ICM Satellite Conference on PDE and Related Topics

TIFR–CAM – August 13-17, 2010

Strong solutions of systems coupling elastic structures

with the Navier-Stokes equations

Jean-Pierre Raymond, Institut de Mathématiques – Toulouse

Collaborators: B. Dehman (Tunis), J. Lequeurre (Toulouse)

Plan of the talk

1. Models of Fluid-Structure-Interaction

- Review of existence and regularity results – Methods for studying these models

2. Systems coupling N.S.E. with damped beam and plate equations

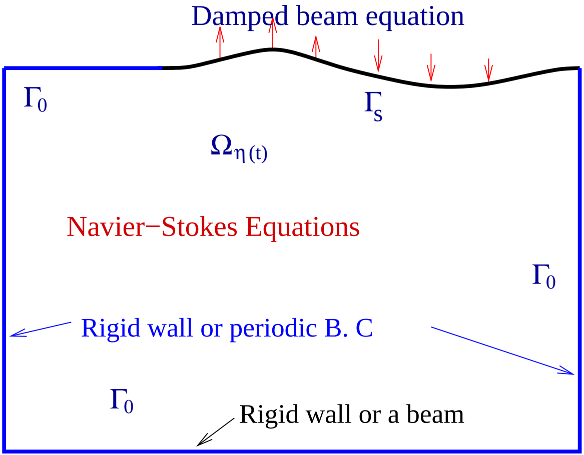
- New way in analyzing the linearized system.

3. N.S.E. coupled with the Lamé system

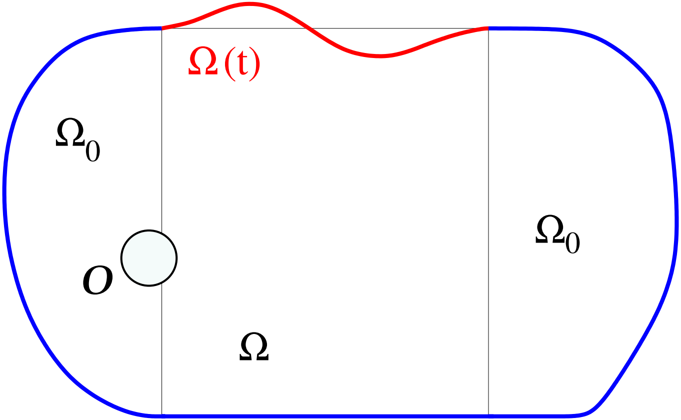
- New regularity results for a linearized model

N.S.E. coupled with a damped beam equation

Beirao da Veiga 04



N.S.E. coupled with a beam or a plate equation



Known results

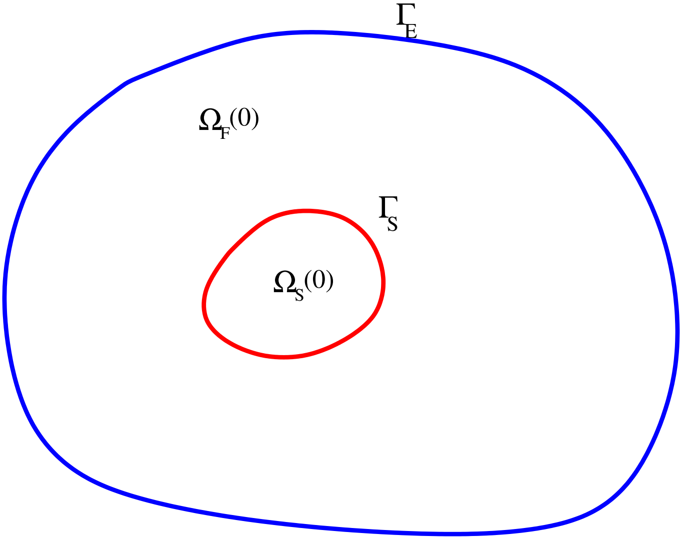
- H. Beirao da Veiga, *On the existence of strong solutions to a coupled fluid-structure evolution system*, JMFM 2004.
- A. Chambolle, B. Desjardins, M. J. Esteban, C. Grandmont, *Existence of weak solutions for unsteady fluid-plate interaction problem*, J. Math. Fluid Mech. 2005.
- C. Grandmont, *Existence of weak solutions for unsteady interaction of a viscous fluid with an elastic plate*, SIMA 08.

New results

- J.-P. Raymond, *Feedback stabilization of a fluid–structure model*, 2009, SICON, to appear.
- J. Lequeurre, *Existence of strong solutions to a fluid–structure system*, 2009, SIMA, to appear.

3D case (work in progress)

N.S.E. coupled with the Lamé system



Several models for the structure

- Rigid body (Takahashi '03, San Martin, Starovoitov, Tucsnak '02, Cumsille, Takahashi '09, ...)

- Deformable body described by a system of O.D.E. (San Martin, Scheid, Takahashi, Tucsnak '08, Court '10)

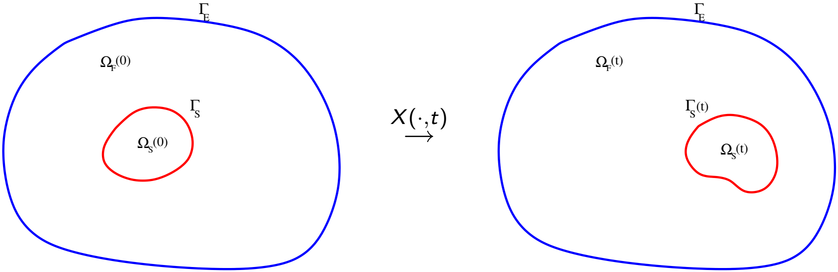
- D. Coutand, S. Shkoller, Elastic structure modeled by the Lamé system , Arch. Rational Mech. Anal. 2005, 25-102.

- G. Avalos, I. Lasiecka, R. Triggiani, Higher regularity of a coupled parabolichyperbolic fluid structure interactive system, Georg. Math. J. ,2008, Linearized model.

Methods for the existence of strong solutions

A change of variable associated with some velocity field ζ

$$X_t(y,t) = \zeta(X(y,t),t), \quad X(y,0) = y, \;\; y \in \Omega_F(0).$$



$\zeta \longmapsto$ fluid velocity or

$\zeta \longmapsto$ velocity of structure deformation.

$$Y(X(y,t),t) = y \quad \text{for all } y \in \Omega_F(0),$$

$$X(Y(x,t),t) = x \quad \text{for all } x \in \Omega_F(t).$$

Make a change of unknowns

$$v(y,t) = u(X(y,t),t) \quad \text{and} \quad q(y,t) = p(X(y,t),t),$$

and

$$U(y,t) = J_Y\, u(X(y,t),t) \quad \text{and} \quad q(y,t) = p(X(y,t),t).$$

The method

- Writing the equation for (U, q) or (v, q), we obtain a strongly nonlinear coupled system.
- Linearize the system.
- Prove regularity results for the linearized system.
- Prove the existence of strong solution to the nonlinear system by a fixed point method.

