

# Incompressible fluids in porous media: On the Muskat problem on a strip

Rafael Orive Illera  
rafael.orive@uam.es

Instituto de Ciencias Matemáticas

Universidad Autónoma de Madrid

Bangalore, August 2010

- 1 The Muskat problem. Motivation
- 2 Some properties of our model.
- 3 Local well-posedness.
- 4 Maximum principle.

- 1 The Muskat problem. Motivation
- 2 Some properties of our model.
- 3 Local well-posedness.
- 4 Maximum principle.

- 1 The Muskat problem. Motivation
- 2 Some properties of our model.
- 3 Local well-posedness.
- 4 Maximum principle.

- 1 The Muskat problem. Motivation
- 2 Some properties of our model.
- 3 Local well-posedness.
- 4 Maximum principle.

# The Muskat problem

We consider the case where we have two incompressible and immiscible fluids on a porous strip. The fluids have the same viscosity but different density, so we impose that  $\rho$  has the following form

$$\rho = \rho^1 \chi_{S^1(t)} + \rho^2 \chi_{S^2(t)}, \quad (1)$$

with  $\rho_1, \rho_2$  some positive constants.

And we are interested in obtain an evolution for the curve  $f$  in a way that we have a weak solution of the transport equation for  $\rho$ .

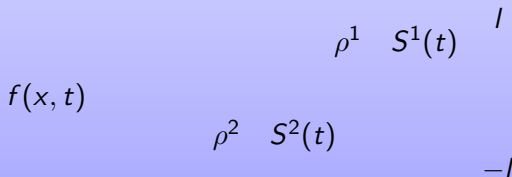


Figure: Physical situation.

# The equations

We consider **Darcy's Law** for an incompressible fluid in a two-dimensional porous medium:

$$\frac{\mu}{\kappa} \mathbf{v} = -\nabla p - (0, g\rho),$$

where  $\mu = 1$  is dynamic viscosity,  $\kappa = 1$  permeability of the medium,  $g = 1$  acceleration due to gravity,  $p$  pressure of the fluid and  $\mathbf{v}$  incompressible field of velocities.

Due to the incompressibility we have

$$\nabla \cdot \mathbf{v} = 0.$$

To close the evolution of this system we need the conservation of mass equation

$$\partial_t \rho + \mathbf{v} \cdot \nabla \rho = 0.$$

# The equations

We consider **Darcy's Law** for an incompressible fluid in a two-dimensional porous medium:

$$\frac{\mu}{\kappa} \mathbf{v} = -\nabla p - (0, g\rho),$$

where  $\mu = 1$  is dynamic viscosity,  $\kappa = 1$  permeability of the medium,  $g = 1$  acceleration due to gravity,  $p$  pressure of the fluid and  $\mathbf{v}$  incompressible field of velocities.

Due to the incompressibility we have

$$\nabla \cdot \mathbf{v} = 0.$$

To close the evolution of this system we need the conservation of mass equation

$$\partial_t \rho + \mathbf{v} \cdot \nabla \rho = 0.$$

# The equations

We consider **Darcy's Law** for an incompressible fluid in a two-dimensional porous medium:

$$\frac{\mu}{\kappa} \mathbf{v} = -\nabla p - (0, g\rho),$$

where  $\mu = 1$  is dynamic viscosity,  $\kappa = 1$  permeability of the medium,  $g = 1$  acceleration due to gravity,  $p$  pressure of the fluid and  $\mathbf{v}$  incompressible field of velocities.

Due to the incompressibility we have

$$\nabla \cdot \mathbf{v} = 0.$$

To close the evolution of this system we need the conservation of mass equation

$$\partial_t \rho + \mathbf{v} \cdot \nabla \rho = 0.$$

In a recent work, Escher and Matioc (2010) consider the problem as a problem in two coupled domains. They write  $u^i = -\rho^i - g\rho^i y$  and obtain the following system of equations:

$$\left\{ \begin{array}{l} \Delta u^i(x, y, t) = 0, \quad \text{in } S^i, \\ \frac{\mu^i}{\kappa_i^i} v^i(x, y, t) = \nabla u^i(x, y, t), \quad \text{in } S^i, \\ u^1(x, f(x, t), t) - u^2(x, f(x, t), t) = g(\rho^2 - \rho^1)f(x, t), \\ \partial_t f(x, t) = (-\partial_x f(x, t), 1) \cdot (\nabla u_i)|_{y=f(x, t)}, \\ + \text{Neumann boundary conditions and initial data} \end{array} \right. \quad (2)$$

If they know  $U^1(x, t) = u^1(x, f(x, t), t)$  and  $f(x, t)$ , they can solve the equation  $\Delta u^1$  in  $S^1(t)$ .

Using  $\partial_n u^1(x, f(x, t), t) = \partial_n u^2(x, f(x, t), t)$  they can solve  $\Delta u_2$ .

Thus, knowing  $f(x, t)$  and  $U^1(x, t)$ , they can be obtain

$$U^2(x, t) = u^2(x, f(x, t), t) = T[f]U^1(x, t).$$

In a recent work, Escher and Matioc (2010) consider the problem as a problem in two coupled domains. They write  $u^i = -p^i - g\rho^i y$  and obtain the following system of equations:

$$\left\{ \begin{array}{l} \Delta u^i(x, y, t) = 0, \quad \text{in } S^i, \\ \frac{\mu^i}{\kappa_i} v^i(x, y, t) = \nabla u^i(x, y, t), \quad \text{in } S^i, \\ u^1(x, f(x, t), t) - u^2(x, f(x, t), t) = g(\rho^2 - \rho^1)f(x, t), \\ \partial_t f(x, t) = (-\partial_x f(x, t), 1) \cdot (\nabla u_i)|_{y=f(x, t)}, \\ + \text{Neumann boundary conditions and initial data} \end{array} \right. \quad (2)$$

If they know  $U^1(x, t) = u^1(x, f(x, t), t)$  and  $f(x, t)$ , they can solve the equation  $\Delta u^1$  in  $S^1(t)$ .

