Oldroyd Model of Viscoelastic Fluids: Some Theoretical and Computational Issues

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Sketch of the Talk

- Problem Description
- Weak Formulation
  - Comments on Existence, Uniqueness and Regularity
  - Nonlocal Compatibility Conditions (behaviour at $t = 0$)
- Finite Element Method
  - Finite element spaces
  - Linearized problem and its role
  - Role of Stokes-Volterra projection
  - Error Analysis
- Concluding Remarks and Extensions
  - Summary
  - Completely Discrete Schemes
  - Two grid methods

Problem Description

The motion of an incompressible fluid in a bounded domain Ω in $\Re^2$:

$$\begin{aligned} \frac{\partial u}{\partial t} + u \cdot \nabla u - \nabla \sigma + \nabla p &= F(x,t),\ x \in \Omega,\ t > 0, \\ \nabla \cdot u &= 0,\ x \in \Omega,\ t > 0, \end{aligned}$$

with appropriate initial and boundary conditions.

- $\sigma = (\sigma_{ik})$: the stress tensor with $tr\sigma = 0$,
- u represents the velocity vector,
- p is the pressure of the fluid
- F is the external force.

Equation of State or Rheological Relation:
The defining relation between σ and the tensor of deformation velocities

$$D = (D_{ik}) = \frac{1}{2}\left(u_{ix_k} + u_{kx_i}\right).$$

Its Role: In fact establishes the type of fluids under consideration.

Examples:

- $\sigma = 0$ corresponds to Euler Equation.
- $\sigma = 2\nu D$ (using Newton's law): Navier-Stokes Equation (with ν: the kinematic coefficient of viscosity)

  This has been a basic model for describing flow at moderate velocities of majority of viscous incompressible fluids encountered in practice.

Models of Viscoelastic Fluids: In the mid-twentieth century, Models have been proposed that take into consideration the prehistory of the flow and are not subject to the Newtonian flow.
One such model was proposed by J. G. Oldroyd [1950].

The defining relation for the Oldroyd Model:

$$\left(1 + \lambda \frac{\partial}{\partial t}\right) \sigma = 2\nu \left(1 + \kappa \nu^{-1} \frac{\partial}{\partial t}\right) D,$$

where λ, ν, k are positive constants with $(\nu - \kappa\lambda^{-1}) \geq 0$.

- λ: relaxation time
- ν: kinematic coefficient of viscosity
- κ: retardation time.

The equation of motion gives rise to the following integro-differential equation:

$$\frac{\partial u}{\partial t} + u \cdot \nabla u - \mu \Delta u - \int_0^t \beta(t-\tau) \Delta u(x,\tau)\, d\tau$$
$$+\nabla p = f(x,t),\ x \in \Omega,\ t > 0, \tag{1}$$

and Incompressibility condition:

$$\nabla \cdot u = 0,\ x \in \Omega,\ t > 0,$$

with Initial and Boundary Conditions

$$u(x,0) = u_0, u = 0,\ \text{on } \partial\Omega, t \geq 0.$$

- boudary $\partial\Omega$,
- $\mu = 2k\lambda^{-1} > 0$
- the kernel $\beta(t) = \gamma \exp(-\delta t)$, where $\gamma = 2\lambda^{-1}(\nu - k\lambda^{-1}) \geq 0$, and $\delta = \lambda^{-1} > 0$.

For details of physical background: D. D. Joseph [1990], Oldroyd [1956], Oskolkov [1989].

Objective:

The Problem (1) is an integral perturbation of the Navier Stokes equations, we would like to investigate:

"How far the results on finite element analysis for the Navier-Stokes equations can be carried over to the present case".