Result by Coutand and Shkoller

If $w_0 \in H^3(\Omega_S)$, $w_1 \in H^2(\Omega_S)$, $u_0 \in H^5(\Omega)$ + C.C., then the solution to the nonlinear coupled system exists for $T^* > 0$,

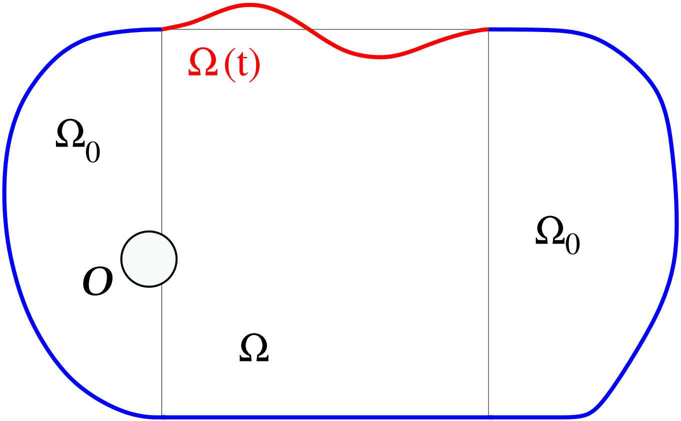
$v \in L^\infty(0,T;H^2(\Omega_F)) \cap W^{2,\infty}(0,T;L^2(\Omega_F))$,
$\int_0^\cdot v \in L^\infty(0,T;H^2(\Omega_F))$ and
$w \in L^\infty(I_T;H^2(\Omega_S)) \cap W^{1,\infty}(I_T;H^1(\Omega_S)) \cap W^{2,\infty}(I_T;L^2(\Omega_S))$.

For the linearized system
$v \in L^2(0,T;H^3(\Omega_F)) \cap H^2(0,T;H^1(\Omega_F))$.

Result by Avalos, Lasiecka, Triggiani '08

If $w_0 \in H^2(\Omega_S)$, $w_1 \in H^1(\Omega_S)$, $u_0 \in H^{3/2}_{\Gamma_E}(\Omega_F)$ + C.C., then the solution to the linearized system obeys
$w \in L^\infty(0,T;H^2(\Omega_S))$, $v \in L^2(0,T;H^2(\Omega_F))$ and
$p \in L^2(0,T;H^1(\Omega_F))$.

2. N.S.E. coupled with a beam or a plate equation



Notation

Domain occupied by the fluid at time t

$$\Omega_{\eta(t)} = \Big\{(x,y) \mid x \in (0,L),\ 0 < y < 1 + \eta(x,t)\Big\} \cup \Omega_0.$$

Boundary corresponding to the structure at time t

$$\Gamma_{s,\eta(t)} = \Big\{(x,y) \mid x \in (0,L),\ y = 1 + \eta(x,t)\Big\}.$$

The space-time domain and boundary

$$\widetilde{Q}_T = \bigcup_{t\in(0,T)} \Omega_{\eta(t)}, \quad \widetilde{\Sigma}_s^T = \bigcup_{t\in(0,T)} \Gamma_{s,\eta(t)}, \quad \Sigma_0^T = \Gamma_0 \times (0,T).$$

We have

$$0 = \int_{\Omega_{\eta(t)}} \operatorname{div} u = \int_{\Gamma_{s,\eta(t)}} u(t) \cdot n(t) = \int_0^L \eta_t = \int_{\Gamma_s} \eta_t ,$$

because

$$n(t) = \left(\frac{-\eta_x}{\sqrt{1+\eta_x^2}}, \frac{1}{\sqrt{1+\eta_x^2}} \right)^T .$$

We choose $\eta_{1,0}$ and $\eta_{2,0}$ the In. Cond. in

$$L^2_0(\Gamma_s) = \Big\{ \eta \mid \int_{\Gamma_s} \eta = 0 \Big\},$$

and we denote by M the orthogonal projection on $L^2_0(\Gamma_s)$.

The fluid equation

$$u_t + (u\cdot\nabla)u - \operatorname{div}\sigma(u,p) = f, \quad \operatorname{div} u = 0 \quad \text{in } \widetilde{Q}_\infty,$$

$$u = \eta_t \vec{e}_2 \ \text{ on } \widetilde{\Sigma}^\infty, \quad u = 0 \ \text{ on } \Sigma_0^\infty, \quad u(0) = u_0 \text{ in } \Omega_0,$$

$$\sigma(u,p) = \nu(\nabla u + \nabla u^T) - p\, I.$$

The structure equation

$$\eta_{tt} - \beta\eta_{xx} - \delta\eta_{txx} + \alpha M\eta_{xxxx} = M(p + H(u,\eta)) \ \text{ on } \Sigma_s^\infty,$$

$$\eta = 0 \quad \text{and} \quad \eta_x = 0 \ \text{ on } \{0, L\} \times (0,\infty)$$

$$\eta(0) = \eta_{1,0} \quad \text{and} \quad \eta_t(0) = \eta_{2,0} \ \text{ in } \Gamma_s,$$

and

$$H(u,\eta) = -\nu(\nabla u + \nabla u^T)(-\eta_x\vec{e}_1 + \vec{e}_2)\cdot\vec{e}_2.$$

We make the change of variable

$$(x,y) \longmapsto (x,z) = \left(x, \frac{y}{1+\eta(x,t)}\right),$$

transforms $\Omega_{\eta(t)}$ onto $\Omega = (0,L)\times(0,1)\cup\Omega_0$. Setting

$$\hat{u}(x,z,t) = u(x,y,t), \quad \hat{p}(x,z,t) = p(x,y,t),$$

the nonlinear system is rewritten in the form

$$\hat{u}_t + (\hat{u}\cdot\nabla)\hat{u} - \nu\Delta\hat{u} - \nabla\hat{p} = \hat{F}(\cdot) + \hat{f}, \quad \mathrm{div}\,\hat{u} = \hat{G}(\eta,\hat{u})$$

$$\hat{u} = \eta_t\vec{e}_2 \ \text{ on } \Sigma_s^\infty, \quad \hat{u} = 0 \ \text{ on } \Sigma_0^\infty, \quad \hat{u}(0) = u_0 \text{ in } \Omega,$$

$$\eta_{tt} - \beta\eta_{xx} - \delta\eta_{txx} + \alpha\eta_{xxxx} = \hat{p} + \hat{H}(\hat{u},\eta) \ \text{ on } \Sigma_s^\infty,$$

$$\eta(0) = \eta_{1,0} \quad \text{and} \quad \eta_t(0) = \eta_{2,0} \ \text{ in } \Gamma_s,$$

where

$$\begin{aligned}
&\hat{F}(\eta, \hat{u}, \nabla \hat{p}) \\
&= -\eta \hat{u}_t + \left(z\eta_t + \nu z \left(\tfrac{\eta_x^2}{1+\eta} - \eta_{xx} \right) \right) \hat{u}_z \\
&\quad + \nu \left(-2z\eta_x \hat{u}_{xz} + \eta \hat{u}_{xx} + \left(\tfrac{z^2\eta_x^2 - \eta}{1+\eta} \right) \hat{u}_{zz} \right) \\
&\quad + z(\eta_x \hat{p}_z - \eta \hat{p}_x)\vec{e}_1 - (1+\eta)\hat{u}_1 \hat{u}_x + (z\eta_x \hat{u}_1 - \hat{u}_2)\hat{u}_z,
\end{aligned}$$

$$\hat{G}(\eta, \hat{u}) = -\eta \hat{u}_{1,x} + z\eta_x \hat{u}_{1,z} = \operatorname{div}\left(-\eta \hat{u}_1 \vec{e}_1 + z\eta_x \hat{u}_1 \vec{e}_2 \right),$$

and

$$\hat{H}(\hat{u}, \hat{p}, \eta) = \nu \left(\tfrac{\eta_x}{1+\eta} \hat{u}_{1,z} + \eta_x \hat{u}_{2,x} - \tfrac{2+\eta_x^2}{1+\eta} \hat{u}_{2,z} \right).$$