Using  $\partial_n u^1(x, f(x, t), t) = \partial_n u^2(x, f(x, t), t)$  they can solve  $\Delta u_2$ .

Thus, knowing  $f(x, t)$  and  $U^1(x, t)$ , they can be obtain

$$U^2(x, t) = u^2(x, f(x, t), t) = T[f]U^1(x, t).$$

If they write two equations in terms of  $f$  and  $U^1$  they close the problem. This can be done using the Dirichlet-to-Neumann operator, denoted by  $G[f]U^1$ , and the operator  $T[f]$ . Finally, they obtain

$$\begin{cases} U^1(x, t) - T[f]U^1(x, t) = g(\rho^2 - \rho^1)f(x, t), \\ \partial_t f(x, t) = G[f]U^1(x, t) \end{cases} \quad (3)$$

where

$$\begin{aligned} G[f]U^1(x, t) &= (-\partial_x f(x, t), 1) \cdot \nabla u^1|_{y=f(x, t)} \\ &= \sqrt{1 + (\partial_x f(x, t))^2} (\partial_n u^1)|_{y=f(x, t)}. \end{aligned}$$

Escher and Matioc prove that, if  $f$  is small enough, they can revert the operator  $I - T[f]$ . Then they obtained an parabolic abstract equation for the interface and local existence for small initial data  $f_0 \in C^{2,\delta}$  when the condition Raileigh-Taylor is satisfied.

If they write two equations in terms of  $f$  and  $U^1$  they close the problem. This can be done using the Dirichlet-to-Neumann operator, denoted by  $G[f]U^1$ , and the operator  $T[f]$ . Finally, they obtain

$$\begin{cases} U^1(x, t) - T[f]U^1(x, t) = g(\rho^2 - \rho^1)f(x, t), \\ \partial_t f(x, t) = G[f]U^1(x, t) \end{cases} \quad (3)$$

where

$$\begin{aligned} G[f]U^1(x, t) &= (-\partial_x f(x, t), 1) \cdot \nabla u^1|_{y=f(x, t)} \\ &= \sqrt{1 + (\partial_x f(x, t))^2} (\partial_n u^1)|_{y=f(x, t)}. \end{aligned}$$

Escher and Matioc prove that, if  $f$  is small enough, they can revert the operator  $I - T[f]$ . Then they obtained an parabolic abstract equation for the interface and local existence for small initial data  $f_0 \in C^{2, \delta}$  when the condition Raileigh-Taylor is satisfied.

# Our interpretation

Taking the curl twice in Darcy's Law, we get

$$\operatorname{curl} \operatorname{curl} v = -\Delta_{x,y} v = -\operatorname{curl} \operatorname{curl} \rho e_2 = -(\partial_y \partial_x \rho, -\partial_x^2 \rho)$$

due to the incompressibility condition. Thus,

$$v = \left( \partial_x (\Delta_{x,y})^{-1} \partial_y \rho, \partial_x (\Delta_{x,y})^{-1} (-\partial_x \rho) \right)$$

This problem is an active scalar problem in the sense that it is equivalent to the following system

$$\begin{cases} \partial_t \rho + v \cdot \nabla \rho = 0 \\ v = \left( (\Delta_{x,y})^{-1} \partial_{xy}^2 \rho, (\Delta_{x,y})^{-1} (-\partial_x^2 \rho) \right) \end{cases} \quad (4)$$

where we need to give sense to the inverse laplacian by means of appropriate boundary conditions if necessary.

# Our interpretation

Taking the curl twice in Darcy's Law, we get

$$\operatorname{curl} \operatorname{curl} v = -\Delta_{x,y} v = -\operatorname{curl} \operatorname{curl} \rho e_2 = -(\partial_y \partial_x \rho, -\partial_x^2 \rho)$$

due to the incompressibility condition. Thus,

$$v = \left( \partial_x (\Delta_{x,y})^{-1} \partial_y \rho, \partial_x (\Delta_{x,y})^{-1} (-\partial_x \rho) \right)$$

This problem is an active scalar problem in the sense that it is equivalent to the following system

$$\begin{cases} \partial_t \rho + v \cdot \nabla \rho = 0 \\ v = \left( (\Delta_{x,y})^{-1} \partial_{xy}^2 \rho, (\Delta_{x,y})^{-1} (-\partial_x^2 \rho) \right) \end{cases} \quad (4)$$

where we need to give sense to the inverse laplacian by means of appropriate boundary conditions if necessary.

Moreover, in our geometry, we have the evolution of  $f$  so the interface moves along with the fluids. We note that  $y(t) = f(x, t)$  and the derivation with respect to  $t$

$$y' = \partial_t f + x' \partial_x f$$

and  $v = (x', y')$ , then, the contour equation satisfies

$$\partial_t f(x) = (-\partial_x f(x), 1) \cdot v|_{y=f(x)}, \quad (5)$$

with  $(-\partial_x f, 1)$  the upper non-unitary normal vector for the interfacial wave.

PROBLEM: How do we resolve  $f$ ? Which is the system?

Moreover, in our geometry, we have the evolution of  $f$  so the interface moves along with the fluids. We note that  $y(t) = f(x, t)$  and the derivation with respect to  $t$

$$y' = \partial_t f + x' \partial_x f$$

and  $v = (x', y')$ , then, the contour equation satisfies

$$\partial_t f(x) = (-\partial_x f(x), 1) \cdot v|_{y=f(x)}, \quad (5)$$

with  $(-\partial_x f, 1)$  the upper non-unitary normal vector for the interfacial wave.

PROBLEM: How do we resolve  $f$ ? Which is the system?

# The whole plane case

In a series of works, Córdoba and Gancedo (2007, 2008) studied when the spatial domain is the whole plane ( $l = \infty$ )..