Spaces:

$$\mathbf{H}_0^1 = [H_0^1]^2, \quad \mathbf{L}^2 = [L^2]^2.$$

Innerproduct on $\mathbf{H}_0^1$

$$(\nabla \phi, \nabla w) = \sum_{i=1}^{2} (\nabla \phi_i, \nabla w_i)$$

and Induced Norm

$$\|\nabla \phi\| = \left(\sum_{i=1}^{2} \|\nabla \phi_i\|^2 \right)^{1/2}.$$

$$\begin{aligned} \mathbf{J}_1 &= \{\phi \in \mathbf{H}_0^1 : \nabla \cdot \phi = 0\} \\ \mathbf{J} &= \{\phi \in \mathbf{L}^2 : \nabla \cdot \phi = 0 \text{ in } \Omega, \phi \cdot n|_{\partial\Omega} = 0\}. \end{aligned}$$

Here, n is the outward normal to $\partial\Omega$.

$P : ((L^2(\Omega))^2) \longrightarrow J$ denotes the **orthogonal projection**.

The **orthogonal complement** $J^{\perp}$ of J in $L^2(\Omega)$ consists of functions ϕ such that $\phi = \nabla p$ for some $p \in H^1(\Omega)/R$.

Weak Formulation:

Find a pair of functions $\{u(t), p(t)\}$ such that

$$\begin{aligned} (u_t, \phi) + \mu(\nabla u, \nabla \phi) &+ (u \cdot \nabla u, \phi) + \int_0^t \beta(t-s)(\nabla u(s), \nabla \phi)\, ds \\ &= (p, \nabla \cdot \phi) + (f, \phi), \quad \forall\, \phi \in H_0^1 \\ (\nabla \cdot u, \chi) &= 0, \qquad\qquad\qquad\qquad \forall\, \chi \in L^2. \end{aligned} \tag{2}$$

Equivalently, find $u(\cdot, t) \in J_1$ such that

$$\begin{aligned} (u_t, \phi) + \mu(\nabla u, \nabla \phi) + (u \cdot \nabla u, \phi) &+ \int_0^t \beta(t-s)(\nabla u(s), \nabla \phi)\, ds \\ &= (f, \phi) \quad \forall\, \phi \in J_1,\ t > 0. \end{aligned} \tag{3}$$

EXISTENCE (Faedo - Galerkin Method)

Denote by $-\tilde{\Delta} = -P\Delta$, the Stokes operator which is selfadjoint, positive definite, closed linear operator on J_1 with domain $H^2 \cap J_1$. It has compact Inverse.

Let $\{\lambda_k\}$ be the sequence of eigenvalues with $0 < \lambda_1 \leq \lambda_2 \leq \cdots, \leq \lambda_k \leq \cdots \lambda_k \to \infty$ as $k \to \infty$, and let $\{\phi^k\}$ be the corresponding eigenvectors of the Stokes operator $-\tilde{\Delta}$, i.e., $-\tilde{\Delta}\phi^k = \lambda_k \phi^k$.

- $\{\phi^k\}$ forms an orthogonal set in J, J_1 and $H^2 \cap J_1$.
- $V_k = \text{span}\{\phi^1, \cdots, \phi^k\}$.

For unique solvability:

Apply Faedo - Galerkin procedure by forcing $u^k(t) \in V_k$ for $t \geq 0$ to satisfy

$$\begin{aligned} (u_t^k, \phi) + (u^k \cdot \nabla u^k, \phi) + \mu(\nabla u^k, \nabla \phi) &+ \int_0^t \beta(t-\tau)(\nabla u^k(\tau), \nabla \phi)\, d\tau \\ &= (f, \phi) \quad \forall\, \phi \in V_k,\ \text{a.}e.\ t > 0, \\ u^k(0) &= P_k u_0. \end{aligned} \tag{4}$$

V_k: finite dimensional $\Longrightarrow$ (4) yields a system of k **nonlinear integro - differential equations**.

- By Picard's Theorem: There exists a unique solution u^k in some neighbourhood $[0, t^*)$. (local existence and uniqueness result).
- For existence of the global solution u^k in $(0, \infty)$, we apply the **continuation argument**, provided $\|u^k(t)\|$ is bounded for $t \in (0, \infty)$.

***A Priori* Bounds + Compactness Arguments $\Longrightarrow$ Existence.**

Existence for finite time $[0, T]$ - Oskolkov [1976], [1989] for $d = 2$, but for $d = 3$ with small initial velocity as well as small external force. (Following Ladyzhenskaya' Analysis)

Regularity results need Compatibility Conditions on data at $t = 0$.