The linearized system (around 0 in 2D) is

$$v_t - \operatorname{div} \sigma(v, p) = f,$$

$$\operatorname{div} v = 0 \quad \text{in } Q_\infty,$$

$$v = \eta_2 \vec{e}_2 \ \text{ on } \Sigma_s^\infty, \quad v = 0 \ \text{ on } \Sigma_0^\infty, \quad v(0) = v_0 \text{ in } \Omega,$$

$$\eta_{1,t} = \eta_2 \quad \text{on } \Sigma_s^\infty,$$

$$\eta_{2,t} - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha\eta_{1,xxxx} = p \ \text{ on } \Sigma_s^\infty,$$

$$+ B.C.$$

$$\eta_1(0) = \eta_{1,0} \quad \text{and} \quad \eta_2(0) = \eta_{2,0} \ \text{ in } \Gamma_s.$$

Observe that

$$v_{1,x} + v_{2,z} = 0 \quad \text{implies} \quad v_{2,z}|_{\Gamma_s} = 0.$$

The linearized system in 3D is

$$v_t - \operatorname{div} \sigma(v, p) = f,$$

$$\operatorname{div} v = 0 \quad \text{in } Q_\infty,$$

$$v = \eta_2 \vec{e}_3 \ \text{ on } \Sigma_s^\infty, \quad v = 0 \ \text{ on } \Sigma_0^\infty, \quad v(0) = v_0 \text{ in } \Omega,$$

$$\eta_{1,t} = \eta_2 \quad \text{on } \Sigma_s^\infty,$$

$$\eta_{2,t} - \beta\Delta_s \eta_1 - \delta\Delta_s \eta_2 + \alpha\Delta_s^2 \eta_1 = p \ \text{ on } \Sigma_s^\infty,$$

$$+\ B.C.$$

$$\eta_1(0) = \eta_{1,0} \quad \text{and} \quad \eta_2(0) = \eta_{2,0} \ \text{ in } \Gamma_s.$$

Definition of the semigroup

$$\mathbf{V}^0_n(\Omega) = \Big\{ y \in \mathbf{L}^2(\Omega) \mid \operatorname{div} y = 0,\ \ y \cdot n = 0 \text{ on } \Gamma \Big\},$$

$$\mathbf{L}^2(\Omega) = \mathbf{V}^0_n(\Omega) \oplus \operatorname{grad} H^1(\Omega),$$

$$P \ : \ \mathbf{L}^2(\Omega) \longmapsto \mathbf{V}^0_n(\Omega).$$

We denote by A_0 either the Stokes operator in $\mathbf{V}^0_n(\Omega)$ with domain

$$\mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega) = \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega) \cap \mathbf{V}^0_n(\Omega),$$

or its extension to $(\mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega))'$ (by extrapolation) as an unbounded operator with domain $\mathbf{V}^0_n(\Omega)$.

Take $f = 0$. The equation satisfied by v can be split

$$Pv_t = A_0 Pv + (-A_0)PD(\eta_2 \vec{e}_2\, \chi_{\Gamma_s}) \quad \text{in} \quad L^2(0, T; (D(A_0))'),$$

$$v(0) = v_0 \quad \text{in} \quad \Omega,$$

$$(I - P)v = (I - P)D(\eta_2 \vec{e}_2\, \chi_{\Gamma_s}) = \nabla q,$$

$$(I - P)v_t(t) = (I - P)D(\eta_{2,t}(t)\vec{e}_2\, \chi_{\Gamma_s}),$$

$$(I - P)v_0 = (I - P)v(0) \qquad \text{i.e.} \qquad v_0 \cdot \vec{e}_2 = \eta_2^0 \quad \text{on} \quad \Gamma_s,$$

and $D(\eta_2 \vec{e}_2\, \chi_{\Gamma_s}) = w$ is the solution to

$$-\nu \Delta w + \nabla \rho = 0, \quad \mathrm{div}\, w = 0 \text{ in } \Omega, \quad w = (\eta_2 \vec{e}_2\, \chi_{\Gamma_s}) \text{ on } \Gamma.$$

Idea: Eliminate $(I - P)v$ in the beam equation.

The pressure $p = \pi - q_t$ where

$$\Delta\pi = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial\pi}{\partial n} = \Delta P v \cdot n \quad \text{on} \quad \Gamma,$$

$$\Delta q(t) = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial q}{\partial n} = \eta_2 \quad \text{on} \quad \Gamma_s, \quad \frac{\partial q}{\partial n} = 0 \quad \text{on} \quad \Gamma_0.$$

Let $N \in \mathcal{L}(L^2_0(\Gamma_s), H^{3/2}(\Omega))$ and $N_0 \in \mathcal{L}(H^{-1/2}(\Gamma), H^1(\Omega))$ be

$$N_0(\Delta P v \cdot n) = \pi \quad \text{and} \quad N(\eta_2) = q,$$

and let $\gamma_s \in \mathcal{L}(H^1(\Omega), L^2_0(\Gamma_s))$ be

$$\gamma_s q = q|_{\Gamma_s} - \frac{1}{|\Gamma_s|}\int_{\Gamma_s} q.$$

For $f = 0$, we rewrite the system in the form

$$Pv_t = A_0 Pv + (-A_0)PD(\eta_2 \vec{e}_2\, \chi_{\Gamma_s}), \quad v(0) = v_0 \quad \text{in} \quad \Omega,$$

$$(I - P)v = (I - P)D(\eta_2 \vec{e}_2\, \chi_{\Gamma_s}) = \nabla q,$$

$$\eta_{1,t} = \eta_2,$$

$$\eta_{2,t} - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha\eta_{1,xxxx}$$
$$= -\gamma_s N(\eta_{2,t}) + \gamma_s N_0(\Delta Pv \cdot n) + f \quad \text{on } \Sigma_\infty^s,$$

$$\eta_1(0) = \eta_{1,0} \quad \text{and} \quad \eta_2(0) = \eta_{2,0} \quad \text{in } \Gamma_s.$$

The equation satisfied by η_2 is

$$(I + \gamma_s N)\eta_{2,t} - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha\eta_{1,xxxx}$$
$$= \gamma_s N_0(\Delta Pv \cdot n) \quad \text{on } \Sigma_\infty^s,$$