They obtained the following nonlinear equation for the interface

$$\partial_t f = \frac{\rho^2 - \rho^1}{2\pi} \text{P.V.} \int_{\mathbb{R}} (\partial_x f(x) - \partial_x f(x - \eta)) \Upsilon(x, \eta, f) d\eta, \quad (6)$$

P.V. denotes principal value at zero and

$$\Upsilon(x, \eta, f) = \frac{\eta}{\eta^2 + (f(x) - f(x - \eta))^2}.$$

In the case with  $\rho_2 > \rho_1 > 0$  (the densest fluid is below) they show well-posedness (2007), a maximum principle for the  $L^\infty$  norm of the interface (2009) and global existence under  $L^\infty$ -conditions in the initial data (2010).

If  $\rho_1 > \rho_2$  they show ill-posedness.

# The whole plane case

In a series of works, Córdoba and Gancedo (2007, 2008) studied when the spatial domain is the whole plane ( $l = \infty$ )..

They obtained the following nonlinear equation for the interface

$$\partial_t f = \frac{\rho^2 - \rho^1}{2\pi} \text{P.V.} \int_{\mathbb{R}} (\partial_x f(x) - \partial_x f(x - \eta)) \Upsilon(x, \eta, f) d\eta, \quad (6)$$

P.V. denotes principal value at zero and

$$\Upsilon(x, \eta, f) = \frac{\eta}{\eta^2 + (f(x) - f(x - \eta))^2}.$$

In the case with  $\rho_2 > \rho_1 > 0$  (the densest fluid is below) they show well-posedness (2007), a maximum principle for the  $L^\infty$  norm of the interface (2009) and global existence under  $L^\infty$ -conditions in the initial data (2010).

If  $\rho_1 > \rho_2$  they show ill-posedness.

# The porous strip case

Remember that

$$v = \left( \partial_x (\Delta_{x,y})^{-1} \partial_y \rho, \partial_x (\Delta_{x,y})^{-1} (-\partial_x \rho) \right)$$

and the contour equation satisfies

$$\partial_t f(x) = \bar{n} \cdot v|_{y=f(x)}.$$

Since

$$\nabla \rho = (\rho^2 - \rho^1) (\partial_x f(x), -1) \delta(y - f(x)).$$

We note that

$$-p = (\Delta_{x,y})^{-1} \partial_y \rho, \quad \Psi = (\Delta_{x,y})^{-1} (-\partial_x \rho)$$

and so  $v_1 = \partial_x(-p)$ ,  $v_2 = \partial_x \Psi$ .  $\Psi$  is the stream function.

We consider homogeneous Dirichlet boundary conditions for  $v$ , *i.e.*  $v(x, \pm l, t) = 0$ . Using separation of variables and Fourier method we have

$$-p = \sum_{n=1}^{\infty} \phi_n(x) \sin(\ell_n(y))$$

and

$$\Psi = \sum_{n=1}^{\infty} \psi_n(x) \sin(\ell_n(y))$$

We denote  $\ell_n(y) = n\pi(y + l)/2l$ .

We solve with respect to  $x$  and conclude that

$$\phi_n(x) = \rho^2 - \rho^1 \int_{\mathbb{R}} \sin(\ell_n(f(\eta))) \frac{e^{-\frac{n\pi|x-\eta|}{2l}}}{n\pi} d\eta.$$

$$\psi_n(x) = \rho^2 - \rho^1 \int_{\mathbb{R}} \partial_x f(\eta) \sin(\ell_n(f(\eta))) \frac{e^{-\frac{n\pi|x-\eta|}{2l}}}{n\pi} d\eta.$$

Since  $v_1 = \partial_x(-\rho)$ ,  $v_2 = \partial_x\Psi$ , the velocity can be written as

$$v = \frac{\rho^2 - \rho^1}{2l} \sum_{n=1}^{\infty} \sin(\ell_n(y)) \int_{\mathbb{R}} \sin(\ell_n(f(\eta))) e^{-\frac{n\pi|x-\eta|}{2l}} \left( -\frac{x-\eta}{|x-\eta|} \right) (1, \partial_x f(\eta)) d\eta$$

We solve with respect to  $x$  and conclude that

$$\phi_n(x) = \rho^2 - \rho^1 \int_{\mathbb{R}} \sin(\ell_n(f(\eta))) \frac{e^{-\frac{n\pi|x-\eta|}{2l}}}{n\pi} d\eta.$$

$$\psi_n(x) = \rho^2 - \rho^1 \int_{\mathbb{R}} \partial_x f(\eta) \sin(\ell_n(f(\eta))) \frac{e^{-\frac{n\pi|x-\eta|}{2l}}}{n\pi} d\eta.$$

Since  $v_1 = \partial_x(-\rho)$ ,  $v_2 = \partial_x\Psi$ , the velocity can be written as

$$v = \frac{\rho^2 - \rho^1}{2l} \sum_{n=1}^{\infty} \sin(\ell_n(y)) \int_{\mathbb{R}} \sin(\ell_n(f(\eta))) e^{-\frac{n\pi|x-\eta|}{2l}} \left( -\frac{x-\eta}{|x-\eta|} \right) (1, \partial_x f(\eta)) d\eta$$

The contour equation satisfies

$$\partial_t f(x) = \bar{n} \cdot \nu|_{y=f(x)} = A[f](x),$$

with  $\bar{n}$  the upper non-unitary normal vector for the interfacial wave. The operator defined by means of Fourier method is

$$\begin{aligned} A[f](x) &= \frac{\rho^2 - \rho^1}{2l} \sum_{n=1}^{\infty} \text{P.V.} \int_{\mathbb{R}} \sin(\ell_n(f(x))) \frac{x - \eta}{|x - \eta|} e^{-\frac{n\pi|x-\eta|}{2l}} \\ &\quad \cdot \sin(\ell_n(f(\eta))) (\partial_x f(x) - \partial_x f(\eta)) d\eta \\ &= \frac{\rho^2 - \rho^1}{2l} \sum_{n=1}^{\infty} \text{P.V.} \int_{\mathbb{R}} \frac{\eta}{|\eta|} e^{-\frac{n\pi|\eta|}{2l}} \sin(\ell_n(f(x))) \\ &\quad \cdot \sin(\ell_n(f(x - \eta))) (\partial_x f(x) - \partial_x f(x - \eta)) d\eta. \end{aligned}$$