However, these **(Nonlocal) Compatibility conditions impose severe restrictions on the data** u_0.
(Almost Uncheckable in Practice).

For Example:

If One of $\|\nabla u_t(t)\|$, $\|\Delta u(t)\|_1$, $\int_0^t \|u_t\|_2^2 \, ds$ or $\int_0^t \|u_{tt}\|^2 \, ds$ remains bounded as $t \to 0$, then there must exist a solution $p(0)$ of

$$\Delta p(0) = -\nabla \cdot (f(0) - u_0 \cdot \nabla u_0) \text{ in } \Omega,$$
$$\nabla p(0) \mid_{\partial\Omega} = (\Delta u_0 + f(0) - u_0 \cdot \nabla u_0) \mid_{\partial\Omega} .$$

Our Emphasis:

- **To Derive regularity results under realistically assumed conditions on the Data** (Behavior at $t \to 0$).
- **To Show Uniform Bounds for all time** t.

Assume:

(A). The initial velocity u_0 and forcing function f satisfy for some constants M_1, M_2

- $u_0 \in J_1$ with $\|u_0\|_1 \leq M_1$, and $\|f\|_{L^\infty(L^2)}, \ \|f_t\|_{L^\infty(L^2)} \leq M_1$.
- $u_0 \in H^2 \cap J_1$ with $\|u_0\|_2 \leq M_2$.

Lemma 1 (Positive Property of the Kernel.) For arbitrary $t^* > 0$ and $\phi \in L^2(0, t^*)$, the following property holds

$$\int_0^{t^*} \left(\int_0^t \beta(t-s)\phi(s)\, ds \right) \phi(t)\, dt \geq 0.$$

Lemma 2 Let $0 \leq \alpha < \min(\delta, \lambda_1 \mu)$. Under the assumptions (A), there is a positive generic constant $K = K(\alpha, \mu, \lambda_1, M_1)$ such that for all $t > 0$

$$\|u(t)\|^2 + (\mu - \frac{\alpha}{\lambda_1}) e^{-2\alpha t} \int_0^t e^{2\alpha\tau} \|\nabla u(\tau)\|^2 \, d\tau \leq K$$

and

$$\|\nabla u(t)\|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha\tau} \|\tilde{\Delta} u(\tau)\|^2 \, d\tau \leq K.$$

Sketch of the proof: Using the notation

$$u_\beta(t) = \int_0^t \beta(t-s)u(s)\,ds,$$

we rewrite (1), in differential form, as

$$u_t - \mu\tilde{\Delta}u + u\cdot\nabla u - \Delta u_\beta = f, \quad t>0, \tag{5}$$

where $\tilde{\Delta}$ is the Stokes operator.

- Form a duality pairing between the above equation and $-\tilde{\Delta}u$.
- Use Sobolev Inequality $\|u\|_{L^4(\Omega)} \le 2^{1/4}\|u\|^{1/2}\|\nabla u\|^{1/2}$.
- As $\|\nabla u\|^2 = (u, -\tilde{\Delta}u) \le \|u\|\|\tilde{\Delta}u\|$, for some constant $\beta_0 > 0$,

$$\beta_0\|\nabla u\|^2 \le \frac{\mu}{3}\|\tilde{\Delta}u\|^2 + \frac{3}{4\mu}\beta_0^2\|u\|^2.$$

Altogether (5) gives

$$\frac{d}{dt}\left(\|\nabla u\|^2 + \frac{1}{\gamma}\|\tilde{\Delta} u_\beta\|^2\right) \; + \; (\beta_0 + \mu\lambda_1 - \frac{C}{\mu^3}\|u\|^2\|\nabla u\|^2)\|\nabla u\|^2 + \frac{2\delta}{\gamma}\|\tilde{\Delta} u_\beta\|^2 \leq \; \frac{3}{\mu}\|f\|^2 + \frac{3}{4\mu}\beta_0^2\|u\|^2 \leq K(\beta_0). \tag{6}$$

- Define $g(t) := \min\{\beta_0 + \mu\lambda_1 - \frac{C}{\mu^3}\|u\|^2\|\nabla u\|^2, 2\delta\}$.
- Set $E(t) = \|\nabla u\|^2 + \frac{1}{\gamma}\|\tilde{\Delta} u_\beta\|^2$.

