2.
\]

Recall that

\[
\hat{u}(\xi, \tau) = \hat{f}_0(\xi) e^{-4\pi^2 c^2 |\xi|^2 \tau}.
\]

Thus, the relation \(\psi\) has to satisfy is

\[
\psi \left( (1 + |\xi|^2)^{-s} \right) |\hat{f}_0(\xi)|^2(1 + |\xi|^2)^s = e^{-8\pi^2 c^2 \tau |\xi|^2} |\hat{f}_0(\xi)|^2.
\]
This is accomplished by defining $\psi(\cdot)$ as

$$
\psi(\lambda) := e^{8\pi^2 c^2 \tau} \lambda e^{-8\pi^2 c^2 \tau / \lambda^{1/s}}, \quad \lambda > 0.
$$

It can be verified that

- $\psi$ is convex, and
- $\lambda \mapsto \psi(\lambda) / \lambda$ is increasing.

Thus,

$$
\psi \left( \frac{\|f_0\|_2^2}{\|f_0\|_{2,s}^2} \right) \leq \frac{\|u(\cdot, \tau)\|_2^2}{\|f_0\|_{2,s}^2}.
$$

Now, let us assume that $f_0 \in M_{\rho,s}$ so that $\|f_0\|_{2,s} \leq \rho$. 
Hence, using the property that $\lambda \mapsto \psi(\lambda)/\lambda$ is an increasing function, we have

$$\frac{\rho^2}{\|f_0\|_2^2} \psi \left( \frac{\|f_0\|_2^2}{\rho^2} \right) \leq \frac{\|f_0\|_{2,s}^2}{\|f_0\|_2^2} \psi \left( \frac{\|f_0\|_2^2}{\|f_0\|_{2,s}^2} \right) \leq \frac{\|u(\cdot, \tau)\|_2^2}{\|f_0\|_2^2}. $$

Thus,

$$\psi \left( \frac{\|f_0\|_2^2}{\rho^2} \right) \leq \frac{\|u(\cdot, \tau)\|_2^2}{\rho^2}$$

so that

$$\|f_0\|_2 \leq \rho \sqrt{\psi^{-1} \left( \frac{\|u(\cdot, \tau)\|_2^2}{\rho^2} \right)}$$

In particular,

$$\omega_\delta(M_\rho, s) \leq \rho \sqrt{\psi^{-1} \left( \frac{\delta^2}{\rho^2} \right)}.$$
It can be seen that

\[ \psi(\lambda) = \lambda \varphi^{-1}(\lambda), \quad \lambda > 0, \]

where

\[ \varphi(\lambda) := \left[ \frac{1}{8\pi^2c^2\tau} \log \left( \frac{e^{8\pi^2c^2\tau}}{\lambda} \right) \right]^{-s}. \]

Also, it can be verified that

\[ M_{\rho,s} = \{ f = [\varphi(A^*A)]^{1/2}h : \|h\|_2 \leq \rho \}, \]

where \( A := A_{0,\tau} \).
In regularization theory for the ill-posed operator equations \( Af = g \), it is known\(^8\) that if a source set is given as

\[
M = \{ f = [\varphi(A^*A)]^{1/2} h : \| h \| \leq \rho \},
\]

where \( \varphi(\cdot) \) is increasing with \( \lim_{\lambda \to 0} \varphi(\lambda) = 0 \) and \( \lambda \mapsto \psi(\lambda) := \lambda \varphi^{-1}(\lambda) \) is convex, then

\[
\omega_\delta(M) \leq \rho \sqrt{\psi^{-1}\left(\frac{\delta^2}{\rho^2}\right)}
\]

and this estimate is sharp.

Thus, for the BHCP: \( A_{0,\tau}f = f_\tau \), the derived estimate for the source set

\[
M_{\rho, s} := \{ f \in H^s(\mathbb{R}^d) : \| f \|_{2, s} \leq \rho \},
\]

is sharp.

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\(^8\) Tautenhahn (1998), Nair, Schock & Tautenhahn (2003)
Case (ii): $\Omega$ is a bounded domain in $\mathbb{R}^d$.

Let $f_0 = u(\cdot, 0)$.

For $0 \leq t < \tau$, define

$$(A_{t, \tau} f)(x) = \sum_{n=1}^{\infty} e^{-\lambda_n^2(\tau-t)} \langle u(\cdot, t_0), \varphi_n \rangle \varphi_n$$

Then equation to be solved is

$$Af = f_\tau := u(\cdot, \tau)$$

where $A := A_{t_0, \tau}$.