We note that  $A[f] \in L^\infty(\mathbb{R})$  and

$$|A[f](x)| \leq \|f\|_{C^2} 8l \left( -\frac{1}{\pi} \log(1 - e^{-\frac{\pi}{2l}}) + \frac{l}{6} \right).$$

The contour equation satisfies

$$\partial_t f(x) = \bar{n} \cdot \nu|_{y=f(x)} = A[f](x),$$

with  $\bar{n}$  the upper non-unitary normal vector for the interfacial wave. The operator defined by means of Fourier method is

$$\begin{aligned} A[f](x) &= \frac{\rho^2 - \rho^1}{2l} \sum_{n=1}^{\infty} \text{P.V.} \int_{\mathbb{R}} \sin(\ell_n(f(x))) \frac{x - \eta}{|x - \eta|} e^{-\frac{n\pi|x-\eta|}{2l}} \\ &\quad \cdot \sin(\ell_n(f(\eta))) (\partial_x f(x) - \partial_x f(\eta)) d\eta \\ &= \frac{\rho^2 - \rho^1}{2l} \sum_{n=1}^{\infty} \text{P.V.} \int_{\mathbb{R}} \frac{\eta}{|\eta|} e^{-\frac{n\pi|\eta|}{2l}} \sin(\ell_n(f(x))) \\ &\quad \cdot \sin(\ell_n(f(x - \eta))) (\partial_x f(x) - \partial_x f(x - \eta)) d\eta. \end{aligned}$$

We note that  $A[f] \in L^\infty(\mathbb{R})$  and

$$|A[f](x)| \leq \|f\|_{C^2} 8l \left( -\frac{1}{\pi} \log \left( 1 - e^{-\frac{\pi}{2l}} \right) + \frac{l}{6} \right).$$

Considering the Euler-De Moivre formulas and summing up the previous series, we obtain the equation for the interfacial wave

$$\partial_t f(x, t) = \frac{\rho^2 - \rho^1}{4l} \text{P.V.} \int_{\mathbb{R}} (\partial_x f(x) - \partial_x f(x - \eta)) \Xi(x, \eta, f) d\eta, \quad (7)$$

where the singular kernel  $\Xi = \Xi_1 - \Xi_2$  is defined as a competition between

$$\Xi_1 = \frac{\sinh\left(\frac{\pi}{2l}\eta\right)}{\cosh\left(\frac{\pi}{2l}\eta\right) - \cos\left(\frac{\pi}{2l}(f(x) - f(x - \eta))\right)},$$

$$\Xi_2 = \frac{\sinh\left(\frac{\pi}{2l}\eta\right)}{\cosh\left(\frac{\pi}{2l}\eta\right) + \cos\left(\frac{\pi}{2l}(f(x) + f(x - \eta))\right)}.$$

The first part of the kernel  $\Xi_1$  corresponds to the singular character of the problem, and the second part  $\Xi_2$  becomes singular when  $f$  reaches the boundaries.

# Some properties

## Definition of weak solution of the mass conservation equation

Let  $v$  be an incompressible field of velocities following Darcy's Law. We define the weak solution of the conservation of mass equation present in (4) as a function satisfying

$$\int_0^T \int_{\mathbb{R}} \int_{-l}^l \rho(x, y, t) \partial_t \phi(x, y, t) + v(x, y, t) \rho(x, y, t) \nabla_{x,y} \phi(x, y, t) dy dx dt = 0 \quad (8)$$

for all  $\phi \in C_c^\infty(\mathbb{R} \times (-l, l) \times (0, T))$ .

## Theorem: The conservation of mass equation

Let  $\rho$  be the function defined in (1). Then  $\rho$  is a weak solution (in the sense of the previous definition) of the conservation of mass equation if and only if  $f$  is a solution of (7).

# Some properties

## Definition of weak solution of the mass conservation equation

Let  $v$  be an incompressible field of velocities following Darcy's Law. We define the weak solution of the conservation of mass equation present in (4) as a function satisfying

$$\int_0^T \int_{\mathbb{R}} \int_{-l}^l \rho(x, y, t) \partial_t \phi(x, y, t) + v(x, y, t) \rho(x, y, t) \nabla_{x,y} \phi(x, y, t) dy dx dt = 0 \quad (8)$$

for all  $\phi \in C_c^\infty(\mathbb{R} \times (-l, l) \times (0, T))$ .

## Theorem: The conservation of mass equation

Let  $\rho$  be the function defined in (1). Then  $\rho$  is a weak solution (in the sense of the previous definition) of the conservation of mass equation if and only if  $f$  is a solution of (7).

# Velocity of the interface

We consider the following limits on  $f(x, t)$ :

$$v^+(x, f(x, t), t) = \lim_{\epsilon \rightarrow 0^+} v(x - \epsilon \partial_x f(x, t), f(x, t) + \epsilon),$$

$$v^-(x, f(x, t), t) = \lim_{\epsilon \rightarrow 0^+} v(x + \epsilon \partial_x f(x, t), f(x, t) - \epsilon).$$

Then,

$$v^\pm(x, f(x, t), t) = \pm \frac{\rho^2 - \rho^1}{2} (1, \partial_x f(x)) \frac{\partial_x f(x)}{1 + (\partial_x f(x))^2} - \bar{\rho} \int_{\mathbb{R}} \Xi(x, \eta, f) (1, \partial_x f(x - \eta)) d\eta.$$

# Mean value conservation

This is a general property for 2D interfaces which are graphs:

$$\int_{\mathbb{R}} f(x, t) dx = \int_{\mathbb{R}} f_0(x) dx.$$

This is immediate because in terms of the stream function we have

$$v = (-\partial_y \Psi, \partial_x \Psi).$$

The equation for the interface is

$$\bar{n} \cdot v = \partial_x f(x) \partial_y \Psi + \partial_x \Psi = \partial_x (\Psi(x, f(x, t), t)).$$

