# Smoothing estimates for dispersive equations

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**ICMPDE Satellite, Bangalore, India**
**16 August 2010**

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This is joint work with **Mitsuru Sugimoto** (Nagoya University)

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  - Comparison principles to relate estimates in lower dimensions. This allows us to treat normal forms;
  - everything together $\Longrightarrow$ comprehensive analysis of smoothing estimates;

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  - Comparison principles to relate estimates in lower dimensions. This allows us to treat normal forms;
  - everything together $\Longrightarrow$ comprehensive analysis of smoothing estimates;
- Related further results: Strichartz type estimates, restriction theorems, equations with time–dependent coefficients or with potentials, critical case of the limiting absorption principle, nonlinear equations;

# General question

For real-valued function $a(\xi) \in \mathbb{R}$ and $D_x = -i\partial_x$, consider evolution equation

$$\begin{cases} (i\partial_t + a(D_x))\, u(t,x) = 0, \quad t \in \mathbb{R},\ x \in \mathbb{R}^n, \\ u(0,x) = \varphi(x). \end{cases}$$

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Basic question: when its solution $u(t,x) = e^{ita(D_x)}\varphi(x)$ has **space-time estimate**

$$\boxed{\|Au(t,x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n_x)}}$$

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since its solution $u(t,x)$ can be expressed as the linear combination of

$$e^{\pm ita(D_x)}\varphi_0, \qquad \frac{e^{\pm ita(D_x)}}{a(D_x)}\varphi_1.$$

The space-time estimate is one of the "fundamental estimates" for (nonlinear) equations that arise in various problems in different sciences:

- $a(\xi)=|\xi|^2 \Longrightarrow$ Schrödinger equation $\boxed{i\partial_t u-\Delta_x u=0}$
- $a(\xi)=|\xi| \Longrightarrow$ Wave equation $\boxed{\partial_t^2 u-\Delta_x u=0}$
- $a(\xi)=\sqrt{|\xi|^2+1} \Longrightarrow$ Relativistic Schrödinger equation

$$\boxed{i\partial_t u+\sqrt{1-\Delta_x}\,u=0}$$

- $a(\xi)=\sqrt{|\xi|^2+1} \Longrightarrow$ Klein–Gordon equation $\boxed{\partial_t^2 u-\Delta_x u+u=0}$
- $a(\xi)=\xi^3$ $(n=1)$ $\Longrightarrow$ Korteweg–de Vries (shallow water waves)

$$\boxed{\partial_t u+\partial_x^3 u+u\partial_x u=0}$$

- $a(\xi)=|\xi|\xi$ $(n=1)$ $\Longrightarrow$ Benjamin–Ono equation (deep water waves)

$$\boxed{\partial_t u-\partial_x|D_x|u+u\partial_x u=0}$$

- $a(\xi) = \xi_1^2 - \xi_2^2$ $(n = 2)$ $\Longrightarrow$ Davey–Stewartson system

$$\boxed{\begin{cases} i\partial_t u - \partial_x^2 u + \partial_y^2 u = c_1 |u|^2 u + c_2 u \partial_x v \\ \partial_x^2 v - \partial_y^2 v = \partial_x |u|^2 \end{cases}}$$

  (shallow water wave in 2D)

Some other higher order/dimensional cases:

- $a(\xi) =$ polynomial of order 3 $(n = 2)$ e.g. Shrira equation

$$\xi_1^3 + \xi_2^3, \quad \xi_1^3 + 3\xi_2^2, \quad \xi_1^2 + \xi_1 \xi_2^2$$

  (gravitational waves in 2D)

- $a(\xi) =$ quadratic form $(n \geq 3)$ $\Longrightarrow$ Zakharov–Schulman equation
  (interaction of small amplitude high frequency wave and acoustic wave)

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Relation between Eq1 and Eq2 = realised as a canonical transform (we will view it as a Fourier integral operator on $\mathbb{R}^n$).

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In turn, we want:

- to find normal forms and operators to reduce equations to normal forms;
- to find operators transforming equations into each other.

Both are done by Fourier integral operators in $\mathbb{R}^n$ $\Longrightarrow$ need global calculus and global weighted estimates for FIOs in $\mathbb{R}^n$ (we developed this in Comm PDE 2006, Math Ann 2006).

