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Two Phase Free Boundary Problems

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Free Boundary Problems

A free boundary problem arises in many areas of physics- free streamlines, jets, Hele-Shaw flows, electromagnetic shaping, gravitational problems etc.... A typical example is the *classical obstacle problem*, where given a function $\psi \in C^2(D)$ with $\psi \leq g$ in the sense that $(\psi - g)^+ \in W_0^{1,2}(D)$ one looks for the minimizer of the functional

$$J(u) = \int_D \left(|\nabla u|^2 + 2fu\right)\, dx \tag{1}$$

on the set

$$K_{g,\psi} = \{u \in W^{1,2}(D) : u - g \in W_0^{1,2}(D), u \geq \psi \text{ in } D\}.$$

J is continuous and strictly convex on the (convex) subset $K_{g,\psi}$ of the Hilbert space $W^{1,2}$ and hence has a unique minimizer in $K_{g,\psi}$.

Physically (or geometrically), one thinks of the graph of u as a membrane attached to a fixed wire which is forced to stay above the graph of ψ. The set $\Lambda := \{u = \psi\}$ is called the *coincidence set* and the boundary $\Gamma := \partial\Lambda \cap D$ is called the *free boundary*. It can be easily seen that u satisfies

$$-\Delta u + f = 0 \quad \text{in} \quad \Omega := \{u > \psi\}, \qquad \Delta u = \Delta\psi \quad \text{a.e on} \quad \Lambda$$

$$-\Delta u + f \geq 0 \quad \text{in} \quad D$$

in the sense of distributions, where $f > 0$ is Lipschitz.

Replacing u by $u - \psi$, we reduce to the obstacle problem with zero obstacle, which corresponds to minimizing the functional

$$J(u) = \int_D \left(|\nabla u|^2 + 2f.u\right)\, dx$$

over the convex subset $K_{g,0} = \{u \in W^{1,2}(D) : u - g \in W_0^{1,2}(D),\ u \geq 0 \ a.e. \text{ in } D\}$, and in this case the associated PDE is

$$\Delta u = f\chi_{\{u>0\}} \quad \text{in} \quad D.$$

Quadrature domains

"Quadrature" means (in latin) "division of land into squares". Mathematically, it refers to "constructive or numerical methods to find areas or more generally integrals". A "quadrature identity" is typically an exact formula for the integral of a harmonic function in terms of a simpler functional like point evaluations. The domain of integration is then called a **quadrature domain**. Classically, a quadrature identity can be written as

$$\int_{\Omega} f\, dA = \sum_{k=1}^{n} c_k f(a_k) \tag{2}$$

where $a_1, \ldots, a_n \in \Omega$, not necessarily distinct. For $\mathbf{n} = \mathbf{1}$, it is known that the discs are the only quadrature domains as in this case the quadrature identity reduces to the mean value property for harmonic functions

$$f(a) = \frac{1}{|B(a,r)|} \int_{B(a,r)} f\, dA.$$

One may thus think of quadrature domain as something obtained by gluing "discs" in potential theoretic way or as some kind of sweeping (balayage) process applied to the measure

$$\mu = \sum_{k=1}^{n} c_k \delta_{a_k}$$

In general, one may replace the Dirac measure by a signed measure μ satisfying suitable conditions. If we denote the Newtonian potential of any signed measure μ as

$$U^{\mu}(x) := \int_{\mathbb{R}^N} G(x-y)d\mu(y), \quad x \in \mathbb{R}^N$$

where

$$G(x) = \begin{cases} \frac{\omega_N}{N}|x|^{2-N} & \text{for} \quad N \geq 3 \\ -\frac{1}{2\pi}\ln|x| & \text{for} \quad N = 2 \end{cases}$$

denotes the Fundamental solution to the Laplace operator, and hence $-\Delta U^{\mu} = \mu$.