We conclude the result,

$$\partial_t \int_{\mathbb{R}} f(x) dx = \int_{\mathbb{R}} \bar{n} \cdot v dx = \int_{\mathbb{R}} \partial_x \Psi(x, f(x, t), t) dx = 0.$$

# Recovering the whole plane case

Let  $f \in H^2_l(\mathbb{R})$  and  $A[f]$  be the operator such that of the evolution of  $f$ . Then,

$$\lim_{l \rightarrow \infty} A[f](x) = \frac{\rho_2 - \rho_1}{2\pi} \text{P.V.} \int_{\mathbb{R}} \frac{\eta(\partial_x f(x) - \partial_x f(x - \eta))}{\eta^2 + (f(x) - f(x - \eta))^2} d\eta,$$

Then, the limit equation as  $l \rightarrow \infty$  is the equation in the whole space worked by Córdoba, Gancedo 2007

$$\partial_t f(x) = \frac{\rho_2 - \rho_1}{2\pi} \text{P.V.} \int_{\mathbb{R}} \frac{\eta(\partial_x f(x) - \partial_x f(x - \eta))}{\eta^2 + (f(x) - f(x - \eta))^2} d\eta.$$

# Linearized equation

We take a perturbation of the state,  $f(t, x) = \epsilon g(t, x)$  and get

$$\begin{aligned} \partial_t g(x) = & \frac{\rho^2 - \rho^1}{4l} \text{P.V.} \int_{\mathbb{R}} (\partial_x g(x) - \partial_x g(x - \eta)) \sinh\left(\frac{\pi}{2l}\eta\right) \\ & \left[ \left( \cosh\left(\frac{\pi}{2l}\eta\right) - \cos\left(\frac{\pi\epsilon}{2l}(g(x) - g(x - \eta))\right) \right)^{-1} \right. \\ & \left. - \left( \cosh\left(\frac{\pi}{2l}\eta\right) + \cos\left(\frac{\pi\epsilon}{2l}(g(x) + g(x - \eta))\right) \right)^{-1} \right] d\eta. \end{aligned}$$

We study the limit when  $\epsilon \rightarrow 0$  of the right term and obtain

$$\partial_t g(x) = \frac{\rho^1 - \rho^2}{2\pi} \text{P.V.} \int_{\mathbb{R}} \frac{\eta}{|\eta|^2} \partial_x g(t, x - \eta) d\eta = \frac{\rho^1 - \rho^2}{2} \Lambda g.$$

Thus, we can recover the same linear operator than in the whole space. Moreover,  $\rho^1 > \rho^2$  case the problem is ill-posed.

# Linearized equation

We take a perturbation of the state,  $f(t, x) = \epsilon g(t, x)$  and get

$$\begin{aligned} \partial_t g(x) = & \frac{\rho^2 - \rho^1}{4l} \text{P.V.} \int_{\mathbb{R}} (\partial_x g(x) - \partial_x g(x - \eta)) \sinh\left(\frac{\pi}{2l}\eta\right) \\ & \left[ \left( \cosh\left(\frac{\pi}{2l}\eta\right) - \cos\left(\frac{\pi\epsilon}{2l}(g(x) - g(x - \eta))\right) \right)^{-1} \right. \\ & \left. - \left( \cosh\left(\frac{\pi}{2l}\eta\right) + \cos\left(\frac{\pi\epsilon}{2l}(g(x) + g(x - \eta))\right) \right)^{-1} \right] d\eta. \end{aligned}$$

We study the limit when  $\epsilon \rightarrow 0$  of the right term and obtain

$$\partial_t g(x) = \frac{\rho^1 - \rho^2}{2\pi} \text{P.V.} \int_{\mathbb{R}} \frac{\eta}{|\eta|^2} \partial_x g(t, x - \eta) d\eta = \frac{\rho^1 - \rho^2}{2} \Lambda g.$$

Thus, we can recover the same linear operator than in the whole space. Moreover,  $\rho^1 > \rho^2$  case the problem is ill-posed.

## Theorem: Well-posedness

If the Rayleigh-Taylor condition is satisfied,  $\rho^2 - \rho^1 > 0$ , and the initial data  $f_0(x) = f(x, 0) \in H^k(\mathbb{R})$ ,  $k \geq 3$ , then there exists a unique solution of (7) with  $f \in C^1(0, T, H^k)$  and  $T = T(f_0)$ .

We prove this result using energy estimates. Define

$$E(t) = \|f\|_{H^3}^2(t) + \|d[f]\|_{L^\infty}(t),$$

where  $d[f] : \mathbb{R}^2 \times \mathbb{R}^+ \mapsto \mathbb{R}^+$  is defined as

$$d[f](x, \eta, t) = \frac{1}{\cosh(\eta) + \cos(f(x) + f(x - \eta))}.$$

## Theorem: Well-posedness

If the Rayleigh-Taylor condition is satisfied,  $\rho^2 - \rho^1 > 0$ , and the initial data  $f_0(x) = f(x, 0) \in H^k(\mathbb{R})$ ,  $k \geq 3$ , then there exists a unique solution of (7) with  $f \in C^1(0, T, H^k)$  and  $T = T(f_0)$ .

We prove this result using energy estimates. Define

$$E(t) = \|f\|_{H^3}^2(t) + \|d[f]\|_{L^\infty}(t),$$

where  $d[f] : \mathbb{R}^2 \times \mathbb{R}^+ \mapsto \mathbb{R}^+$  is defined as

$$d[f](x, \eta, t) = \frac{1}{\cosh(\eta) + \cos(f(x) + f(x - \eta))}.$$

## Theorem: Well-posedness

If the Rayleigh-Taylor condition is satisfied,  $\rho^2 - \rho^1 > 0$ , and the initial data  $f_0(x) = f(x, 0) \in H^k(\mathbb{R})$ ,  $k \geq 3$ , then there exists a unique solution of (7) with  $f \in C^1(0, T, H^k)$  and  $T = T(f_0)$ .