We can rewrite the system in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{A}\begin{pmatrix} Pv \\ \eta_1 \\ \eta_2 \end{pmatrix},$$

with

$$\mathcal{A} =$$

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (I+\gamma_s N)^{-1} \end{pmatrix}\begin{pmatrix} A_0 & 0 & (-A_0)PD \\ 0 & 0 & I \\ \gamma_s N_0(\Delta(\cdot)\cdot n) & A_{\alpha,\beta} & \delta\Delta_s \end{pmatrix}$$

$$D(A_{\alpha,\beta}) = H^4(\Gamma_s)\cap H^2_0(\Gamma_s)\cap L^2_0(\Gamma_s), \quad A_{\alpha,\beta}\eta = \beta\eta_{xx} - \alpha\eta_{xxxx},$$

and in $H = \mathbf{V}_n^0(\Omega) \times (H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)) \times L_0^2(\Gamma_s)$, we have

$$D(\mathcal{A}) = \Big\{(Pv, \eta_1, \eta_2) \in \mathbf{V}_n^2(\Omega) \times (H^4 \cap H_0^2 \cap L_0^2)(\Gamma_s) \times (H_0^2 \cap L_0^2)(\Gamma_s) \\ \mid Pv|_\Gamma = -\nabla_\tau (I-P) D(\eta_2\, \vec{e}_2\, \chi_{\Gamma_s})\Big\}.$$

The compatibility condition

$$Pv|_\Gamma = -\nabla_\tau (I-P) D(\eta_2 \vec{e}_2\, \chi_{\Gamma_s})$$

with

$$(I-P)v = (I-P)D(\eta_2 \vec{e}_2\, \chi_{\Gamma_s})$$

is equivalent to

$$v = \eta_2\, \vec{e}_2 \chi_{\Gamma_s} \quad \text{on} \quad \Gamma.$$

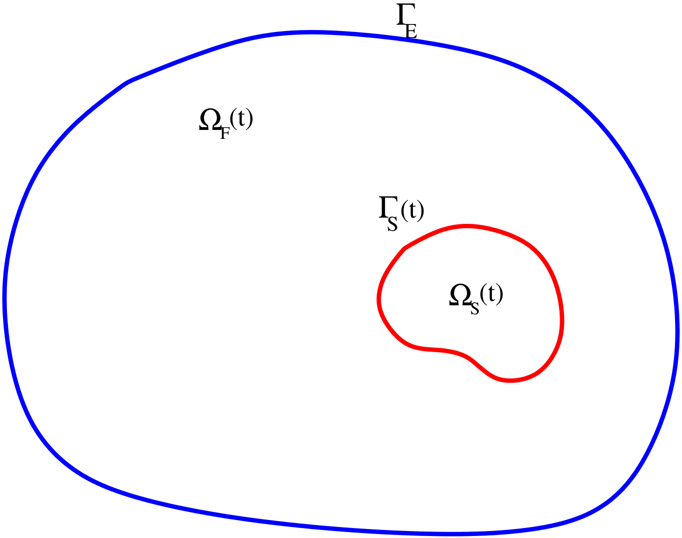
Theorem. (JPR, 09) The operator $(\mathcal{A}, D(\mathcal{A}))$ is the infinitesimal generator of an analytic semigroup on $H = \mathbf{V}_n^0(\Omega) \times (H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)) \times L_0^2(\Gamma_s)$ and its resolvent is compact.

Moreover if $u_0 \in H^1(\Omega)$, $\eta_{1,0} \in H^3(\Gamma_s) \cap H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)$, $\eta_{2,0} \in H_0^1(\Gamma_s) \cap L_0^2(\Gamma_s)$, satisfying some C.C., the solution to the linearized system belongs to $H^{2,1}(\Omega \times (0, T)) \times H^{4,2}(\Sigma_s) \times H^{2,1}(\Sigma_s)$.

Theorem. (2D, Lequeure '10) For all $u_0 \in H^1(\Omega)$, $\eta_{1,0} \in H^3(\Gamma_s) \cap H^2_0(\Gamma_s) \cap L^2_0(\Gamma_s)$, $\eta_{2,0} \in H^1_0(\Gamma_s) \cap L^2_0(\Gamma_s)$, satisfying some C.C., there exists $T^* > 0$ such that the system coupling the N.S.E. with the damped beam eq. admits a unique solution in $H^{2,1} \times H^{4,2}(\Sigma_s) \times H^{2,1}(\Sigma_s)$.

The extension to the 3D case is in progress.

3. N.S.E. coupled with the Lamé system



The symmetric stress tensor σ and the strain tensor ε

$$\sigma(w) = \lambda \operatorname{tr} \varepsilon(w)\, I + 2\mu\, \varepsilon(w),$$

$$\varepsilon(w) = \frac{1}{2}\left(\nabla w + (\nabla w)^T\right).$$

The stress tensor σ of the fluid

$$\sigma(u,p) = \nu\left(\nabla u + (\nabla u)^T\right) - pI.$$

The flow of the fluid velocity

$$X(\cdot,t)\ :\ \Omega_F(0) \longmapsto \Omega_F(t),$$

$$X_t(y,t) = u(X(y,t),t), \quad X(y,0) = y, \qquad y \in \Omega_F(0).$$

The fluid equation

$$u_t + (u \cdot \nabla) u - \operatorname{div} \sigma(u,p) = 0, \quad \operatorname{div} u = 0 \quad \text{in } \Omega_F(t),\ t > 0$$

$$u = 0 \ \text{ on } \Gamma_E \times (0,T) = \Sigma_E, \quad u(0) = u_0 \text{ in } \Omega_F(0),$$

$$u(X(y,t),t) = w_t(y,t) \quad \text{on } \Sigma_S. \quad \text{(eq. of velocities)}$$

The Lamé system

$$w_{tt} - \operatorname{div} \sigma(w) =$$

$$w_{tt} - \mu \Delta w + (\lambda + \mu) \nabla(\operatorname{div} w) = 0 \ \text{ in } \Omega_S(0) \times (0,T),$$

$$w(0) = w_0 \quad \text{and} \quad w_t(0) = w_1 \ \text{ in } \Omega_S(0),$$

$$\sigma(w) n = \sigma(u,p) \circ X \operatorname{cof}(\nabla X) n = \sigma(u,p) \circ X\, \tilde{n} \ \text{ on } \Sigma_S.$$

The nonlinear model in the cylindrical domain $\Omega_F(0) \times (0, T)$

The Lagrangian velocity and the Lagrangian pressure

$$v(y,t) = u(X(y,t),t) \quad \text{and} \quad q(y,t) = p(X(y,t),t),$$

$$X(Y(x,t),t) = x, \quad x \in \Omega_F(t),$$

$$Y(X(y,t),t) = y, \quad y \in \Omega_F(0).$$

$$v_t - \nu\nabla\,\left(J_Y^T \nabla v + \nabla v^T J_Y\right)\cdot J_Y + J_Y^T \nabla q = 0,$$

$$\nabla\, v \cdot J_Y = 0 \quad \text{in } \Omega_F(0) \times (0, T),$$

$$v = 0 \ \text{ on } \Sigma_E, \quad v(0) = u_0 \text{ in } \Omega_F(0),$$

$$v(y,t) = w_t(y,t) \quad \text{on } \Sigma_S,$$

$$\sigma(w)n = \nu\left(J_Y^T \nabla v + \nabla v^T J_Y\right)\tilde{n} - q\tilde{n} \ \text{ on } \Sigma_S.$$