Thus, the problem is to find a domain Ω with the property that the support of μ is a subset of Ω and such that

$$\int_\Omega h dx = \int h d\mu \tag{3}$$

for all harmonic integrable functions h over the set Ω; which is equivalent to the following identity

$$U^{\chi_\Omega} = U^\mu, \quad \text{and} \quad \nabla U^{\chi_\Omega} = \nabla U^\mu \quad \text{in} \qquad I\!R^N \setminus \Omega, \tag{4}$$

Equivalently, to find (u, Ω) such that

$$\left.\begin{aligned} -\Delta u = \mu - \chi_\Omega \quad \text{in} \quad I\!R^N \\ u = 0,\ \nabla u = 0 \quad \text{in} \quad I\!R^N \setminus \Omega, \end{aligned}\right\} \tag{5}$$

where $u = U^\mu - U^{\chi_\Omega}$. This is a one-phase problem!

If we require in (3) or/and (4)

$$\int_\Omega h dx \geq \int h d\mu, \qquad\qquad U^{\chi_\Omega} \leq U^\mu \quad \text{in} \quad I\!R^N, \tag{6}$$

where the first inequality holds for all integrable subharmonic functions h, then we have $u \geq 0$ in Ω in (5).

Natural generalization-

(a)Two-phase obstacle problem : Given any two constants, λ_+ and λ_-, to find a solution of

$$\left.\begin{aligned} \Delta u &= \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}} \quad \text{in} \quad D \\ u &= g \quad \text{on} \quad D \end{aligned}\right\} \tag{7}$$

in a bounded domain D and $g \in W^{1,2}(D)$.

(b)"Two phase" Quadrature domain......

Known: There exists a locally bounded solution to the two-phase obstacle problem.

• Lot of interest and work on proving the regularity of free boundary- Cafarrelli-Alt, Gustafson-Shagholian, Sakai....

• The existence is generally assumed and usually proved using the variational methods by minimizing a suitable functional.

For Ω an open connected set (maybe unbounded) in $I\!R^N$, with reasonably smooth boundary, consider the functional

$$J_\Omega(u) = \int_\Omega \left\{ \frac{1}{2}|\nabla u|^2 - f(x)u^+ + h(x)u^- \right\} dx \tag{8}$$

where $f,\, h \in L^\infty(\Omega)$ are such that $supp(f^+)\cap\Omega \neq \emptyset$ and $supp(h^+)\cap \Omega \neq \emptyset$.

The functional J_Ω can be "split" into two one-phase functionals

$$J_+(u) = \int_\Omega \left\{ \frac{1}{2}|\nabla u|^2 - f(x)u^+ \right\} dx \tag{9}$$

and

$$J_-(u) = \int_\Omega \left\{ \frac{1}{2}|\nabla u|^2 + h(x)u^- \right\} dx \tag{10}$$

in the set $W^+ := \{u \in W^{1,2}(\Omega) : u - g^+ \in W^{1,2}(\Omega),\ u \geq 0 \text{ in } \Omega\}$ and $W^- := \{u \in W^{1,2}(\Omega) : u = -g^- \in W^{1,2}(\Omega),\ u \leq 0 \text{ in } \Omega\}$.

Let U_+ and U_- denote (local) minimizers of the functionals J_+ and J_- respectively, which solve

$$\left.\begin{array}{lll} -\Delta u & = & f(x)\chi_{\{u>0\}} \\ u & \geq & 0 \quad \text{in} \quad \Omega \\ u & = & g^+ \quad \text{on} \quad \partial\Omega \end{array}\right\} \tag{11}$$

and

$$\left.\begin{array}{lll} -\Delta u & = & -h(x)\chi_{\{u<0\}} \\ u & \leq & 0 \quad \text{in} \quad \Omega \\ u & = & -g^- \quad \text{on} \quad \partial\Omega \end{array}\right\} \tag{12}$$

respectively.