We prove this result using energy estimates. Define

$$E(t) = \|f\|_{H^3}^2(t) + \|d[f]\|_{L^\infty}(t),$$

where  $d[f] : \mathbb{R}^2 \times \mathbb{R}^+ \mapsto \mathbb{R}^+$  is defined as

$$d[f](x, \eta, t) = \frac{1}{\cosh(\eta) + \cos(f(x) + f(x - \eta))}.$$

Step 1.  $L^2$  estimate. Note

$$\frac{1}{2} \frac{d}{dt} \|f\|_{L^2}^2 = \int_{\mathbb{R}} f(x) \sum_{n=1}^{\infty} \int_{\mathbb{R}} \frac{\eta}{|\eta|} e^{-n|\eta|} \sin(\ell_n(f(x))) \sin(\ell_n(f(x-\eta))) \cdot (\partial_x f(x) - \partial_x f(x-\eta)) d\eta dx.$$

We use the following identity

$$\partial_x f(x) - \partial_x f(x-\eta) = \int_0^1 \partial_x^2 f(x + (s-1)\eta) \eta ds. \quad (9)$$

Thus, we conclude

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|f\|_{L^2}^2 &\leq \int_0^1 \int_{\mathbb{R}} \sum_{n=1}^{\infty} \int_{\mathbb{R}} |f(x)| |\partial_x^2 f(x + (s-1)\eta)| |\eta| e^{-n|\eta|} d\eta dx ds \\ &\leq \|f\|_{L^2} \|\partial_x^2 f\|_{L^2} \sum_{n=1}^{\infty} 2 \int_0^{\infty} \eta e^{-n\eta} d\eta = \frac{\pi^2}{3} \|f\|_{L^2} \|\partial_x^2 f\|_{L^2}. \end{aligned}$$

**Step 2.**  $H^3$  estimate. First, we identify the problems

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_x^3 f\|_{L^2}^2 &= \int_{\mathbb{R}} \partial_x^3 f(x) \text{P.V.} \int_{\mathbb{R}} \partial_x (\partial_x^3 f(x) - \partial_x^3 f(x-\eta)) \Xi(x, \eta) d\eta dx \\ &+ 3 \int_{\mathbb{R}} \partial_x^3 f(x) \text{P.V.} \int_{\mathbb{R}} (\partial_x^3 f(x) - \partial_x^3 f(x-\eta)) \partial_x \Xi(x, \eta) d\eta dx \\ &+ 3 \int_{\mathbb{R}} \partial_x^3 f(x) \text{P.V.} \int_{\mathbb{R}} (\partial_x^2 f(x) - \partial_x^2 f(x-\eta)) \partial_x^2 \Xi(x, \eta) d\eta dx \\ &+ \int_{\mathbb{R}} \partial_x^3 f(x) \text{P.V.} \int_{\mathbb{R}} (\partial_x f(x) - \partial_x f(x-\eta)) \partial_x^3 \Xi(x, \eta) d\eta dx. \end{aligned}$$

We note that  $\Xi = \Xi_1 - \Xi_2$ ,  $\Xi_1$  where has order  $0(\eta^{-1})$ . Then, the derivatives with respect to  $x$  have the same order.

$$\begin{aligned}
& \int_{\mathbb{R}} \partial_x^3 f(x) \text{P.V.} \int_{\mathbb{R}} \partial_x (\partial_x^3 f(x) - \partial_x^3 f(x - \eta)) \Xi(x, \eta) d\eta dx \\
&= -\frac{1}{2} \int_{\mathbb{R}} (\partial_x^3 f(x))^2 \text{P.V.} \int_{\mathbb{R}} \partial_x \Xi(x, \eta) d\eta dx \\
&+ \int_{\mathbb{R}} \partial_x^3 f(x) \text{P.V.} \int_{\mathbb{R}} \partial_x^3 f(x) - \partial_x^3 f(x - \eta) \partial_\eta \Xi(x, \eta) d\eta dx
\end{aligned}$$

The first integral has order  $0(\eta^{-1})$  but the second has order  $0(\eta^{-2})$ . Note that the integrals with order  $0(\eta^{-1})$  can be integrated in P.V. but the order  $0(\eta^{-2})$  are NOT integrated.

What does it happen?

$$\begin{aligned} & \int_{\mathbb{R}} (\partial_x^3 f(x))^2 \text{P.V.} \int_{\mathbb{R}} \partial_x \Xi(x, \eta) d\eta dx \\ & \leq c(l) \|\partial_x^3 f\|_{L^2}^2 (\|f\|_{H^3}^8 + 1 + \|f\|_{H^3} \|d[f]\|_{L^\infty}^2) \\ & + \int_{\mathbb{R}} |\partial_x^3 f(x)|^2 \frac{\partial_x f(x) \partial_x^2 f(x)}{(1 + (\partial_x f(x))^2)^2} \text{P.V.} \int_{B(0,1)} \frac{\sinh(\eta) \eta^2 d\eta}{4 \sinh^4(\frac{\eta}{2})} dx. \end{aligned}$$

In more singular case,

$$\begin{aligned} & \int_{\mathbb{R}} \partial_x^3 f(x) \text{P.V.} \int_{\mathbb{R}} \partial_x^3 f(x) - \partial_x^3 f(x - \eta) \partial_\eta \Xi(x, \eta) d\eta dx \\ & \leq c(l) \|\partial_x^3 f\|_{L^2}^2 (\|f\|_{H^3}^8 + 1 + \|f\|_{H^3} \|d[f]\|_{L^\infty}^2) \\ & - \frac{1}{4} \int_{\mathbb{R}} \text{P.V.} \int_{\mathbb{R}} \frac{(\partial_x^3 f(x) - \partial_x^3 f(\eta))^2 ((x - \eta)^2 + (f(x) - f(\eta))^2)}{(\cosh(x - \eta) - \cos(f(x) - f(\eta)))^2} d\eta dx \end{aligned}$$