The nonlinear model in the cylindrical domain $\Omega_F(0) \times (0, T)$ for a general transformation X and Y

$$v_t - \operatorname{div} \sigma(v, p) = f, \qquad \operatorname{div} v = g \quad \text{in } \Omega_F(0) \times (0, T),$$

$$v = 0 \;\; \text{on } \Sigma_E, \quad v(0) = u_0 \text{ in } \Omega_F(0),$$

$$v(y, t) = w_t(y, t) \quad \text{on } \Sigma_S,$$

and

$$\sigma(w)n + h = \sigma(v, q)n \;\; \text{on } \Sigma_S,$$

$$X_t(y, t) = \theta(y)\, v(y, t), \quad X(y, 0) = y, \quad y \in \Omega_S(0).$$

where

$$
\begin{aligned}
f &= f(v, q) \\
&= \mathcal{F}\left(\frac{\partial^2 Y_k}{\partial x_j^2}, \frac{\partial v_i}{\partial y_k}; \frac{\partial Y_l}{\partial x_j}, \frac{\partial Y_k}{\partial x_j} \frac{\partial^2 v_i}{\partial y_l \partial y_k}; \frac{\partial Y_j}{\partial t}, \frac{\partial v_i}{\partial y_j}; \right. \\
&\qquad\qquad\qquad\qquad \left. \frac{\partial Y_k}{\partial x_j}, v_j, \frac{\partial v_i}{\partial y_k}; \frac{\partial Y_j}{\partial x_i}, \frac{\partial q}{\partial y_j} \right), \\
g &= g(v) = (I - J_Y^T) : \nabla v, \\
h &= h(v, q) = \mathcal{H}\left(\frac{\partial Y_j}{\partial x_i}, \frac{\partial v_i}{\partial y_k}; \frac{\partial Y_j}{\partial x_i} \frac{\partial v_k}{\partial y_j}; \frac{\partial Y_j}{\partial x_i}, q \right) \\
&= \nu(J_Y^T \nabla v + (\nabla v)^T J_Y)n - \nu(\nabla v + (\nabla v)^T)n + q(n - \operatorname{cof}(\nabla X)n).
\end{aligned}
$$

There is a better way for writing $h(v, q)$.

The linearized model

$$v_t - \operatorname{div} \sigma(v, q) = f,$$

$$\operatorname{div} v = g \quad \text{in } \Omega_F(0),\ t > 0$$

$$v = 0 \ \text{ on } \Sigma_E, \quad v(0) = u_0 \text{ in } \Omega_F(0),$$

$$v(y, t) = w_t(y, t), \quad \text{on } \Sigma_S,$$

$$w_{tt} - \mu \Delta w + (\lambda + \mu) \nabla(\operatorname{div} w) = 0 \ \text{ in } \Omega_S(0) \times (0, T),$$

$$w(0) = w_0 \quad \text{and} \quad w_t(0) = w_1 \ \text{ in } \Omega_S(0),$$

$$\sigma(w) n + h = \sigma(v, q) n \ \text{ on } \Sigma_S.$$

Idea for proving regularity results useful for studying the nonlinear system

- Prove optimal regularity results for the Lamé system

- Prove optimal regularity results for the Stokes system

- Determine the spaces in which we can prove the existence of a fixed point

A new writing of the linearized model

$$v_t - \operatorname{div}\sigma(v,q) = f,$$

$$\operatorname{div} v = g \quad \text{in } \Omega_F(0),\ t>0$$

$$v = 0 \ \text{ on } \Sigma_E, \quad v(0) = u_0 \text{ in } \Omega_F(0),$$

$$w(y,t) = w_0 + \int_0^t v, \quad \text{on } \Sigma_S,$$

$$w_{tt} - \mu\Delta w + (\lambda+\mu)\nabla(\operatorname{div} w) = 0 \ \text{ in } \Omega_S(0)\times(0,T),$$

$$w(0) = w_0 \quad \text{and} \quad w_t(0) = w_1 \ \text{ in } \Omega_S(0),$$

$$\sigma(w)n + h = \sigma(v,q)n \ \text{ on } \Sigma_S.$$

Analysis of the Lamé system

$$w_{tt} - \mu\Delta w + (\lambda + \mu)\nabla(\operatorname{div} w) = F \quad \text{in } \Omega_S \times (0, T),$$

$$w = G = w_0|_{\Gamma_S} + \int_0^t v \quad \text{on } \Sigma_S,$$

$$w(0) = w_0 \quad \text{and} \quad w_t(0) = w_1 \quad \text{in } \Omega_S.$$

Theorem. (B. Dehman, JPR, '10)

If $G \in H^{r+1}(\Sigma_S)$, $w_0 \in H^{r+1}(\Omega_S)$, $w_1 \in H^r(\Omega_S)$ + C.C., then

$$\sigma(w) \in H^r(\Sigma_S).$$

Theorem. (B. Dehman, JPR, '10)

If

$$\begin{aligned}
&F \in L^1(0,T;H^2(\Omega_S)) \cap W^{1,1}(0,T;H^1(\Omega_S)) \cap W^{2,1}(0,T;L^2(\Omega_S)),\\
&w_0 \in H^3(\Omega_S), \quad w_1 \in H^2(\Omega_S),\\
&G \in H^3(\Sigma_S), \quad G|_{t=0} = w_0|_{\Gamma_S}, \quad G_t|_{t=0} = v_0|_{\Gamma_S} = w_1|_{\Gamma_S}\\
&\text{and} \quad G_{tt}|_{t=0} = (\Delta w_0 + F(0))|_{\Gamma_S},
\end{aligned}$$

then the solution to the Lamé system satisfies

$$\begin{aligned}
w \in\ & C([0,T];H^3(\Omega_S)) \cap C^1([0,T];H^2(\Omega_S))\\
&\cap C^2([0,T];H^1(\Omega_S)) \cap C^3([0,T];L^2(\Omega_S))
\end{aligned}$$

and

$$\sigma(w) \in H^2(\Sigma_S).$$

Idea of proof. (Lasiecka, Lions, Triggiani, waves '86)

If

$$F \in L^1(0,T;L^2(\Omega_S)), \quad w_0 \in H^1(\Omega_S), \quad w_1 \in L^2(\Omega_S),$$
$$G \in H^1(\Sigma_S), \quad G|_{t=0} = w_0|_{\Gamma_S},$$

then the solution to the Lamé system satisfies

$$w \in C([0,T];H^1(\Omega_S)) \cap C^1([0,T];L^2(\Omega_S)),$$
$$\frac{\partial w}{\partial n},\ \operatorname{div} w|_{\Sigma_S} \text{ and } \sigma(w) \in L^2(\Sigma_S).$$

The proof is based on energy estimates with a multiplier $(T-t)H \cdot \nabla w$, where $H = n$ on Γ_S.