From (6)

$$E(t) \leq e^{-\int_0^t g(\tau)\,d\tau}\|\nabla u_0\|^2 + K\int_0^t e^{-\int_s^t g(\tau)\,d\tau}\,ds. \tag{7}$$

- Use $\alpha T_0 \leq \int_t^{t+T_0} g(s)\,ds$ for $T_0 > 0$ and $-g(t) > -2\delta \;\; \forall t > 0$.

Remark. As has been claimed in papers like Oskolkov et al.[a], Lin et al.[b], it is not possible to derive uniform estimate for Dirichlet norm, just by following the arguments in Navier-Stokes equations, as we can not apply the Uniform Gronwall inequality.

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[a] On dynamical systems generated by initial-boundary value problems for the equations of motion of linear viscoelastic fluids - N.A. Karzeeva, A.A. Kotsiolis and A.P. Oskolkov, Proc. Steklov Inst. Math., 1991.

[b] Finite element approximation for the viscoelastic fluid motion prbloem - Y. He, Y. Lin, S. Shen, W Sun and R.Tait, J. Comp. Appl. Math., 2003.

Theorem 1. There is a constant $K = K(M_1, M_2)$ such that for $0 < \alpha < \min(\delta, \lambda_1\mu)$ the following estimates hold for all time $t > 0$:

$$\|u(t)\|_2^2 + \|u_t(t)\|^2 + \|p(t)\|_{H^1/R}^2 \leq K$$
$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\|u_t\|_1^2\, ds \leq K,$$

and

$$\tau^*(t)\|u_t\|_1^2 \leq K,$$

where, $\tau^*(t) = \min(t, 1)$.
Moreover, for all time $t > 0$

$$e^{-2\alpha t}\int_0^t \sigma(s)\left(\|u_t\|_2^2 + \|u_{tt}\|^2 + \|p_t\|_{H^1/R}^2\right)\, ds \leq K,$$

where, $\sigma(t) = \tau^*(t)e^{2\alpha t}$.

Hints: Use $e^{\alpha t}u_t$, $-\sigma(t)\tilde{\Delta}u_t$ and special care must be taken to avoid using Gronwall Lemma.

Following the analysis of Ladyzhenskaya (for NS equations), and the above *a priori* bounds, it is possible to prove existence of global strong solutions for all $t > 0$, when $d = 2$ (and for $d = 3$ with small initial velocity u_0 and small forcing function f).

Finite Element Method

Galerkin Method:

Introduce two finite dimensional spaces H_h and L_h (h positive parameter tending to zero) of H_0^1 and L^2, respectively, approximating velocity vector and the pressure satisfying the following approximation properties:

Properties (B)

- For each $v \in H_0^1 \cap H^2$ and $q \in H^1/R$ there exist approximations $i_h v \in H_h$ and $j_h q \in L_h$ such that the standard consistency conditions holds

$$\begin{aligned} \|v - i_h v\| + h\|\nabla(v - i_h v)\| &\leq K_0 h^2 \|v\|_2, \\ \|q - j_h q\|_{L^2/R} &\leq K_0 h \|q\|_{H^1/R} \end{aligned}$$

- The spaces H_h and L_h should satisfy the usual stability condition (LBB - condition)

$$|(q_h, \nabla \cdot \phi_h)| \geq K_0 \|\nabla \phi_h\| \, \|q_h\|_{L^2/N_h}, \quad K_0 > 0.$$

  The above condition is essential for computation of the pressure.
  Here,

$$N_h = \{q_h \in L_h : (q_h, \nabla_h \cdot \phi_h) = 0, \ \forall \ \phi_h \in H_h.\}$$

- **To deal with the nonlinearity, we assume the following inverse hypothesis** for $v_h \in H_h$ (needs quasi-uniformity condition)

$$\|\nabla v_h\| \leq K_0 h^{-1} \|v_h\|.$$

Example:

- Ω be a convex polygon in $\Re^2$.
- Let $\{\mathcal{T}_h\}$ be a family of finite decomposition of the domain Ω into 2- simplexes K with diameter h_K. Let $h = \max_{K\in\mathcal{T}_h} h_K$.
- Assume that this family of triangulations is regular and it satisfies the quasi-uniformity condition, see Ciarlet [1978].
- Let $P_r(K)$ denote the space of all polynomials of degree less than or equal to r.

