On 3TGFEA for Singularly Perturbed Transient Convection Diffusion Equation

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Indian Institute of Technology, Kanpur

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Outline of the talk:

- Introduction to Singularly Perturbed Differential Equations
- Significance of Singularly Perturbed Problems
- Continuous Problem
- Solution Methodology
- Error Estimates for the problem
- Numerical Results
- Refrences

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Singularly Perturbed Differential Equation

$$-\epsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x)$$

- Boundary Layer
- Reduced System
- Singularly Perturbed Differential Equation
  - $\rightsquigarrow$ ϵ is multiplied with the highest order derivative terms
  - $\rightsquigarrow$ as $\epsilon \longrightarrow 0$, the reduced system does not satisfy the boundary conditions, in general
- Ludwig Prandtl - 'Third International Congress of Mathematicians', 1904

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Significance of SPP

- Theoretical explanation of the physical phenomenon of the boundary layer
- Hydrodynamics
- Fluid Mechanics, Fluid Dynamics
- Elasticity Theory
- Chemical Reactor Theory
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Analytical Vs Numerical Approach

Analytical methods can provide exact solution. But !

$\longrightarrow$ Complex domain

$\longrightarrow$ Nonolinear equations

$\longrightarrow$ Complicated variable co-efficients

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Time-dependent Singularly Perturbed Convection-Diffusion problem

- $L_\epsilon u(x,t) \equiv -\epsilon u_{xx}(x,t) + b(x,t)u_x(x,t) + c(x,t)u(x,t) + d(x,t)u_t(x,t) = f(x,t); \quad (x,t) \in \Omega = [0,1] \times [0,T] \quad (1)$

  $\longrightarrow$ initial conditions $\quad u(x,0) = u_0(x); \quad 0 \le x \le 1$

  $\longrightarrow$ and boundary conditions
  $u(0,t) = q_0(t), \quad u(1,t) = q_1(t); \quad 0 \le t \le T$

  $\longrightarrow$ $b,\ c,\ d$ and f are sufficiently smooth.

  $\longrightarrow$ $0 < \beta \le b(x,t) \le \beta^*, \ \ 0 < \gamma \le c(x,t) \le \gamma^*, \ \ 0 < \delta \le d(x,t) \le \delta^*$ on Ω.

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Motivation for error estimates:

- Standard finite element methods are USUALLY UNSATISFACTORY !!!

- Consider the problem
  $Lu := -\epsilon u'' + b(x)u' + c(x)u = f(x)$ (2)

  $\longrightarrow$ $u(0) = u(1) = 0$

  $\longrightarrow$ $c(x) - b'(x)/2 \geq \alpha^* > 0 \qquad \forall\ x \in \Omega = [0, 1]$

  $\longrightarrow$ $\|u - u_h\|_1 \leq \frac{c\beta h}{\epsilon}|u|_2$

  $\longrightarrow$ $|u|_2 = O(\epsilon^{-3/2})$

  $\longrightarrow$ when $\epsilon \rightarrow 0$!!!

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Piecewise Uniform Convergent Finite Element Schemes

- The idea of using special kind of meshes for solving the SPPDE was given by Bakhvalov.
- Bakhvalov's idea:
  Assume that we have an exponential boundary layer at $x = 0$, so the boundary layer function is $y = exp(-\beta x/\epsilon)$, for some fixed β. Bakhvalov's idea is to use an equidistant y-grid near $y = 1$ (corresponding to $x = 0$), then to map this grid back to the x-axis by means of the boundary layer function.
- For example grid points x_i near $x = 0$ are defined by $exp(\frac{-\beta x_i}{\epsilon}) = 1 - (\frac{i}{N})$ (using (N+1) mesh points)
  $\Rightarrow x_i = -(\frac{\epsilon}{\beta}) log(1 - (\frac{i}{N}))$
- Meshes based on such logarithmic functions are called Bakhvalov meshes.