Note that the operator $A$ is compact, positive and self adjoint, with singular values (which are in fact eigenvalues)

$$\sigma_n := e^{-\lambda_n^2(\tau-t_0)}, \quad n \in \mathbb{N}.$$
Assume first that $t_0 > 0$. As in the previous case, we consider the source set as

$$M_\rho := \{ u(\cdot, t_0) \in L^2(\Omega) : \| u(\cdot, 0) \|_2 \leq \rho \},$$

i.e.,

$$M_\rho = \{ A_{0,t_0} f : \| f \|_2 \leq \rho \}.$$

Recall that

$$u(\cdot, t_0) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 t_0} \langle f_0, \varphi_n \rangle \varphi_n.$$
Using Hölder’s inequality with $p \rangle 1$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$
\| u(\cdot, t_0) \|_2^2 = \sum_{n=1}^{\infty} e^{-2\lambda_n^2 t_0} |\langle f_0, \varphi_n \rangle|^2
\leq \left( \sum_{n=1}^{\infty} e^{-2p\lambda_n^2 t_0} |\langle f_0, \varphi_n \rangle|^2 \right)^{1/p} \left( \sum_{n=1}^{\infty} |\langle f_0, \varphi_n \rangle|^2 \right)^{1/q}.
$$

Then, taking $p = \tau / t_0$, it follows that

$$
\| u(\cdot, t_0) \|_2 \leq \| g \|_2^{t_0/\tau} \| f_0 \|_2^{1-t_0/\tau}.
$$

Hence, we obtain

$$
\omega_\delta(M_\rho) \leq \rho^{1-t_0/\tau} \delta^{t_0/\tau}.
$$
Since
\[ f_0 := u(\cdot, 0) = \sum_{n=1}^{\infty} e^{\lambda_n^2 t_0} \langle u(\cdot, t_0), \varphi_n \rangle \varphi_n \]
we have
\[ u(\cdot, t_0) \in M_\rho \iff \| f_0 \|_2 \leq \rho \]
\[ \iff \sum_{n=1}^{\infty} e^{2\lambda_n^2 t_0} | \langle u(\cdot, t_0), \varphi_n \rangle |^2 \leq \rho^2. \]

Thus,
\[ M_\rho = \{ f \in L^2(\Omega) : \sum_{n=1}^{\infty} e^{2\lambda_n^2 t_0} | \langle f, \varphi_n \rangle |^2 \leq \rho^2 \}. \]

In view of this, one look for estimates under a less restrictive assumption.
Thus, consider the source set

\[ \tilde{M}_\rho := \{ f \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^2 |\langle f, \varphi_n \rangle|^2 \leq \rho^2 \}. \]

We note the following:

- The requirement \( u(\cdot, t_0) \in \tilde{M}_\rho \) is less restrictive than the requirement \( u(\cdot, t_0) \in \tilde{M}_\rho \).

- \( \tilde{M}_\rho \) is independent of \( t_0 \).

Hence, the estimates associated with \( \tilde{M} \) would be applicable to the case of \( t_0 = 0 \) as well.
It can be seen that

\[ \sigma_n := e^{-\lambda_n^2(\tau-t_0)} \iff \lambda_n = \left[ \frac{1}{\tau - t_0} \ln \left( \frac{1}{\sigma_n} \right) \right]^{1/2}. \]

Hence, we have

\[ M_\rho = \left\{ f : \sum_{n=1}^{\infty} \frac{\left| \langle f, \varphi_n \rangle \right|^2}{\sigma_n^{2\mu}} \leq \rho^2 \right\}, \quad \mu := \frac{t_0}{\tau - t_0}. \]

and

\[ \tilde{M}_\rho = \left\{ f : \sum_{n=1}^{\infty} \frac{\left| \langle f, \varphi_n \rangle \right|^2}{\varphi(\sigma_n)^2} \leq \rho^2 \right\} \]

where

\[ \varphi(\sigma_n) := \left[ \frac{1}{\tau - t_0} \ln \left( \frac{1}{\sigma_n} \right) \right]^{-1/2}, \quad n \in \mathbb{N}. \]

Note that \( \{ \varphi(\sigma_n) \} \) converges to 0 more slowly than \( \{ \sigma_n^{\mu} \} \).
Associated with the source set $\tilde{M}_\rho$ we have the following stability estimate\(^9\).

\[ \omega_\delta(\tilde{M}_\rho) \leq \rho \psi^{-1}\left(\frac{\delta}{\rho}\right), \]

where $\psi(\lambda) := \lambda \varphi^{-1}(\lambda)$.

It can be seen that

\[ \rho \psi^{-1}\left(\frac{\delta}{\rho}\right) = \rho \sqrt{\tau - t_0} \left[ \ln\left(\frac{\rho}{\delta}\right) \right]^{-1/2} \left[ 1 + o(1) \right]. \]

\(^9\)Nair, Schock and Tautenhahn (2003)
Regularization

Suppose the available data is noisy, i.e., we have $\tilde{g}$ in place of $g := u(\cdot, \tau)$ with

$$\|g - \tilde{g}\|_2 \leq \delta$$

for some known noise level $\delta > 0$. Then one would like to have a regularized solution $\tilde{f}$ in place of $f := u(\cdot, t_0)$ such that

$$\|f - \tilde{f}\|_2 \leq c_0 \rho \psi^{-1}\left(\frac{\delta}{\rho}\right)$$

whenever $u(\cdot, t_0) \in \tilde{M}_\rho$.

The above result has been proved in the context of Tikhonov regularization\textsuperscript{10} and Lavrentiev regularization\textsuperscript{11} by appropriate choices of the regularization parameter.

\textsuperscript{10}Nair, Schock and Tautenhahn (2003)
\textsuperscript{11}Nair and Tautenhahn (2004)
References


References