The finite dimensional spaces (see, Girault and Raviart [1980])

$$\begin{aligned} H_h &= \{v_h \in \left(C^0(\Omega)\right)^2 \cap H_0^1 : v_h|_k \in (P_2(K))^2 \ \forall\ K \in \mathcal{T}_h\} \\ L_h &= \{q_h \in L^2(\Omega) : q_h|_K \in P_0(K) \ \forall\ K \in \mathcal{T}_h\} \end{aligned}$$

satisfy the properties B.
For defining the Galerkin approximations set for $v, w, \phi \in H_0^1$,

$$a(v,\phi) = (\nabla v, \nabla \phi)$$

and

$$b(v,w,\phi) = \frac{1}{2}(v\cdot\nabla w, \phi) - \frac{1}{2}(v\cdot\nabla\phi, w).$$

Note that the operator $b(\cdot,\cdot,\cdot)$ preserves the antisymmetric properties of the original nonlinear term that is

$$b(v_h, w_h, w_h) = 0,\ \forall\, v_h, w_h \in H_h.$$

The **discrete analogue of the weak formulation** (2) now read as: Find $u_h(t) \in H_h$ and $p_h(t) \in L_h$ such that $u_h(0) = u_{0,h}$ and for $t > 0$

$$\begin{aligned}(u_{ht}, \phi_h) + \mu a(u_h, \phi_h) &+ b(u_h, u_h, \phi_h) = (p_h, \nabla \cdot \phi_h) \\ &- \int_0^t \beta(t-s) a(u_h(s), \phi_h)\, ds \quad \forall\, \phi_h \in H_h, \\ (\nabla \cdot u_h, \chi_h) &= 0 \quad \forall\, \chi_h \in L_h\end{aligned}$$

where $u_{0,h} \in H_h$ is a suitable approximation of $u_0 \in J_1$.

Discrete space analogous to J_1: (Impose the discrete incompressibility condition on H_h)

$$J_h = \{v_h \in H_h : (\chi_h, \nabla \cdot v_h) = 0,\ \forall \chi_h \in L_h.\}$$

Note that the space J_h is not a subspace of J_1. **This noncofirmity will show up in the error analysis.**

Galerkin approximation $u_h(t)$:

Find $u_h(t) \in J_h$ such that $u_h(0) = u_{0h}$ and for $t > 0$

$$\begin{aligned}(u_{ht}, \phi_h) \; &+ \; \mu a(u_h, \phi_h) + \int_0^t \beta(t-s) a(u_h(s), \phi_h)\, ds \\ &= \; -b(u_h, u_h, \phi_h) + (f, \phi_h) \quad \forall\, \phi_h \in J_h.\end{aligned}$$

Approximation $p_h(t) \in L_h$:

Can be found out by solving the following system

$$\begin{aligned}(p_h, \nabla \cdot \phi_h) \; &= \; (u_{ht}, \phi_h) + \mu a(u_h, \phi_h) + \int_0^t \beta(t-s) a(u_h(s), \phi_h)\, ds \\ &+ \; b(u_h, u_h, \phi_h) - (f, \phi_h) \quad\quad \forall\, \phi_h \in H_h.\end{aligned}$$

Error Analysis

Effect of J_h being not a subspace of J_1:

$$(u_t, \phi_h) + \mu a(u, \phi_h) + \int_0^t \beta(t-s) a(u(s), \phi_h)\, ds$$
$$= -b(u, u, \phi_h) + (f, \phi_h) + (p, \nabla \cdot \phi_h)\ \forall\ \phi \in J_h.$$