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Piecewise Uniform grids leading to Convergent Finite Element Schemes

- Bakhvalov used the mesh generating function as
$$\lambda(t) = \begin{cases} \psi(t) := -A\epsilon \; ln(1 - t/q), & for \quad t \in [0, \tau], \\ \pi(t) := \psi(\tau) + \psi'(\tau(t-\tau)), & for \quad t \in [\tau, 1], \end{cases}$$
for a boundary layer at x=0.

- Here A and q are user-chosen positive parameters and the point τ satisfies
$\psi(\tau) + \psi'(\tau)(1-\tau) = 1.$

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- Theorem: The simple upwind scheme, applied to SPPDE (2) on a Bakhvalov type mesh is uniformly convergent with respect to the singular perturbation parameter i.e

$$\max_{i|u(x_i)-u_i|\leq Ch}$$

Piecewise Uniform grids leading to Convergent Finite Element Schemes

- Shishkin's piecewise equidistant meshes
- Shishkin's idea was:
  "Is it possible to prove nodal uniform convergence of a method that is not exponentially fitted, if $\frac{N}{2}$ points are placed equidistantly in each of the intervals
  $[0, 1-\sigma]$ and $[1-\sigma, 1]$ for some σ".
- Shishkin chooses $\sigma = \min\{\frac{1}{2}, K\epsilon logN\}$ where K is a constant independent of ϵ and N.

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  "Is it possible to prove nodal uniform convergence of a method that is not exponentially fitted, if $\frac{N}{2}$ points are placed equidistantly in each of the intervals
  $[0, 1-\sigma]$ and $[1-\sigma, 1]$ for some σ".
- Shishkin chooses $\sigma = \min\{\frac{1}{2}, K\epsilon logN\}$ where K is a constant independent of ϵ and N.

Some convergence estimates for FEM on a Shishkin mesh

- Theorem: The simple upwind scheme when applied to the SPPDE(2) on Shishkin mesh, is nodally uniformly convergent i.e.
  $\max_i |u(x_i) - u_i| \leq CN^{-1} ln(N)$.

- Theorem: Let u^I be the piecewise linear interpolant of the exact solution u of the SPPDE (2) on Shishkin mesh, then
  $\|u - u^I\|_\infty \leq C(N^{-1} ln(N))^2$
  $\|(u - u^I)(x)\| \leq CN^{-2} \quad$ for $x \in [0, 1-\sigma]$
  and $\|u - u^I\|_\epsilon \leq CN^{-1} ln(N)$,
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Streamline Diffusion Method(SUPG):

- Global error bound [U. Navert]:
  Assume $\epsilon \leq h,\ \tau \leq Ch,\ \gamma \leq C^{'}h$ for some sufficiently small coefficient $C^{'}$. Then

  $$|||u - u_{h,\tau}|||_Q \leq Ch^{3/2}|u|_{H^2(Q)}$$

  where

  $\longrightarrow$ $$|||w|||_Q^2 := \epsilon \sum_{T\in Q} \| \nabla w\|_{L_2(T)}^2 + \sum_{T\in Q} \gamma \|w_\beta\|_{L_2(T)}^2 + \|w^2\|_{L_2(Q)}; \quad w \in H^1(T) \quad \forall\ T \in Q$$

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Three Step Taylor Galerkin Finite Element Scheme

- Taylor series expansion:

  $u(t+\Delta t) = u(t) + \Delta t u_t(t) + \frac{\Delta t^2}{2} u_{tt}(t) + \frac{\Delta t^3}{6} u_{ttt}(t) + O(\Delta t^4)$

- Consider the simple case of Heat Equation $u_t = -\epsilon u_{xx}$
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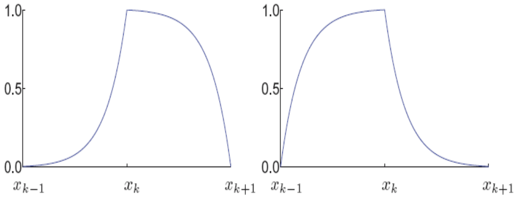
$\overline{L}$ and $\overline{L}^*$ splines:



$\overline{L}-splines$ and $\overline{L}^*-splines$ for $\overline{b}=1$ and $\frac{\varepsilon}{h}=0.2$

Some comments on Three-step TGFEM

$\longrightarrow$ Proposed by J. Donea

$\longrightarrow$ Third order accurate in time

$\longrightarrow$ No higher order spatial derivatives !!!

$\longrightarrow$ Time discretization is carried out prior to the space distretization

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- $L_\epsilon u(x,t) \equiv -\epsilon u_{xx}(x,t) + b(x,t)u_x(x,t) + c(x,t)u(x,t) + d(x,t)u_t(x,t) = f(x,t); \quad (x,t) \in \Omega = [0,1] \times [0,T] \quad (1)$

  $\longrightarrow$ initial conditions $\quad u(x,0) = u_0(x); \quad 0 \leq x \leq 1$

  $\longrightarrow$ and boundary conditions
  $u(0,t) = q_0(t), \quad u(1,t) = q_1(t); \quad 0 \leq t \leq T$

  $\longrightarrow$ $b,\ c,\ d$ and f are sufficiently smooth.

  $\longrightarrow$ $0 < \beta \leq b(x,t) \leq \beta^*, \ \ 0 < \gamma \leq c(x,t) \leq \gamma^*, \ \ 0 < \delta \leq d(x,t) \leq \delta^*$ on Ω.

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Error Estimation for the Three-step TGFEM

- Compatibility conditions:

  $u_0(0) = q_0(0), \quad u_0(1) = q_1(0)$

  $\Rightarrow |u(x,t) - u_0(x)| \leq Ct,$

  $$|u(x,t) - q_0(t)| \leq C(1-x) \quad \forall \ (x,t) \in \bar{\Omega}$$

- [*Bobisud*] Solution can be written as:

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- The operator L given by equation (1) satisfies the maximum principle.
- $|u_t(x,t)| \leq C; \qquad (x,t) \in \bar{\Omega}$
  - ↪ $|u_t(x,0)| = \frac{f(x,0)}{d(x,0)} \quad \leq C_1; \quad 0 \leq x \leq 1$
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  - ↪ $L_1(u_t)(x,t) = f_t - c_t u \quad \leq C_2 \quad on\ \bar{\Omega}$
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  $|u_x(x,t)| \leq C(1+\epsilon^{-1}e^{-\beta(1-x)/\epsilon}); \quad (x,t) \in \bar{\Omega}$

- $|u_{tt}(x,t)| \leq C; \qquad (x,t) \in \bar{\Omega}$

  ↪ On sides, $x=0$, $x=1$, $u_{tt}=0$

  On side $t=0$, $u=0$, $u_x=0$, $u_{xx}=0$

  ↪ $\mathrm{u}_t = -f/d \quad \Rightarrow |u_{tx}| \leq C, \quad |u_{txx}| \leq C$

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- $|u_{xt}(x,t)| \leq C(1+\epsilon^{-1}e^{-\beta(1-x)/\epsilon}); \quad (x,t) \in \bar{\Omega}$
  - ↪ differentiate equation (1) w.r.t. x
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  - ↪ differentiate equation (1) w.r.t. x
  - ↪ $-\epsilon(u_{xx})_x + bu_{xx} = h(x,t) \quad on \ \bar{\Omega}$
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- Theorem: The solution $u(x,t)$ satisfies

  $|(\frac{\partial}{\partial x})^k (\frac{\partial}{\partial t})^m u(x,t)| \leq C(1 + \epsilon^{-k} e^{-\beta(1-x)/\epsilon}); \;\; (x,t) \in \bar{\Omega}$

  where $0 \leq k \leq 1$ and $0 \leq k + m \leq 3$
- We will use the approximation:

  $\int_0^1 d(x,t) u_t(x,t) \psi^i(x,t) \approx h d(x_i,t) u_t(x_i,t)$

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- $|u_{ttt}(x,t)| \leq C; \qquad (x,t) \in \bar{\Omega}$
- $|u_{xtt}(x,t)| \leq C(1 + \epsilon^{-1} e^{-\beta(1-x)/\epsilon}); \quad (x,t) \in \bar{\Omega}$
- Theorem: The solution $u(x,t)$ satisfies

  $|(\frac{\partial}{\partial x})^k (\frac{\partial}{\partial t})^m u(x,t)| \leq C(1 + \epsilon^{-k} e^{-\beta(1-x)/\epsilon}); \quad (x,t) \in \bar{\Omega}$

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- $|(1, \psi_i^m) - h| \leq Ch^2$

$\hookrightarrow$ $(1,\psi_i^m) = h[1 + g(\rho_{i+1}^m) - g(\rho_i^m)]; \quad \rho_i^m = \frac{b_i^m h}{\epsilon}$

$$g(z) = \begin{cases} (e^z - 1)^{-1} - z^{-1}, & if \quad z \neq 0 \\ \frac{-1}{2}, & if \quad z = 0 \end{cases}$$

$\hookrightarrow$ $|(1, \psi_i^m) - h| = h|(\rho_{i+1}^m - \rho_i^m)| \, |g^{'}(\eta_i)|$

$\hookrightarrow$ $|g^{'}(.)| \leq C \quad and \quad |g^{'}(z)| \leq C|z|^{-2} \quad for \quad z \neq 0$

$\hookrightarrow$ Case-1:
$h \leq \epsilon \Rightarrow |(1, \psi_i^m) - h| \leq Ch|(b_{i+1}^m - b_i^m)|h/\epsilon \quad \leq Ch^2$

$\hookrightarrow$ Case-2:
$h \geq \epsilon \Rightarrow |(1, \psi_i^m) - h| \leq Ch|(b_{i+1}^m - b_i^m)|\frac{h}{\epsilon}(\frac{\epsilon}{h})^2 \quad \leq Ch^2$

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- Theorem: If the coefficients of the problem (1) are smooth enough, then

$$|u(x_i, t^m) - y_{i,m}| \le C(h + k^3) \qquad \forall\ i, m.$$

- Proof: Consider the operator

$$(L_{h,k}v)_{i,m} = h^{-1}\{\bar{B}(v^m, \psi_i^m) + hc_i^m v_i^m + hd_i^m((v_i^{m+1} - v_i^m)/k) - h(k/2)d_i^m(v_{tt})_i^m - h(k^2/6)d_i^m(v_{ttt})_i^m\} \qquad (3)$$

where $\bar{B}(v, w) = (\epsilon v_x, w_x) + (\bar{b}v_x, w)$

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$$\Rightarrow (L_{h,k}e)_{i,m} \le C\int_{x_{i-1}}^{x_{i+1}} \epsilon^{-1} e^{-\beta(1-x)/\epsilon} dx + Ch + Ck^3$$

Applying the notion of barrier function and maximum principle argument proves the result.

$$\approx \{((\bar{b}-b)u_x^m - c^m u^m + c_i^m u_i^m + f^m - f_i^m, \psi_i^m) + O(h^2)\} - (d^m u_t^m, \psi_i^m) + hd_i^m((u_t)_i^m + (k/2)(u_{tt})_i^m + (k^2/6)(u_{ttt})_i^m) - h(k/2)d_i^m(u_{tt})_i^m - h(k^2/6)d_i^m(u_{ttt})_i^m + O(hk^3)$$

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Stability conditions:

Theorem: Consider the operator $L_{h,k}$ defined by (3). If the mesh widths h and k are chosen so that the inequality

$$kh^{-1}\left(\frac{\alpha^*}{1-e^{\frac{-b_{i,m}h}{\epsilon}}}+\frac{\alpha^*}{e^{\frac{b_{i+1,m}h}{\epsilon}}-1}-\beta h\right)<\delta$$

is satisfied, then the discrete operator $L_{h,k}$ satisfies the discrete maximum principle.