Main Result

PROPOSITION 1 *Assume that* $\Omega \subset I\!R^n$ *is a bounded domain. The functional* J_Ω *defined by (8) attains its minimum in*

$$S := \{u \in W^{1,2}_g(\Omega) : U_- \le u \le U_+\}.$$

The fact that we consider the set S is because of the following–

THEOREM 1 *Let* u *be a minimizer of the functional* J_Ω *in* $W^{1,2}_g(\Omega)$. *Then, the support of* u *must be a subset of* $supp(U_+) \cup supp(U_-)$. *In fact,*

$$supp(u^+) \subseteq supp(U_+), \tag{13}$$

$$supp(u^-) \subseteq supp(U_-) \tag{14}$$

where $u^+(x) := \max\{u(x), 0\}$ *and* $u^-(x) := -\min\{u(x), 0\}$.

Proof of the Proposition

First, note that S is a closed set and that J_Ω is lower semi continuous. Moreover, for any $u \in S$,

$$\begin{aligned} J_\Omega(u) &= \int_\Omega \left\{ \frac{1}{2}|\nabla u|^2 - f(x)u^+ + h(x)u^- \right\} dx \\ &\geq \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \int_\Omega |f(x)|u^+\, dx - \int_\Omega |h(x)|u^-\, dx \\ &\geq \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - c(f,h)\int_\Omega (u^+ + u^-)\, dx \\ &= \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - c(f,h)\int_\Omega |u|\, dx \\ &\geq \frac{1}{2}||\nabla u||^2 - c(f,h)C_1||\nabla u|| \qquad (15) \end{aligned}$$

where the last inequality follows from Poincaré inequality. Thus J is coercive in S. Hence, J attains it infimum in S.

• In case Ω is bounded the proof in above proposition goes through even in the set $W_g^{1,2}(\Omega)$. Thus in case of bounded domains, Theorem 1 can be considered as a qualitative result which gives a smaller set in which one obtains the minimizer for J_Ω. We may thus possibly obtain more than one minimizers in case the minimizers of J_+ or J_- are not unique.

• Theorem 1 is more crucial in case Ω is unbounded, where in general it may be difficult to prove that the functional J_Ω is bounded from below. In case one can find conditions so that the minimizers of J_+, J_- have compact support then, in these situations, Theorem 1 implies that the support of minimizer is contained in the union of supports of U_+ and U_-.

An immediate corollary of above two results is-

COROLLARY 0.1 *Let the functions f, h be such that*

$$\left.\begin{array}{l} f,\, h \in L^\infty(I\!R^N) \\ \\ supp(f^+) \text{ and } supp(h^-) \text{ are nonempty and compact} \\ \\ f \leq \text{const.} < 0 \text{ holds outside a compact set} \\ \\ h \geq \text{const.} > 0 \text{ holds outside a compact set.} \end{array}\right\} \tag{16}$$

Then, the functional

$$J_{I\!R^N}(u) := \int\limits_{I\!R^N} (\frac{1}{2}|\nabla u|^2 - f u^+ + h u^-)\, dx \tag{17}$$

has a minimizer in S (and hence in $W^{1,2}(I\!R^N)$).

Application: Quadrature domain with two measures

The concept of "two phase version" of quadrature domain is formulated using the pde (rather than using potential theory) as follows:

Given two positive measures $\mu_\pm$ with disjoint, compact supports and non-negative functions $\lambda_\pm$, we look for a function u satisfying

$$\left.\begin{aligned} -\Delta u &= (\mu_+ - \lambda_+\chi_{\{u>0\}}) - (\mu_- - \lambda_-\chi_{\{u<0\}}) \quad \text{in} \quad I\!R^N, \\ &supp \mu_\pm \subset \{\pm u > 0\}. \end{aligned}\right\} \tag{18}$$

Example: In the special case of (18) when $\lambda_\pm = 1$ and the measures $\mu_\pm = 0$, it can be verified that the function

$$u(x) = \frac{(x_1^+)^2}{2} - \frac{(x_1^-)^2}{2} \tag{19}$$

where $x_1^\pm(x) := \max(\pm x_1, 0)$ is a global solution to the equation

$$\Delta u = \chi_{\{u>0\}} - \chi_{\{u<0\}} \quad \text{in} \quad I\!R^N. \tag{20}$$

Consider the equation

$$-\Delta u = (\mu_+ - \lambda_+)\chi_{\{u>0\}} - (\mu_- - \lambda_-)\chi_{\{u<0\}} \tag{21}$$

in $I\!R^N$ where μ_+, μ_- are positive Radon measures with compact support.