Thus, we conclude that

$$\frac{d}{dt} \|\partial_x^3 f\|_{L^2}^2 \leq C(l)(E(t)^5 + 1).$$

In more singular case,

$$\begin{aligned} & \int_{\mathbb{R}} \partial_x^3 f(x) \text{P.V.} \int_{\mathbb{R}} \partial_x^3 f(x) - \partial_x^3 f(x - \eta) \partial_\eta \Xi(x, \eta) d\eta dx \\ & \leq c(l) \|\partial_x^3 f\|_{L^2}^2 (\|f\|_{H^3}^8 + 1 + \|f\|_{H^3} \|d[f]\|_{L^\infty}^2) \\ & - \frac{1}{4} \int_{\mathbb{R}} \text{P.V.} \int_{\mathbb{R}} \frac{(\partial_x^3 f(x) - \partial_x^3 f(\eta))^2 ((x - \eta)^2 + (f(x) - f(\eta))^2)}{(\cosh(x - \eta) - \cos(f(x) - f(\eta)))^2} d\eta dx \end{aligned}$$

Thus, we conclude that

$$\frac{d}{dt} \|\partial_x^3 f\|_{L^2}^2 \leq C(l)(E(t)^5 + 1).$$

**Step 3.** Estimate of  $\|d[f]\|_{L^\infty}$ . We have that

$$\begin{aligned}\frac{d}{dt}d[f] &= d[f]^2 \sin(f(x) + f(x - \eta))(\partial_t f(x) + \partial_t f(x - \eta)) \\ &\leq c(l)d[f]^2 \|\partial_t f\|_{L^\infty} \leq c(l)d[f]^2 \|f\|_{H^3}.\end{aligned}$$

Due to the previous expressions we obtain

$$\frac{d}{dt}d[f] \leq c(l)d[f]\|d[f]\|_{L^\infty}\|f\|_{H^3},$$

and integrating in time we conclude that

$$d[f](t+h) \leq d[f](t)e^{\int_t^{t+h} c(l)\|d[f]\|_{L^\infty}\|f\|_{H^3}(s)ds}.$$

Finally we have that

$$\frac{d}{dt}\|d[f]\|_{L^\infty} = \lim_{h \rightarrow 0} \frac{\|d[f]\|_{L^\infty}(t+h) - \|d[f]\|_{L^\infty}(t)}{h} \leq c(l)(E+1)^3.$$

**Step 4.** Due to the previous estimates, we obtain the following bound

$$\frac{d}{dt}E \leq c(I)(E + 1)^5.$$

With this 'a priori' bounds we can obtain using the energy methods the local existence of classical solutions.

In order to prove the uniqueness we suppose that we have two solutions in  $H^3$  with the same initial data,  $f_1$  and  $f_2$ . Then, we obtain the following expression for the  $L^2$  norm of  $f = f_1 - f_2$ :

$$\frac{d}{dt} \|f\|_{L^2}^2 \leq c(\|f_1\|_{H^3}, \|d[f_1]\|_{L^\infty}, \|f_2\|_{H^3}, \|d[f_2]\|_{L^\infty}) \|f\|_{L^2}^2,$$

and applying Gronwall's Inequality the uniqueness holds.

**Step 4.** Due to the previous estimates, we obtain the following bound

$$\frac{d}{dt}E \leq c(l)(E + 1)^5.$$

With this 'a priori' bounds we can obtain using the energy methods the local existence of classical solutions.

In order to prove the uniqueness we suppose that we have two solutions in  $H_l^3$  with the same initial data,  $f_1$  and  $f_2$ . Then, we obtain the following expression for the  $L^2$  norm of  $f = f_1 - f_2$ :

$$\frac{d}{dt} \|f\|_{L^2}^2 \leq c(\|f_1\|_{H^3}, \|d[f_1]\|_{L^\infty}, \|f_2\|_{H^3}, \|d[f_2]\|_{L^\infty}) \|f\|_{L^2}^2,$$

and applying Gronwall's Inequality the uniqueness holds.

# Maximum Principle

## Theorem: Maximum principle

Let  $f$  be the unique classical solution of (7) under the same assumptions of the Well-Posedness Theorem. Then,  $f$  satisfies that

$$\|f\|_{L^\infty}(t) \leq \|f_0\|_{L^\infty}.$$

Due to the smoothness of  $f$ , we have that

$M(t) = \max_x f(x) = f(x_t)$  is Lipschitz. Then, using Rademacher Theorem we have that  $M$  is differentiable almost everywhere.

Thus,

$$\begin{aligned} M'(t) &= \text{P.V.} \int_{\mathbb{R}} (\partial_x f(x_t) - \partial_x f(x_t - \eta)) \Xi(x_t, \eta, f) d\eta \\ &= \text{P.V.} \int_{\mathbb{R}} \partial_\eta f(x_t - \eta) (\Xi_1(x_t, \eta, f) - \Xi_2(x_t, \eta, f)) d\eta = h_1 + h_2. \end{aligned}$$

# Maximum Principle

## Theorem: Maximum principle

Let  $f$  be the unique classical solution of (7) under the same assumptions of the Well-Posedness Theorem. Then,  $f$  satisfies that

$$\|f\|_{L^\infty}(t) \leq \|f_0\|_{L^\infty}.$$

Due to the smoothness of  $f$ , we have that

$M(t) = \max_x f(x) = f(x_t)$  is Lipschitz. Then, using Rademacher Theorem we have that  $M$  is differentiable almost everywhere.

Thus,

$$\begin{aligned} M'(t) &= \text{P.V.} \int_{\mathbb{R}} (\partial_x f(x_t) - \partial_x f(x_t - \eta)) \Xi(x_t, \eta, f) d\eta \\ &= \text{P.V.} \int_{\mathbb{R}} \partial_\eta f(x_t - \eta) (\Xi_1(x_t, \eta, f) - \Xi_2(x_t, \eta, f)) d\eta = h_1 + h_2. \end{aligned}$$

# Maximum Principle

## Theorem: Maximum principle

Let  $f$  be the unique classical solution of (7) under the same assumptions of the Well-Posedness Theorem. Then,  $f$  satisfies that

$$\|f\|_{L^\infty}(t) \leq \|f_0\|_{L^\infty}.$$

Due to the smoothness of  $f$ , we have that

$M(t) = \max_x f(x) = f(x_t)$  is Lipschitz. Then, using Rademacher Theorem we have that  $M$  is differentiable almost everywhere.