However, usual FIOs of the form

$$Tu(x) = \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi)\widehat{u}(\xi) d\xi$$

work best if the phase $\phi(x,\xi)$ is essentially of order one in $\xi$ (e.g. homogeneous, or SG-order one, etc.) Thus, we can reduce equation of order $m$ to a normal form of the same order $m$. How to relate equations of different orders?

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For example, we can relate estimates for wave, Schrödinger, and KdV equations, in particular showing that they are in fact equivalent.

# Smoothing effect for Schrödinger equation

As one model, let us first consider the following Schrödinger equation:

$$\begin{cases} (i\partial_t + \triangle_x)\, u(t,x) = 0, \\ \qquad\qquad u(0,x) = \varphi(x) \in L^2(\mathbb{R}^n). \end{cases}$$

By Plancherel's theorem, we know that the solution $u(t,x) = e^{it\triangle}\varphi(x)$ preserves the $L^2$–norm of initial data $\varphi$, that is, we have

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One objective: to relate this smoothing effect for Schrödinger equation to other equations; or can we even use it to understand other (a-priori completely unrelated) equations? e.g.

does it imply gain of 1 derivative for KdV? – why should it?

- In the case $n \geq 2$, we have the smoothing estimate (still for Schrödinger)

$$\boxed{\|Au\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n_x)}}$$

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where $A$ is one of the following operators:

$$[1]\ A = \langle x\rangle^{-s}|D_x|^{1/2} \qquad (s > 1/2)$$

$$[2]\ A = |x|^{\alpha-1}|D_x|^{\alpha} \qquad (1 - n/2 < \alpha < 1/2)$$

$$[3]\ A = \langle x\rangle^{-1}\langle D_x\rangle^{1/2} \qquad (n > 2)$$

* The type [1] was given by Ben-Artzi & Klainerman 1992 ($n \geq 3$) and Chihara 2002 ($n \geq 2$).

* The type [2] was given by Kato & Yajima 1989 ($n \geq 3$, $0 \leq \alpha < 1/2$), Sugimoto 1998 ($n \geq 2$, $1 - n/2 < \alpha < 1/2$). It is not true for $\alpha = 1/2$ (Watanabe 1991).

* The type [3] was given by Kato & Yajima 1989. Also Walther (1999).

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$$\sup_{\mathrm{Im}\, z>0} |(R(z)A^* f, A^* f)| \leq C||f||^2_{L^2(\mathbb{R}^n)} \qquad \text{(Resolvent estimate)}$$

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$$\mathrm{Im}\,(R(\rho^2 + i0)f, f) = \frac{1}{4(2\pi)^{n-1}\rho}||\widehat{f}|_{\rho\mathbb{S}^{n-1}}||^2_{L^2(\rho\mathbb{S}^{n-1})}.$$

By duality, both of these estimates easily imply the smoothing estimate

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Our strategy instead:

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Our strategy instead: geometric analysis, relying on the development of the global analysis of Fourier integral operators in $\mathbb{R}^n$, and weighted estimates for these operators in Sobolev spaces.

# Some history

The smoothing effect of evolution PDEs has been studied for more than 25 years. Below is a small extraction of references:

- Kato (1983): local gain of one derivative for KdV;
- Gain of half derivative for Schrödinger: Ben-Artzi & Devinatz, Constantin & Saut, Sjölin, Vega, Kato & Yajima, Ben-Artzi & Klainerman, etc;
- Analysis of Carleson problem (1979): Vega, Sjölin, Dahlberg & Kenig, Kenig & Ruiz, Carbery, Cowling, Bourgain, Moyua & Vargas & Vega, Tao, Sanghyuk Lee;
- Smoothing for different dispersive equations: Kenig & Ponce & Vega, Walther, Heinig & Wang, Ben-Artzi & Devinatz, Ben-Artzi & Nemirovsky (relativistic Schrödinger), Linares & Ponce, Watanabe, Hoshiro, Chihara, Sugimoto, Ben-Artzi & Koch & Saut, etc;
- Smoothing on manifolds: Doi, Burq, Burq & Gerard & Tzvetkov, etc;
- Schrödinger with potentials: huge literature;
- Our approach (jointly with Sugimoto) is different: to study relations among equations rather than particular equations.