The following theorem was proved in [GS](Gustafsson B., Shahgholian H., *Existence and geometric properties of solutions of a free boundary problem in potential theory, J. Reine Angew. Math. 473 (1996), 137-179.*)

THEOREM 2 *(Theorem 4.7 in [GS]) Let* μ *be a positive measure which is concentrated to a ball* $B_R = B(x_0, R)$ *to the extent that*

$$\left.\begin{array}{rcl} \mu(B_R^c) & = & 0 \\ \mu(B_R) & > & \left(b + \frac{Nc}{3R}\right) 6^N |B_R|. \end{array}\right\} \tag{22}$$

Then, for any h, g, there exists a weak solution u of

$$\Delta u + (\mu - h)\mathcal{L}^N \mid \Omega = g\mathcal{H}^{N-1} \mid \partial\Omega \tag{23}$$

where $\Omega = \{u > 0\}$ which satisfies

$$supp(\mu) \subset \Omega. \tag{24}$$

Moreover, if we define the measure $\nu := h\mathcal{L}^N \mid \Omega + g\mathcal{H}^{N-1} \mid \partial\Omega$ and denote the Newtonian potential of ν by U^ν then

$$U^\nu \ \leq \ U^\mu \tag{25}$$

$$|\nabla U^\nu| \leq \text{const.} < \infty. \tag{26}$$

In fact, we have $B_{3R} \subset \Omega$.

The following lemma was crucial in the proof of the above theorem.

LEMMA 0.1 *(Lemma 4.6, [GS]) Let* $0 \le \psi \in L^\infty(\mathbb{R}^n)$ *be a non-increasing, radially symmetric function satisfying* $\int \psi \, dx = 1$. *Then, for* μ *a positive measure with compact support,*

$$\Omega \in \mathcal{Q}(\mu * \psi; h, g) \Longrightarrow \Omega \in \mathcal{Q}(\mu; h, g). \tag{27}$$

Moreover, if $U^\nu \le U^{\mu * \psi}$ *then* $U^\nu \le U^\mu$.

Here, $\mathcal{Q}(\mu; h, g)$ denotes the set of quadrature domains for μ with given densities g and h. Precisely, with notations as in Theorem 2, a set $O \in \mathcal{Q}(\mu; h, g)$ if O is a bounded open set in $\mathbb{R}^N$ such that

$$supp(\mu) \subset O,$$

$$U^\nu = U^\mu \quad \text{on} \quad \mathbb{R}^N \setminus O \text{ where } \nu := h\mathcal{L}^N \mid O + g\mathcal{H}^{N-1} \mid \partial O.$$

Lemma 0.1 essentially says that we can replace a positive measure with compact support by a suitable mollifier. Then, the weak solution corresponding to the minimization of the functional

$$J_{f,g}(u) := \int_{I\!R^N} (\frac{1}{2}|\nabla u|^2 - fu + g^2\chi_{\{u>0\}})\, dx \qquad (28)$$

with $f = \mu * \psi - \lambda$ is also a (weak) solution of

$$\left.\begin{array}{c} \Delta u = \lambda - \mu \quad \text{in} \quad \{u > 0\} \\ u = 0, \ |\nabla u| = g \quad \text{in} \quad \partial\{u > 0\} \\ supp(\mu) \subset \{u > 0\}. \end{array}\right\} \qquad (29)$$

In our case, the one phase functionals J_+, J_- defined by (9), (10) respectively are simpler with $g = 0$.