Thus,

$$\begin{aligned} M'(t) &= \text{P.V.} \int_{\mathbb{R}} (\partial_x f(x_t) - \partial_x f(x_t - \eta)) \Xi(x_t, \eta, f) d\eta \\ &= \text{P.V.} \int_{\mathbb{R}} \partial_\eta f(x_t - \eta) (\Xi_1(x_t, \eta, f) - \Xi_2(x_t, \eta, f)) d\eta = I_1 + I_2. \end{aligned}$$

Let us introduce the following notation:

$$\theta = \frac{f(x_t) - f(x_t - \eta)}{2}, \quad \bar{\theta} = \frac{f(x_t) + f(x_t - \eta)}{2}.$$

Thus,  $\Xi_1(x_t, \eta, f) = \Xi_1(\eta, \theta)$  and  $\Xi_2(x_t, \eta, f) = \Xi_2(\eta, \bar{\theta})$ .

We calculate to obtain

$$M'(t) = -4M(t) + \text{P.V.} \int_{\mathbb{R}} \frac{\cot(\bar{\theta})}{\cosh^2(\eta/2) \tanh^2(\eta/2) + \cot^2(\bar{\theta})} \frac{1}{\cosh^2(\eta/2) \tanh^2(\eta/2) + \cot^2(\bar{\theta})} d\eta \\ - \text{P.V.} \int_{\mathbb{R}} \frac{\tan(\theta)}{\cosh^2(\eta/2) \tanh^2(\eta/2) + \tan^2(\theta)} \frac{1}{\cosh^2(\eta/2) \tanh^2(\eta/2) + \tan^2(\theta)} d\eta. \quad (10)$$

Let us introduce the following notation:

$$\theta = \frac{f(x_t) - f(x_t - \eta)}{2}, \quad \bar{\theta} = \frac{f(x_t) + f(x_t - \eta)}{2}.$$

Thus,  $\Xi_1(x_t, \eta, f) = \Xi_1(\eta, \theta)$  and  $\Xi_2(x_t, \eta, f) = \Xi_2(\eta, \bar{\theta})$ .

We calculate to obtain

$$\begin{aligned} M'(t) = & -4M(t) + \text{P.V.} \int_{\mathbb{R}} \frac{\cot(\bar{\theta})}{\cosh^2(\eta/2) \tanh^2(\eta/2) + \cot^2(\bar{\theta})} \frac{1}{\cosh^2(\eta/2) \tanh^2(\eta/2) + \cot^2(\bar{\theta})} d\eta \\ & - \text{P.V.} \int_{\mathbb{R}} \frac{\tan(\theta)}{\cosh^2(\eta/2) \tanh^2(\eta/2) + \tan^2(\theta)} \frac{1}{\cosh^2(\eta/2) \tanh^2(\eta/2) + \tan^2(\theta)} d\eta. \quad (10) \end{aligned}$$

Due to the definition of  $\bar{\theta}$

$$\cot(\bar{\theta}) = \tan\left(\frac{\pi}{2} - \bar{\theta}\right) = \tan\left(\frac{\pi}{2} - f(x_t) + \theta\right),$$

and we use the equality

$$\tan\left(\frac{\pi}{2} - f(x_t) + \theta\right) = \frac{\tan\left(\frac{\pi}{2} - f(x_t)\right) + \tan(\theta)}{1 - \tan\left(\frac{\pi}{2} - f(x_t)\right) \tan(\theta)}.$$

Moreover, we can write that

$$4M(t) = \int_{\mathbb{R}} \frac{1}{\cosh^2\left(\frac{\eta}{2}\right)} \frac{\tan\left(\frac{\pi}{2} - f(x_t)\right)}{\tan^2\left(\frac{\pi}{2} - f(x_t)\right) + \tanh^2\left(\frac{\eta}{2}\right)} d\eta.$$

By notational convenience we use the notation  $\sigma = \frac{\pi}{2} - f(x_t)$ . So, from (10), we prove the maximum principle because

$$\begin{aligned} & \frac{(\tan(\sigma) + \tan(\theta))(1 - \tan(\sigma) \tan(\theta))}{(\tan(\sigma) + \tan(\theta))^2 + (1 - \tan(\sigma) \tan(\theta))^2 \tanh^2\left(\frac{\eta}{2}\right)} \\ & - \frac{\tan(\sigma)}{\tan^2(\sigma) + \tanh^2\left(\frac{\eta}{2}\right)} - \frac{\tan(\theta)}{\tan^2(\theta) + \tanh^2\left(\frac{\eta}{2}\right)} \leq 0. \end{aligned}$$

# References

This work is with strong collaboration of Rafael Granero and, also, D. Córdoba and F. Gancedo: *On the Muskat problem on a strip*, preprint.

P. Constantin, D. Córdoba, F. Gancedo, R. M. Strain, *On the global existence for the Muskat problem*. arXiv:1007.3744, (2010)

D. Córdoba, F. Gancedo, *Contour dynamics of incompressible 3-D fluids in a porous medium with different densities*. Comm. Math. Phys. 273 (2007), no. 2, 445–471.

D. Córdoba, F. Gancedo, *A maximum principle for the Muskat problem for fluids with different densities*. Comm. Math. Phys. 286 (2009), no. 2, 681–696.

D. Córdoba, F. Gancedo, R. Orive, *A note on the interface dynamics for convection in porous media*. Physica D **237** (10-12), (2008), pp. 1488–1497.

J. Escher, B-V. Matioc, *On the parabolicity of the Muskat problem: Well-posedness, fingering, and stability results*, arXiv:1005.2512v1, (2010).