Explicit LSFEM with Linear Splines

- LSFEM seeks the solutions (u, p), from suitable spaces, which minimizes the quadratic functional:
  $I(u,p) = \int_0^1 (\| d(x,t)u_t - \epsilon p_x + b(x)u_x + c(x,t)u$
  - f(x,t) $\|^2_{0,\Omega} + \|u_x + p\|^2_{0,\Omega})ds$

- Finite Element Formulation

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  - $f(x,t) \|_{0,\Omega}^2 + \|u_x + p\|_{0,\Omega}^2) ds$

- Finite Element Formulation

Numerical Results: Linear case:-

- We consider equation (1) with $0 \leq x \leq 1, \quad 0 \leq t \leq 1,$

  $\hookrightarrow$ $b(x,t) = -1, \quad c(x,t) = 0, \; d(x,t) = 1,$

  $\hookrightarrow$ The source term is so chosen to satisfy the Analytical solution:
  $u(x,t) = \frac{e^{(\frac{-x}{\epsilon})} - e^{(\frac{-1}{\epsilon})}}{1 - e^{(\frac{-1}{\epsilon})}} sin(2t) + 2xcos(\frac{\pi x}{2}) sin(t)$

  $\hookrightarrow$ The ϵ-uniform maximum nodal error is defined by
  $E_{N,\delta t} = \max_{\epsilon} E_{\epsilon,N,\delta t}$

  $\hookrightarrow$ The numerical ϵ-uniform rate of convergence is given by
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- We consider equation (1) with $0 \leq x \leq 1, \;\; 0 \leq t \leq 1,$

  $\hookrightarrow$ $b(x,t) = -1, \quad c(x,t) = 0, \; d(x,t) = 1,$

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  $u(x,t) = \frac{e^{(\frac{-x}{\epsilon})} - e^{(\frac{-1}{\epsilon})}}{1 - e^{(\frac{-1}{\epsilon})}} sin(2t) + 2xcos(\frac{\pi x}{2}) sin(t)$

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Numerical maximum errors (E_ϵ^N) and numerical rate of convergence (p_N) for 3TGFEM with exponential fitted splines:

ϵ	$N = 32$	$N = 64$	$N = 128$	$N = 256$
2^{-2}	0.151E-03	0.373E-04	0.937E-05	0.237E-05
	3.03	2.99	0.00	
2^{-4}	0.496E-03	0.125E-03	0.313E-04	0.782E-05
	1.38	4.60	0.42	
2^{-6}	0.986E-03	0.283E-03	0.883E-04	0.266E-04
	1.25	3.53	0.23	
2^{-8}	0.139E-02	0.313E-03	0.883E-04	0.266E-04
	1.49	3.63	0.23	
2^{-10}	0.155E-02	0.405E-03	0.988E-04	0.266E-04
	1.34	1.41	1.31	
2^{-12}	0.155E-02	0.405E-03	0.104E-03	0.266E-04
	1.34	1.36	1.36	
2^{-14}	0.155E-02	0.405E-03	0.104E-03	0.266E-04
	1.34	1.36	1.36	
$E_{N,\delta t}$	0.155E-02	0.405E-03	0.104E-03	0.266E-04
p_N	1.34	1.36	1.36	

Numerical maximum errors (E_ϵ^N) and numerical rate of convergence (p_N) for LSFEM with linear splines:

ϵ	$N = 32$	$N = 64$	$N = 128$	$N = 256$
2^{-2}	0.307E-03	0.165E-03	0.360E-04	0.967E-05
	0.62	3.31	2.20	
2^{-4}	0.165E-02	0.423E-03	0.107E-03	0.269E-04
	1.36	1.37	2.74	
2^{-6}	0.824E-02	0.143E-02	0.332E-03	0.963E-04
	1.75	1.46	3.96	
2^{-8}	0.106E-01	0.157E-02	0.332E-03	0.963E-04
	1.91	1.55	3.96	
2^{-10}	0.112E-01	0.165E-02	0.332E-03	0.963E-04
	1.92	1.60	1.24	
2^{-12}	0.114E-01	0.167E-02	0.332E-03	0.963E-04
	1.92	1.62	1.24	
2^{-14}	0.114E-01	0.168E-02	0.332E-03	0.963E-04
	1.92	1.62	1.24	
$E_{N,\delta t}$	0.114E-01	0.168E-02	0.332E-03	0.963E-04
p_N	1.92	1.62	1.24	

Numerical maximum errors (E_ϵ^N) and numerical rate of convergence (p_N) for 3TGFEM with linear splines:

ϵ	$N = 32$	$N = 64$	$N = 128$	$N = 256$
2^{-2}	0.319E-03	0.810E-04	0.202E-04	0.505E-05
	5.80	1.35	1.64	
2^{-4}	0.433E-02	0.107E-02	0.267E-03	0.668E-04
	1.40	1.39	3.67	
2^{-6}	0.303E-01	0.933E-02	0.286E-02	0.857E-03
	1.18	1.18	1.20	
2^{-8}	0.390E-01	0.122E-01	0.406E-02	0.130E-02
	1.16	1.10	1.14	
2^{-10}	0.412E-01	0.129E-01	0.435E-02	0.141E-02
	1.16	1.08	1.13	
2^{-12}	0.418E-01	0.130E-01	0.442E-02	0.143E-02
	1.16	1.08	1.13	
2^{-14}	0.419E-01	0.131E-01	0.444E-02	0.144E-02
	1.16	1.08	1.13	
$E_{N,\delta t}$	0.419E-01	0.131E-01	0.444E-02	0.144E-02
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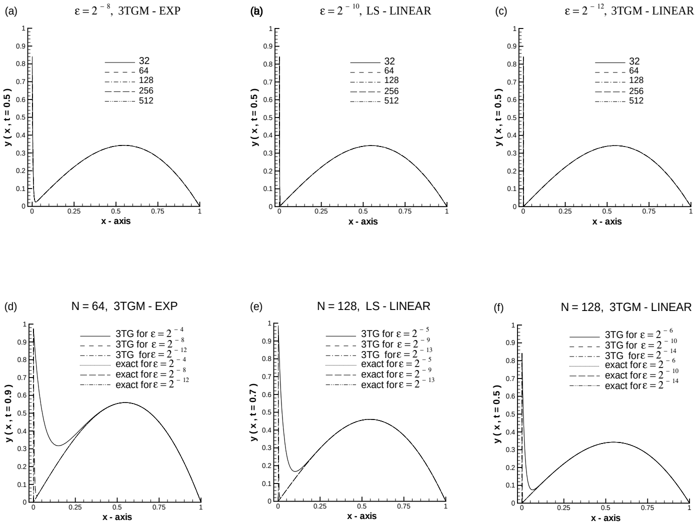
Comparison between 3TGFEM with exponentially fitted splines, Explicit LSFEM and 3TGFEM with linear splines:



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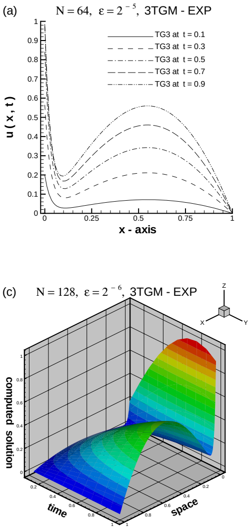