THEOREM 3 *Assume that the measures μ_+ and μ_- are sufficiently concentrated in some balls $B(x_1, R_1)$ and $B(x_2, R_2)$ respectively in the sense defined in (22). Then, there exists a (weak) solution to the equation (21).*

REMARK 0.1 Observe that while for the 1-phase case we have $supp(\mu_+) \subset u_1$ and $supp(\mu_-) \subset u_2$ (see [GS]), it is not clear from the above proof that $supp(\mu_+) \subset supp(u^+)$ and $supp(\mu_-) \subset supp(u^-)$ where u is the minimizer of the (corresponding) 2-phase problem.

Example : Let $\lambda_\pm = 1$, $\mu^+ := m\chi_{B(e_1,1/2)}$, $\mu^+ := 2\chi_{B(-e_1,1/2)}$ where m is a positive integer. By Theorem 2, for $m = 2$, both measures give rise to one phase quadrature domains. Hence our theorem implies existence of solution for the two phase problem. Vary m!

A direct proof by approximation:

THEOREM 4 *Let* $f := \mu - \lambda$ *where* μ *is a Radon measure with compact support, say denoted by* D *and let* φ_i *be a sequence of smooth functions with compact support such that* $\varphi_i \to \mu$ *(as measures i.e.,* $\int\limits_{I\!R^N} \varphi_i \phi \, dx \to \int\limits_{I\!R^N} \phi \, d\mu(x)$*). Let* u_i *be the minimizer for the functional*

$$J_i(u) := \int\limits_{I\!R^N} \frac{1}{2}|\nabla u|^2 - (\varphi_i(x) - \lambda(x))u^+ \, dx. \tag{30}$$

Then there exists a compact set K *such that* $supp(u_i) \subset K$ *for all* i*.*

Sketch of proof:

• Assume $\varphi_i \to \mu$ and $supp(\varphi_i)$, $supp(\mu) \subset B(0, R_1)$. Let u_i be solution to

$$\left.\begin{array}{lll} \Delta u_i = \lambda(x) - \varphi_i(x) & \text{in} & D_i := \{u_i > 0\}, \\ u = 0,\ |\nabla u| = 0 & \text{on} & \partial D_i \end{array}\right\}. \qquad (31)$$

Then, for each i, $supp(u_i) := D_i$ is compact.

• The Newtonian potential of φ_i, defined by $G_i := G * \varphi_i$ then

$$\Delta G_i = -\varphi_i \quad \text{in} \quad I\!R^N.$$

Therefore,

$$\begin{array}{rcl} \Delta(u_i - G_i) & = & \lambda(x) \geq \lambda_0 > 0 \quad \text{in} \quad D_i \\ (u_i - G_i) & \leq & 0 \quad \text{on} \quad \partial D_i \end{array}$$

implies by maximum principle that

$$u_i \leq G_i \quad \text{in} \quad D_i. \qquad (32)$$

In particular, u_i's are bounded by the Newtonian potential G^μ of μ.

• There exists an R_0 sufficiently large such that $D_i \subset B(0, R_0)$ for all large i–if not, choose $p_i \in \partial D_i$ such that $p_i \notin B(0, r_i)$ and $|p_i| \to \infty$. The function

$$w_i(x) := u_i(x) - \lambda_0 \frac{|x - p_i|^2}{2n} \quad \text{in} \quad \Omega_i = D_i \setminus B(0, R_1) \tag{33}$$

satisfies the equation

$$\Delta w_i = \Delta u_i - \lambda_0 = \lambda(x) - \lambda_0 \geq 0 \quad \text{in} \quad \Omega_i. \tag{34}$$

Therefore, by the maximum principle, w_i achieves its maximum on the boundary $\partial\Omega_i$, which implies

$$\max_{\partial B(0,R_1)} u_i(x) \geq d(p_i, \partial B(0, R_1)) \to \infty \quad \text{as} \quad i \to \infty \tag{35}$$

a contradiction!