# Dispersive equations with homogeneous symbols

For simplicity, let us consider general equations with homogeneous symbols:

$$\begin{cases} (i\partial_t + a(D_x))\, u(t,x) = 0, \\ u(0,x) = \varphi(x) \in L^2(\mathbb{R}^n) \end{cases}$$

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where $a(\xi)$ consists of only principal part $a_m(\xi)$ and is dispersive in the following sense

$$\boxed{\textbf{(H)} \quad a(\xi) = a_m(\xi), \quad \nabla a_m(\xi) \neq 0 \quad (\xi \neq 0)}$$

- $a_m(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$ real-valued
- $a_m(\lambda\xi) = \lambda^m a_m(\xi)$ $(\lambda > 0, \xi \neq 0)$

The dispersive assumption (H) means that the classical orbit, that is, the solution curve to

$$\begin{cases} \dot{x}(t) = (\nabla a)(\xi(t)), \quad \dot{\xi}(t) = 0 \\ x(0) = 0, \quad \xi(0) = \xi_0, \end{cases}$$

escapes to infinity as $t \to \infty$.

We may also consider more general dispersive equations

$$\begin{cases} (i\partial_t + a(D_x))\, u(t,x) = 0, \\ u(0,x) = \varphi(x). \end{cases}$$

Let $a_m(\xi) \in C^\infty(\mathbb{R}^n \backslash 0)$: $a_m(\lambda\xi) = \lambda^m a_m(\xi)$, $\lambda > 0, \xi \neq 0$. So $a_m$ is the homogeneous principal part of $a$. Let us assume one of the following:

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(H) $a(\xi) = a_m(\xi)$, and $\nabla a_m(\xi) \neq 0$ for $\xi \neq 0$.
(L) $a(\xi) \in C^\infty(\mathbb{R}^n)$, $\nabla a(\xi) \neq 0$ for all $\xi$, and $a - a_m \in S^{m-1}$.

For example, $a(\xi) = \xi_1^3 + \cdots + \xi_n^3 + \xi_1$ satisfies (L) with $m = 3$.

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For example, $a(\xi) = \xi_1^3 + \cdots + \xi_n^3 + \xi_1$ satisfies (L) with $m = 3$.

There are ways to generalise these conditions further, including classes of non–dispersive equations. Dispersive condition sometimes breaks down (e.g. often for systems), but our method gives some (quite sharp) smoothing estimates in these cases as well.

We will skip these for simplicity (but later present an “invariant estimate”).

# Global normal forms

Canonical transforms allow us to reduce general estimates to estimates in the following normal forms (of $a(D_x)$). Such transformations may be microlocal in $\xi$, but they are global in $x$. Without going much into detail, we have:

In general, $a(\xi)$ does not have to be homogeneous (as under assumption (L)). The non–homogeneous behaviour is absorbed in the phase function - this is another difference with usual FIOs.

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$\boxed{\mathsf{non-principal\ \ type:}\quad \nabla a(\xi_0)=0}$ In this case we can transform $a(D)$ to some normal forms determined by the Hessian $D_\xi^2 a(\xi_0)$, etc.

In general, $a(\xi)$ does not have to be homogeneous (as under assumption (L)). The non–homogeneous behaviour is absorbed in the phase function - this is another difference with usual FIOs.

# Global normal forms

Canonical transforms allow us to reduce general estimates to estimates in the following normal forms (of $a(D_x)$). Such transformations may be microlocal in $\xi$, but they are global in $x$. Without going much into detail, we have:

$\boxed{\text{elliptic}: \quad a(\xi) \neq 0}$ In this case we can transform $a(D)$ to $|D_1|^m$

$\boxed{\text{principal type}: \quad \nabla a(\xi) \neq 0}$ Here we can transform $a(D)$ to $|D_1|^{m-1} D_2$

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In general, $a(\xi)$ does not have to be homogeneous (as under assumption (L)). The non–homogeneous behaviour is absorbed in the phase function - this is another difference with usual FIOs.