Numerical Results: Nonlinear case:-

- Again consider equation (1) with $0 \le x \le 1, \;\; 0 \le t \le 1,$
  - $\hookrightarrow$ $b(x,t) = 1, \quad c(x,t) = 0, \; d(x,t) = 1,$
  - $\hookrightarrow$ The source term $f(x,t) = e^{-u} - 1$
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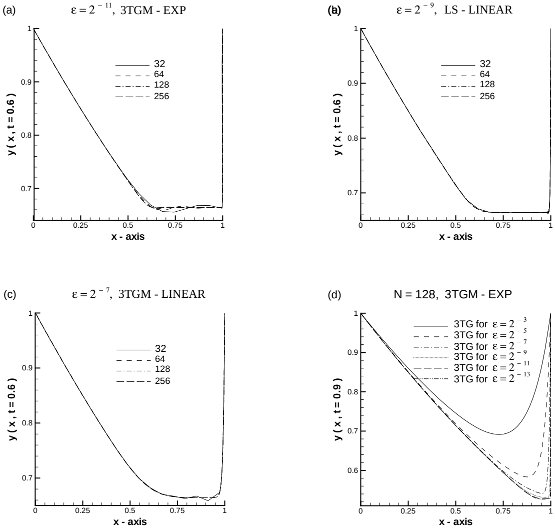

Numerical Results: 2 Dimensional case:-

- We consider the two dimensional SPP with $\Omega = [0,1]^2, \quad 0 \leq t \leq 1,$

  $\hookrightarrow$ $u_t - \epsilon(u_{xx} + u_{yy}) + u_x + u_y = 2$

  $\hookrightarrow$ Initial and boundary conditions are defined so as to satisfy the exact solution.

  $\hookrightarrow$ Analytical solution is given as:
  $u(x,y,t) = (1 - e^{\frac{-1}{\epsilon}})(x+y) + 2e^{\frac{-1}{\epsilon}} t - e^{\frac{x-1}{\epsilon}} - e^{\frac{y-1}{\epsilon}}$

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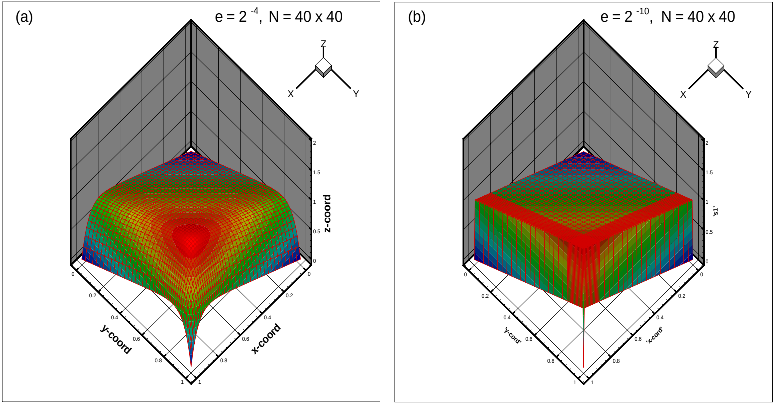
3TGFEM solution profile for the 2D case:

Table: Pointwise error table in absolute norm for $\epsilon = 2^{-10}$ for different grids at different time levels:

ϵ	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	$t = 1.0$
ϵ^{-4}	5.24740251E-11	5.56839019E-11	5.45672396E-11	5.39133183E-11	5.37245803E-11
ϵ^{-6}	1.77016376E-07	1.77587602E-07	1.77599635E-07	1.77599954E-07	1.77599961E-07
ϵ^{-8}	0.000429735896	0.000429736591	0.000429736591	0.000429736591	0.000429736591
ϵ^{-10}	0.00542082742	0.00542082742	0.00542082742	0.00542082742	0.00542082742

3TGFEM with exponentially fitted splines:



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THANK YOU.