Direct comparision between exact solution u and the Galerkin approximation u_h does not, in general, yield optimal estimates.
So there is a need to look for **an appropriate auxiliary (intermediate) function.**

Split the error e as

$$e := u - u_h = (u - v_h) + (v_h - u_h) = \xi + \eta,$$

The intermediate solution v_h is a finite element approximation to a linearized Oldroyd model:

$$\begin{aligned}(v_{ht}, \phi_h) \ &+\ \mu a(v_h, \phi_h) + \int_0^t \beta(t-s) a(v_h(s), \phi_h) \\ &=\ -b(u, u, \phi_h) + (f, \phi_h) \quad \forall\ \phi_h \in J_h.\end{aligned}$$

- $\xi = (u - v_h)$: **the error committed by approximating a linearized Oldroyd model.**
- $\eta = (v_h - u_h)$: **the error due to nonlinearity of the equation** (Dissociate the effect of nonlinearity)

Equation in ξ:

$$(\xi_t, \phi_h) + \mu a(\xi, \phi_h) + \int_0^t \beta(t-s) a(\xi(s), \phi_h)\, ds$$
$$= (p, \nabla \cdot \phi_h),\ \phi \in J_h.$$

(More like a linear Parabolic Integro - Differential Equation).

- **Useful Estimate.**

$$\int_0^t e^{2\alpha s} \|\xi(s)\|^2\, ds \leq K \tau^*(t) h^4,$$

where $\tau^*(t) = \min(t, 1)$.

Use of duality argument for parabolic integro-differential equations, Pani and Sinha (2000).

Estimates of ξ in $L^\infty(L^2)$ and $L^\infty(H^1)$- norms:

Again introduce the following auxiliary projection $V_h : [0,\infty) \to J_h$

$$\mu a(u - V_h u, \phi_h) + \int_0^t \beta(t-s) a(u(s) - V_h u(s), \phi_h) \\ = (p, \nabla \cdot \phi_h), \; \forall \, \phi_h \in J_h$$

and call it as **Stokes - Volterra projection**.
Decompose the error ξ:

$$\xi = (u - V_h u) + (V_h u - v_h) = \zeta + \theta.$$

First, derive error bounds for ζ, then for θ in terms of ζ.

Lemma 3. Assume that the conditions (A) and (B) are satisfied. Then there is a constant C such that

$$\|(u - V_h u)(t)\|^2 + h^2 \|\nabla(u - V_h u)(t)\|^2 \leq Ch^4.$$

Moreover, the error in the time derivative satisfies

$$\|(u - V_h u)_t(t)\|^2 + h^2 \|\nabla(u - V_h u)_t(t)\|^2 \\ \leq Ch^4 \sum_{j=0}^{1} \left(\|\frac{\partial^j}{\partial t^j}(\tilde{\Delta} u)\|^2 + \|\frac{\partial^j}{\partial t^j} \nabla p\|^2 \right) \\ + Ch^4 \int_0^t \left(\|\tilde{\Delta} u\|^2 + \|\nabla p\|^2 \right) \, ds.$$

Equation in θ:

$$(\theta_t, \phi_h) + \mu a(\theta, \phi_h) + \int_0^t \beta(t-s) a(\theta(s), \phi_h)\, ds$$
$$= -(\zeta_t, \nabla \cdot \phi_h),\ \phi \in J_h,$$

Where $\zeta_t = (u - V_h u)_t$.

Estimates in θ:
Need to introduce

- $\sigma(t) = \tau^*(t) e^{2\alpha t}$ to get ride of **nonlocal compatibility** conditions.
- $\tilde{\theta}(t) := \int_0^t \theta(s)\, ds$ to avoid direct use of Gronwall's Lemma.