Thus, we first have

• to establish estimates for model cases and to relate them to each other;

• to develop a global theory of FIOs including operators with phases coming from these canonical transformations.

We consider ("Fourier integral") operators

$$\boxed{Tu(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\phi(y,\xi))} a(x,y,\xi)u(y)d\xi\, dy}$$

real phase function $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. ΨDOs: $\phi(y,\xi) = -y\cdot\xi$.

Similar results for the adjoint. In particular, this includes operators of the form

$$Su(x) = \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi)\widehat{u}(\xi)d\xi,$$

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Local $L^2, L^p$ estimates

$L^2$: Eskin, Hörmander (e.g. for non-degenerate operators)

$L^p$: Peral, Beals, Miyachi, Stein, $\cdots$, Seeger-Sogge-Stein, R., for FIOs and wave-type equations.

$\mathbb{C}$-phase: $L^2$: Melin & Sjöstrand, Hörmander, $L^p$: R. dependence on geometry.

# Fourier integrals globally on $L^2(\mathbb{R}^n)$

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For example, in a typical application to smoothing problem, the canonical transform has the phase of the form $x\cdot\xi - y\cdot\psi(\xi)$, so $\phi(y,\xi) = y\cdot\psi(\xi)$, where $\psi$ is positively homogeneous of order one. But then

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• Boulkhemair (2000): conditions in Sobolev–Kato spaces, which relax conditions on the differentiability of amplitudes, but similar assumptions on $\partial_\xi \partial_\xi \phi$ to be uniformly bounded, as above.

# Results under weaker conditions

Question: What are minimal requirements for global $L^2$, Sobolev, weighted Sobolev estimates and for the global calculus?

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We assume (R., Sugimoto 2002–2006 FIOs; 2009 calculus): on $\operatorname{supp} a$

(C1) $\quad |\det \partial_y \partial_\xi \phi(y,\xi)| \geq C > 0, \quad \forall (y,\xi) \in \mathbb{R}^n \times \mathbb{R}^n.$

(C2) $\quad |\partial_y^\alpha \partial_\xi \phi(y,\xi)| \leq C_\alpha,\ |\partial_y \partial_\xi^\beta \phi(y,\xi)| \leq C_\beta$
$\forall (y,\xi) \in \mathbb{R}^n \times \mathbb{R}^n, \quad 1 \leq |\alpha|, |\beta| \leq 2n+2.$

Note that condition (C1) is a global "local graph condition", necessary even for local $L^2$-bounds for $S^0_{1,0}$. We have microlocal versions too.

Difference with previous results: in (C2) now we take only mixed derivatives.

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Further recent work:

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Further recent work: R.-Sugimoto (global FIO calculus), Andrews-R. (SG-FIO calculus), Coriasco-R. (global $L^p$ estimates), R.-Smith, R.-Matsuyama, R.-Wirth (dispersive estimates)

**THEOREM 1** *Let operator $T$ be defined by*

$$\boxed{Tu(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(\phi(x,\xi)-y\cdot\xi)} a(x,y,\xi)u(y)\,dy\,d\xi}$$

*Let the phase $\phi = \phi(x,\xi) \in C^\infty$ for all $|\alpha|, |\beta| \geq 1$ satisfy*

$$|\det \partial_x \partial_\xi \phi(x,\xi)| \geq C_0 > 0,\ |\partial_x^\alpha \phi(x,\xi)| \leq C_\alpha \langle \xi \rangle,\ |\partial_x^\alpha \partial_\xi^\beta \phi(x,\xi)| \leq C_{\alpha\beta}.$$

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*Assume also one of the following:*

*(1) For all $\alpha$, $\beta$, and $\gamma$, $|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma a(x,y,\xi)| \leq C_{\alpha\beta\gamma}\langle x\rangle^{m_1}\langle y\rangle^{m_2-|\beta|}\langle\xi\rangle^{m_3}$, and for all $|\beta| \geq 1$, $|\partial_\xi^\beta\phi(x,\xi)| \leq C_\beta\langle x\rangle$.*