Boundary estimate by the energy

$$\mu \int_0^T \int_{\Gamma_S} \left| \frac{\partial w}{\partial n} \right|^2 + \frac{\mu + \lambda}{2} \int_0^T \int_{\Gamma_S} |\operatorname{div} w|^2 \leq CE(0).$$

To improve the regularity

$$\frac{\partial w}{\partial n},\ \operatorname{div} w|_{\Sigma_S} \text{ and } \sigma(w) \in H^1(\Sigma_S),$$

the idea is to write the equation satisfied by w_t, it gives

$$\frac{\partial w}{\partial n},\ \operatorname{div} w|_{\Sigma_S},\ \text{ and } \sigma(w) \in H^1(0,T;L^2(\Gamma_S)).$$

By writing the equation satisfied by Hw, where $H = \Sigma_i h_i \partial_{x_i}$ with $\Sigma_i h_i n_i = 0$, we prove that

$$\frac{\partial w}{\partial n},\ \operatorname{div} w|_{\Sigma_S} \text{ and } \sigma(w) \in L^2(0,T;H^1(\Gamma_S)).$$

First consequence (Avalos, Lasiecka, Triggiani '08) If $w_0 \in H^2(\Omega_S)$, $w_1 \in H^1(\Omega_S)$, $u_0 \in H^{3/2}_{\Gamma_E}(\Omega_F)$, then the solution to the linearized system obeys $w \in L^\infty(0,T; H^2(\Omega_S))$, $v \in L^2(0,T; H^2(\Omega_F))$ and $p \in L^2(0,T; H^1(\Omega_F))$.

Idea of the proof.

If $w_0 \in H^2(\Omega_S)$, $w_1 \in H^1(\Omega_S)$, and $\int_0^t v \in H^2(\Sigma_S)$, then $\sigma(w) \in H^1(\Sigma_S)$.

If $u_0 \in H^{3/2}_{\Gamma_E}(\Omega)$ + some C.C., since $\sigma(w) \in H^1(\Sigma_S)$, then $v \in L^2(0,T; H^{5/2}(\Omega_F)) \cap H^{5/4}(0,T; L^2(\Omega_F))$.

Analysis of the Stokes system

$$v_t - \operatorname{div} \sigma(v, q) = f,$$

$$\operatorname{div} v = g \quad \text{in } \Omega_F \times (0, T),$$

$$v = 0 \ \text{ on } \Sigma_E, \quad v(0) = v_0 \quad \text{in } \Omega_F,$$

$$\sigma(v, q)n = \sigma(w) + h = \widehat{h} \ \text{ on } \Sigma_S.$$

With the intial data

$$\begin{aligned} f(0) = -\nu \left(\Delta Y_k \, \partial_k v_0 + \partial_j Y_\ell \partial_j Y_k \partial^2_{j,k} v_0 - \Delta v_0 \right) \\ - \partial_t Y_j \, \partial_j v_0 - \partial_j Y_k \; v_{0,j} \, \partial_k v_0, \end{aligned}$$

$$g(0) = 0, \quad g_t(0) = -J^T_{Y,t} : \nabla v_0 = -I : \nabla v_0 = 0,$$

$$\begin{aligned} g_{tt}(0) &= -J^T_{Y,tt} : \nabla v_0 - J^T_{Y,t} : \nabla v_t(0) \\ &= -2 \nabla v_0^T : \nabla v_0 - I : \nabla v_t(0) = -2 \nabla v_0^T : \nabla v_0, \end{aligned}$$

$$\widehat{h}(0) = \sigma(w_0), \quad \widehat{h}_t(0) = \sigma(w_1) + \nu (J^T_{Y,t} \nabla v_0 + (\nabla v_0)^T J_{Y,t}).$$

Theorem.

If

$$f \in L^2(I_T; H^{3/2}(\Omega_F)) \cap H^{3/2}(I_T; L^2(\Omega_F)) \\ \qquad + L^2(I_T; H^{5/2}(\Omega_F)) \cap H^{1/2}(I_T; H^2(\Omega_F)) \cap H^1(I_T; H^1(\Omega_F)),$$

$f(0)$ is regular,

$$v_0 \in H^5(\Omega_F) \cap H^1_{\Gamma_E}(\Omega_F), \quad \operatorname{div} v_0 = 0,$$

$$\widehat{h} \in L^2(I_T; H^2(\Gamma_S)) \cap H^{3/2}(I_T; H^{1/2}(\Gamma_S)) \cap H^{7/4}(I_T; L^2(\Gamma_S)),$$

$$g \in L^2(I_T; H^{5/2}(\Omega_F)) \cap H^{3/2}(I_T; H^1(\Omega_F)) \cap H^2(I_T; L^2(\Omega_F)),$$

$$g(0) = 0, \quad g_t(0) = 0,$$

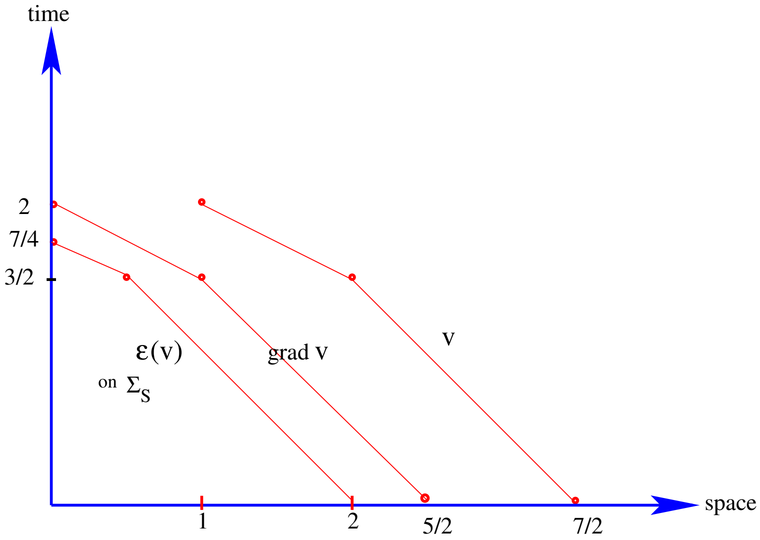
and if f, $\widehat{h}$, v_0 obeys some C.C. at $t = 0$,

then the solution to the Stokes system satisfies

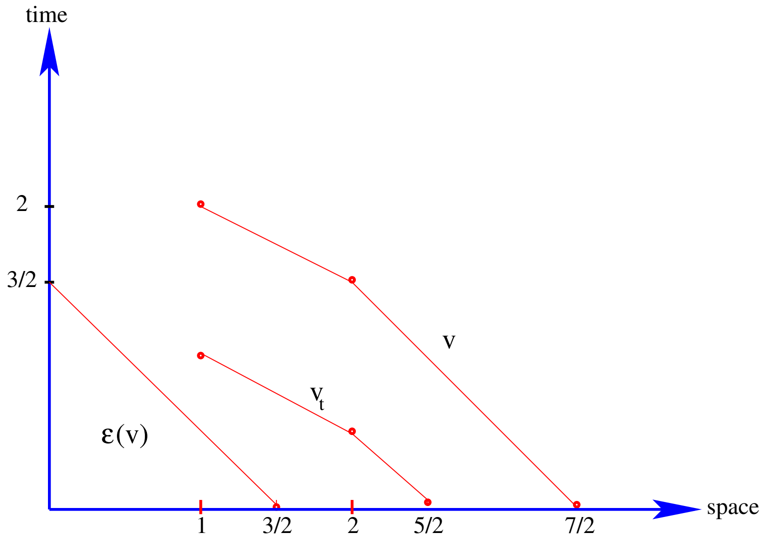
$$v \in L^2(0, T; H^{7/2}(\Omega_F)) \cap H^{3/2}(0, T; H^2(\Omega_F)) \cap H^2(0, T; H^1(\Omega_F))$$

∇q belongs to the same space as f.