Estimates of ξ: (Altogether)

$$\|\xi(t)\| + h\|\nabla \xi(t)\| \leq K h^2.$$

Estimates of $\eta = v_h - u_h$ (the effect of nonlinearity):

$$\|\eta(t)\| + h\|\nabla \eta(t)\| \leq K h^2.$$

Hints: Error equation in η

$$(\eta_t, \phi_h) + \mu a(\eta, \phi_h) + \int_0^t \beta(t-s) a(\eta(s), \phi_h)\, ds$$
$$= b(u_h, u_h, \phi_h) - b(u, u, \phi_h).$$

Hint:

Use $\phi_h = e^{2\alpha t}\eta$, Sobolev inequality and approximation property.

$$\begin{aligned}\|\hat{\eta}\|^2 \;&\leq\; C\left(\|\eta(0)\|^2 + \int_0^t e^{2\alpha s}\|\xi(s)\|^2\,ds\right)\\ &+\; C\int_0^t \left(\|\nabla u\|\|\tilde{\Delta} u\|\right)\|\hat{\eta}(s)\|^2\,ds.\end{aligned}$$

A direct use of Gronwall's Lemma yields:

$$\|\hat{\eta}\|^2 \;\leq\; Kh^4 \exp\left(C\int_0^t \|\nabla u\|\|\tilde{\Delta} u\|\,ds\right) \leq Kh^4.$$

Error estimates for Velocity Vector e:

$$e = (u - u_h) = \xi + \eta$$

$$\|(u - u_h)(t)\| + h\|\nabla(u - u_h)(t)\| \leq Kh^2.$$

Estimates of $p(t) - p_h(t)$:

$$\|(p - p_h)(t)\| \leq \frac{K}{(\tau^*)^{1/2}}h.$$

To Summarise:

- **New a priori bounds and uniform bounds.**
- **For convergence:** first dissociate the **effect of nonlinearity:**

$$e = u - u_h = (u - v_h) + (v_h - u_h) = \xi + \eta,$$

Amiya K. Pani, Jin Yun Yuan, Semidiscrete finite element Galerkin approximations to the equations of motion arising in the Oldroyd model. IMA J. Numer. Anal. 25 (2005), no. 4, 750–782.

Estimate of ξ:

Using newly introduced **Stokes - Volterra Projection** $V_h u$, decompose the error ξ:

$$\xi = (u - V_h u) + (V_h u - v_h) = \zeta + \theta,$$

- **Convergence results are obtained with out nonlocal compatibility conditions (under realistically assumed conditions on u_0).**

Difficulties:

- Construction of finite element spaces satisfying LBB- condition.
- Imposition of discrete incompressibility condition.
  In order to avoid this, use penalty method or artificial compressibility condition.

Extensions

- Under the conditions,

$$\int_0^t e^{2\alpha t}\|f(\tau)\|^2\,d\tau \le M_1, \text{ and } \|\nabla u_0\|^2 \le M_1$$

  the following uniform boundedness property can be proved
  (a step towards the dynamics of this system)

$$\|\nabla u(t)\| \le C(\alpha,\mu,\delta,\lambda_1,M_1)\ \forall\ t>0.$$

- With $f, f_t \in L^\infty(L^2(\Omega))$, $u_0 \in H^2 \cap J_1$, we have

$$\|(u-u_h)(t)\| \le K(t)h^2,$$

  and

$$\|(p-p_h)(t)\| \le \frac{K(t)}{(\tau^*)^{1/2}}h.$$

  Estimates are similar to the results derived by Heywood and Rannacher (SIAM J. Numer. Anal. 1986).

- As in Hill and Suli (IMA J. Numer. Anal. 2000) for Navier-Stokes Equation, estimates are derived when $u_0 \in J_1$.

- Most of these results are valid under the assumption that the data are small when $d = 3$.
- Time discretization and its effect (using Semigroup theoretic approach, we have proved some results).
  Pani, Jin, Pedro : Backward Euler for the full system,
  SIAM J. Numer. Anal., 44 (2006), pp. 804-825.
- Two grid methods in combination with the nonlinear Galerkin scheme (Extension of some results by Temam and his group in the context of NSE).
  Cannon, Ewing *et al.* (Spectral Galerkin approximation for periodic problem) J. Engrg. Sci. (1999).
- Study of dynamics (existence of global attractor and its approximations) is in progress

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