*(2) For all $\alpha$, $\beta$, and $\gamma$, $|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma a(x,y,\xi)| \leq C_{\alpha\beta\gamma}\langle x\rangle^{m_1-|\alpha|}\langle y\rangle^{m_2}\langle\xi\rangle^{m_3}$, and for all $\alpha$ and $|\beta| \geq 1$, $|\partial_x^\alpha\partial_\xi^\beta\phi(x,\xi)| \leq C_{\alpha\beta}\langle x\rangle^{1-|\alpha|}$.*

*Then $T$ is bounded from $H^{s_1,s_2}(\mathbb{R}^n)$ to $H^{s_1-m_1-m_2,s_2-m_3}(\mathbb{R}^n)$, for all $s_1, s_2 \in \mathbb{R}^n$.*

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• The answer is: yes, we can! For this, we introduce what we call comparison principles for evolution PDEs.

• Altogether, this allows us to relate smoothing estimates for a-priori completely unrelated equations. Indeed, what is the relation between (linearised) tsunamis, quantum chemistry, signals in internet cables, and sun radiation?

But smoothing estimates for them turn out to be closely related (equivalent)!

# Comparison Principle

For solutions $u(t,x) = e^{itf(D_x)}\varphi(x)$ and $v(t,x) = e^{itg(D_x)}\varphi(x)$ to two equations

$$\begin{cases} (i\partial_t + f(D_x))\, u(t,x) = 0, \\ \qquad\qquad u(0,x) = \varphi(x) \end{cases} \quad \text{and} \quad \begin{cases} (i\partial_t + g(D_x))\, v(t,x) = 0, \\ \qquad\qquad v(0,x) = \varphi(x) \end{cases}$$

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comparison principle is the general rule to derive time-space estimate for $u$

$$\left\| w(x)\sigma(D_x)\, \overbrace{e^{itf(D_x)}\varphi(x)}^{u(t,x)} \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n_x)}$$

from time-space estimate for $v$

$$\left\| w(x)\tau(D_x)\, \underbrace{e^{itg(D_x)}\varphi(x)}_{v(t,x)} \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n_x)}.$$

Thus, if we know an estimate for $v(t,x)$, how to get an estimate for $u(t,x)$?

**Theorem.** *Let* $f, g \in C^1(\mathbb{R})$ *be real-valued and strictly monotone. If* $\sigma, \tau \in C^0(\mathbb{R})$ *satisfy*

$$\boxed{\frac{|\sigma(\xi)|}{|f'(\xi)|^{1/2}} \leq A \frac{|\tau(\xi)|}{|g'(\xi)|^{1/2}}}$$

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*for all* $x, \tilde{x} \in \mathbb{R}$. *Consequently,*

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We can keep track of constants. For example, for $n \geq 3$ and $m > 0$, we have

$$\left\| |x|^{-1}|D_x|^{m/2-1}e^{it|D_x|^m}\varphi(x)\right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq \sqrt{\frac{2\pi}{m(n-2)}}\|\varphi\|_{L^2(\mathbb{R}^n_x)},$$

where the constant $\sqrt{\frac{2\pi}{m(n-2)}}$ is sharp (for $m = 2$ this was calculated by B. Simon 1989 using Kato's theory).

For example, comparison principle immediately gives

$$\sqrt{m}\left\| |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} = \sqrt{l}\left\| |D_x|^{(l-1)/2} e^{it|D_x|^l} \varphi(\widetilde{x}) \right\|_{L^2(\mathbb{R}_t)}$$

for all $l, m > 0$ and $x, \tilde{x} \in \mathbb{R}$. Here $\operatorname{supp} \widehat{\varphi} \subset [0, +\infty)$ or $(-\infty, 0]$. With $l = 1$, we get

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Hence, multiplying by $\langle x\rangle^{-s}$ with $s > 1/2$ and integrating, we get

$$\boxed{\left\| \langle x\rangle^{-s} |D_x|^{(m-1)/2} e^{it|D_x|^m}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}_x)} \le C\|\varphi\|_{L^2(\mathbb{R}_x)}}$$

which is the smoothing estimate in the 1D model case. We have the full range of such estimates in 1D, 2D, and radially symmetric model cases, and comprehensive relations among them.