Regularity of g and h



Regularity of f and ∇q



More generally

If

$$f \in L^2(I_T; H^{s-1/2}(\Omega_F)) \cap H^{s-1/2}(I_T; L^2(\Omega_F))$$
$$+L^2(I_T; H^{s+1/2}(\Omega_F)) \cap H^{s-3/2}(I_T; H^2(\Omega_F)) \cap H^{s-1}(I_T; H^1(\Omega_F)),$$

$f(0)$ is regular,

$$v_0 \in H^{s+3}(\Omega_F) \cap H^1_{\Gamma_E}(\Omega_F), \quad \operatorname{div} v_0 = 0,$$

$$\widehat{h} \in L^2(I_T; H^s(\Gamma_S)) \cap H^{s-1/2}(I_T; H^{1/2}(\Gamma_S)) \cap H^{s-1/4}(I_T; L^2(\Gamma_S)),$$

$$g \in L^2(I_T; H^{s+1/2}(\Omega_F)) \cap H^{s-1/2}(I_T; H^1(\Omega_F)) \cap H^s(I_T; L^2(\Omega_F)),$$

$$g(0) = 0, \quad g_t(0) = 0,$$

and if f, $\widehat{h}$, v_0 obeys some C.C. at $t = 0$,

then the solution to the Stokes system satisfies

$$v \in L^2(I_T; H^{s+3/2}(\Omega_F)) \cap H^{s-1/2}(I_T; H^2(\Omega_F)) \cap H^s(I_T; H^1(\Omega_F))$$

∇q belongs to the same space as f.

We look for the solution v in the form $v = \widetilde{v} + \widehat{v}$ where

$$\operatorname{div} \widetilde{v} = g \quad \text{and} \quad \operatorname{div} \widehat{v} = 0 \qquad \text{in } \Omega_F \times (0, T).$$

Equation for $\widehat{v}$

$$\widehat{v}_t - \operatorname{div} \sigma(\widehat{v}, \widehat{q}) = f,$$

$$\operatorname{div} \widehat{v} = 0 \qquad \text{in } \Omega_F \times (0, T),$$

$$\widehat{v} = 0 \ \text{ on } \Sigma_E, \quad \widehat{v}(0) = v_0 \qquad \text{in } \Omega_F,$$

$$\sigma(\widehat{v}, \widehat{q})n = \sigma(w) + h = \widehat{h} \ \text{ on } \Sigma_S.$$

Maximal space regularity for $\widehat{v}$ (Grubb, Solonnikov, 91)

If

$$f \in L^2(0,T;H^{3/2}(\Omega_F)) \cap H^{3/4}(0,T;L^2(\Omega_F)), \quad f(0) \text{ is regular},$$

$$v_0 \in H^1_{\Gamma_E}(\Omega_F) \cap H^{5/2}(\Omega_F), \quad \operatorname{div} v_0 = 0,$$

$$\widehat{h} \in L^2(0,T;H^2(\Gamma_S)) \cap H^1(0,T;L^2(\Gamma_S)),$$

$$2\nu\,(\varepsilon(v_0)n)\cdot\tau = \widehat{h}(0)n\cdot\tau = \sigma(w_0)n\cdot\tau \quad \text{on } \Gamma_S,$$

+ an additional C.C. for f

then the solution to the Stokes system satisfies

$$\widehat{v} \in L^2(0,T;H^{7/2}(\Omega_F)) \cap H^{7/4}(0,T;L^2(\Omega_F)),$$

$$\widehat{q} \in L^2(0,T;H^{5/2}(\Omega_F)) \cap H^{3/4}(0,T;H^1(\Omega_F)).$$

With the compatibility condition

$$2\nu\,(\varepsilon(v_0)n)\cdot\tau = h(0)n\cdot\tau = \sigma(w_0)n\cdot\tau,$$

we define q_0 as the solution to

$$\begin{aligned}
&\Delta q_0 = \operatorname{div} f(0) \quad \text{in } \Omega_F,\\
&\frac{\partial q_0}{\partial n} = f(0)\cdot n + 2\nu \operatorname{div}\varepsilon(v_0)\cdot n \ \ \text{on } \Gamma_E,\\
&q_0 = 2\nu\,(\varepsilon(v_0)n)\cdot n \ \ \text{on } \Gamma_S.
\end{aligned}$$

We can verify that

$$\widehat{q}(0) = q_0.$$

Maximal time regularity for $\widehat{v}$

We set

$$\widehat{v}_t = \widehat{u} \quad \text{and} \quad \widehat{q}_t = \widehat{p}.$$

$$\widehat{u}_t - \operatorname{div} \sigma(\widehat{u}, \widehat{p}) = f_t,$$

$$\operatorname{div} \widehat{u} = 0 \quad \text{in } \Omega_F \times (0, T),$$

$$\widehat{u} = 0 \ \text{ on } \Sigma_E, \quad \widehat{u}(0) = \operatorname{div} \sigma(v_0, q_0) + f(0) \quad \text{in } \Omega_F,$$

$$\sigma(u, p)n = \widehat{h}_t \ \text{ on } \Gamma_S \times (0, T).$$

Maximal regularity for $\widehat{u}$

If

$$f_t \in L^2(0,T;H^{1/2}(\Omega_F)) \cap H^{1/4}(0,T;L^2(\Omega_F)),$$

$$\widehat{u}(0) \in H^1_{\Gamma_E}(\Omega_F) \cap H^{3/2}(\Omega_F),$$

$$\widehat{h}_t \in L^2(0,T;H^1(\Gamma_S)) \cap H^{1/2}(0,T;L^2(\Gamma_S)),$$

$$2\nu\,(\varepsilon(\widehat{u}(0))n)\cdot\tau = \widehat{h}_t(0)n\cdot\tau = \sigma(w_1)n\cdot\tau \quad \text{on } \Gamma_S,$$

then the solution to the Stokes system satisfies

$$\widehat{u} \in L^2(0,T;H^{5/2}(\Omega_F)) \cap H^{5/4}(0,T;L^2(\Omega_F))$$

and

$$v \in H^1(0,T;H^{5/2}(\Omega_F)) \cap H^{9/4}(0,T;L^2(\Omega_F)).$$

We have not obtained the desired result, but we have not used

$$f_t \in H^{1/2}(0, T; L^2(\Omega_F))$$

and

$$\widehat{h}_t \in H^{1/2}(0, T; H^{1/2}(\Gamma_S)) \cap H^{3/4}(0, T; L^2(\Gamma_S)).$$

To recover the desired regularity we proceed by interpolation.

New time derivative

We set

$$\widehat{u}_t = \zeta \quad \text{and} \quad \widehat{p}_t = \chi.$$

$$\zeta_t - \operatorname{div}\sigma(\zeta, \chi) = \widehat{f}_{tt},$$

$$\operatorname{div}\zeta = 0 \quad \text{in } \Omega_F \times (0, T),$$

$$\zeta = 0 \ \text{ on } \Sigma_E, \quad \zeta(0) = \operatorname{div}\sigma(\widehat{u}_0, \widehat{p}_0) + \widehat{f}_t(0) \quad \text{in } \Omega_F,$$

$$\sigma(\zeta, \chi)n = \widehat{h}_{tt} \ \text{ on } \Sigma_S \times (0, T).$$

Maximal regularity for ζ

If

$$\begin{aligned} &\widehat{f}_{tt} \in L^2(0,T;L^2(\Omega_F)), \\ &\zeta(0) \in H^1_{\Gamma_E}(\Omega_F), \\ &\widehat{h}_{tt} \in L^2(0,T;H^{1/2}(\Gamma_S)) \cap H^{1/4}(0,T;L^2(\Gamma_S)), \end{aligned}$$

then the solution to the Stokes system satisfies

$$\zeta \in L^2(0,T;H^2(\Omega_F)) \cap H^1(0,T;L^2(\Omega_F)).$$

We recover the final estimate for $\widehat{v}$ by interpolation

$$\begin{aligned} \widehat{v} \in\ & H^{3/2}(0,T;H^2(\Omega_F)) \cap H^{5/2}(0,T;L^2(\Omega_F)) \\ &\hookrightarrow H^{3/2}(0,T;H^2(\Omega_F)) \cap H^2(0,T;H^1(\Omega_F)). \end{aligned}$$

Fixed point method

Let (v_0, w_0, w_1) satisfy the regularity assumptions and the compatibility conditions. Let $\overline{v}$ and $\overline{q}$ be such that $(\overline{v}, \overline{q})(0) = (v_0, q_0)$ and

$$\begin{aligned}
&\overline{v} \in L^2(0,T;H^{7/2}(\Omega_F)) \cap H^{3/2}(0,T;H^2(\Omega_F)) \cap H^2(0,T;H^1(\Omega_F)) \\
&\nabla\overline{q} \in L^2(I_T;H^{3/2}(\Omega_F)) \cap H^{3/2}(I_T;L^2(\Omega_F)) \\
&\qquad +L^2(I_T;H^{5/2}(\Omega_F)) \cap H^{1/2}(I_T;H^2(\Omega_F)) \cap H^1(I_T;H^1(\Omega_F)).
\end{aligned}$$

We consider the solution (v, q, w) to the system

$$v_t - \operatorname{div} \sigma(v, q) = \overline{f} = f(\overline{v}, \overline{q}),$$

$$\operatorname{div} v = \overline{g} = g(\overline{v}) \quad \text{ in } \Omega_F \times (0, T),$$

$$v = 0 \ \text{ on } \Sigma_E, \quad v(0) = v_0,$$

$$\sigma(v, q)n = \sigma(w) + h(\overline{v}, \overline{q}) \ \text{ on } \Sigma_S,$$

$$w_{tt} - \mu \Delta w + (\lambda + \mu)\nabla(\operatorname{div} w) = 0 \ \text{ in } \Omega_S \times (0, T),$$

$$w = w_0|_{\Gamma_S} + \int_0^t v, \quad \text{ in } \Gamma_S \times (0, T) = \Sigma_S,$$

$$w(0) = w_0 \quad \text{and} \quad w_t(0) = w_1 \ \text{ in } \Omega_S,$$

$$\overline{X}_t(y, t) = \theta(y)\, \overline{v}(y, t), \quad \overline{X}_t(y, 0) = y,$$

$$\overline{u} = \overline{v} \circ \overline{Y}, \quad \overline{p} = \overline{q} \circ \overline{Y}.$$

For $T^* > 0$ small enough, depending only on (u_0, w_0, w_1), the mapping

$$M \ : \ (\overline{v}, \overline{q}) \longmapsto (v, q)$$

admits a unique fixed point in the space of functions $(\overline{v}, \overline{q})$ belonging to the space above defined, that is

$$\begin{aligned} & v \in L^2(0, T; H^{7/2}(\Omega_F)) \cap H^{3/2}(0, T; H^2(\Omega_F)) \cap H^2(0, T; H^1(\Omega_F)) \\ & \nabla q \in L^2(I_T; H^{3/2}(\Omega_F)) \cap H^{3/2}(I_T; L^2(\Omega_F)) \\ & \qquad + L^2(I_T; H^{5/2}(\Omega_F)) \cap H^{1/2}(I_T; H^2(\Omega_F)) \cap H^1(I_T; H^1(\Omega_F)). \end{aligned}$$

and satisfying the condition $(v, q)(0) = (v_0, q_0)$.

References

H. Beirao da Veiga, *On the existence of strong solutions to a coupled fluid-structure evolution system*, JMFM 6 (2004), 21-52.

A. Chambolle, B. Desjardins, M. J. Esteban, and C. Grandmont, *Existence of weak solutions for unsteady fluid-plate interaction problem*, J. Math. Fluid Mech. 7 (2005), 368-404.

C. Grandmont, *Existence of weak solutions for unsteady interaction of a viscous fluid with an elastic plate*, SIMA, 2008.

J. Lequeurre, *Existence of strong solutions to a fluid–structure system*, SIMA, to appear.

J-P. Raymond, *Feedback stabilization of a fluid–structure model*, SICON, to appear.

G. Avalos, I. Lasiecka, R. Triggiani, Higher regularity of a coupled parabolichyperbolic fluid structure interactive system, Georg. Math. J. 15(3) (2008).

M. Boulakia, S. Guerrero, *Regular solutions of a problem coupling a compressible fluid and an elastic structure*, 2010.

M. Boulakia, *Existence of weak solutions for the three dimensional motion of an emastic structure in an incompressible fluid*, J. Math. Fluid Mech. 9 (2007), 262-294.

C. Conca, J. San Martin, M. Tucsnak, *Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid*, CPDE 25 (2000), 1019-1042.

D. Coutand, S. Shkoller, Motion of an elastic solid inside an incompressible viscous fluid, ARMA, 2005.

B. Dehman, J-P. Raymond, *Control of the Lamé system*, in progress.

J-P. Raymond, *Stokes and Oseen equations with a nonhomogeneous divergence condition*, DCDS-B, 2010.

J-P. Raymond, *Strong solutions to a system coupling the Navier-Stokes equations and the Lamé system*, in progress.

J. San Martin, J-F. Scheid, T. Takahashi, M. Tucsnak, *An initial and boundary value problem modeling of fish-like swimming,* Arch. Rational Mech. Anal. 188 (2008),429-455.

T. Takahashi, *Analysis of strong solutions for the equations modelling the motion of a rigid-fluid system in a bounded domain,* Adv. Differ. Equ. 8 (2003),1499-1532.

T. Takahashi, M. Tucsnak, *Global strong solutions for the two-dimensional motion of an infinite cylinder in a viscous fluid*, J. Math. Fluid Mech. 6 (2004) 53-77.

End

Thank you