**Theorem.** *Let* $f(\xi,\eta), g(\xi,\eta) \in C^1(\mathbb{R}^2)$ *be real-valued and strictly monotone in* $\xi \in \mathbb{R}$ *for each fixed* $\eta \in \mathbb{R}$. *If* $\sigma, \tau \in C^0(\mathbb{R}^2)$ *satisfy*

$$\boxed{\frac{|\sigma(\xi,\eta)|}{|\partial f/\partial\xi(\xi,\eta)|^{1/2}} \leq A \frac{|\tau(\xi,\eta)|}{|\partial g/\partial\xi(\xi,\eta)|^{1/2}}}$$

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*for all* $x, \tilde{x} \in \mathbb{R}$. *Especially, for any* $w(x)$, *we have*

$$\|w(x)\sigma(D_x,D_y)e^{itf(D_x,D_y)}\varphi(x,y)\|_{L^2(\mathbb{R}_t\times\mathbb{R}^2_{x,y})} \leq A\|w(x)\tau(D_x,D_y)e^{itg(D_x,D_y)}\varphi(x,y)\|_{L^2(\mathbb{R}_t\times\mathbb{R}^2_{x,y})}.$$

**Theorem.** *Let* $f, g \in C^1(\mathbb{R}_+)$ *be real-valued and strictly monotone. If* $\sigma, \tau \in C^0(\mathbb{R}_+)$ *satisfy*

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$$\boxed{\frac{|\sigma(\rho)|}{|f'(\rho)|^{1/2}} \le A \frac{|\tau(\rho)|}{|g'(\rho)|^{1/2}}}$$

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$$\|\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t)} \le A\|\tau(|D_x|)e^{itg(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t)}$$

*for all $x \in \mathbb{R}^n$. Especially, for any $w(x)$, we have*

$$\|w(x)\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \le A\|w(x)\tau(|D_x|)e^{itg(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)}.$$

Since our normal forms are in 1D (elliptic case) and in 2D (non-elliptic case), and some estimates reduce to the radially symmetric case, these comparison principles cover all normal forms and all types that we need. There are also extensions of these principles to equations with time–dependent coefficients and to equations with potentials (the latter is joint work with M. Ben-Artzi and M. Sugimoto).

Since the comparison principle works in both ways, we can conclude, in particular, that the smoothing estimates for the wave ($m = 1$), Schrödinger ($m = 2$) and KdV ($m = 3$) equations are all equivalent to each other.

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As another example, we can relate Klein–Gordon, relativistic Schrödinger, standard Schrödinger, and wave equations among each other by the comparison principle that yields

$$\begin{aligned}\left\| e^{-it\sqrt{1-\Delta_x}}\varphi(x)\right\|_{L^2(\mathbb{R}_t)} &= \sqrt{2}\left\| \langle D_x\rangle^{1/2} e^{it\Delta_x}\varphi(\tilde{x})\right\|_{L^2(\mathbb{R}_t)} \\ &= \left\| |D_x|^{-1/2}\langle D_x\rangle^{1/2} e^{\pm it\sqrt{-\Delta_x}}\varphi(\tilde{x})\right\|_{L^2(\mathbb{R}_t)}.\end{aligned}$$

This implies the equivalence of the corresponding smoothing estimates.

All this applies to all model cases, and to smoothing properties with weights of all the types mentioned above.

In comparison principle we can take any further norm with respect to $x$.

**Corollary.** *Let functions $f, g, \sigma, \tau$ be as in the comparison principle. Let $0 < p \leq \infty$. Then, for any measurable function $w$ on $\mathbb{R}^n$, we have the estimate*

$$\|w(x)\chi(|D_x|)\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\|_{L^p(\mathbb{R}^n_x, L^2(\mathbb{R}_t))} \leq A\|w(x)\chi(|D_x|)\tau(|D_x|)e^{itg(|D_x|)}\varphi(x)\|_{L^p(\mathbb{R}^n_x, L^2(\mathbb{R}_t))}.$$

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Equivalence: for all $0 < p \le \infty$, quantities $||e^{it\sqrt{-\Delta}}\varphi||_{L^p(\mathbb{R}^n_x, L^2(\mathbb{R}_t))}$, $|||D_x|^{1/2}e^{-it\Delta}\varphi||_{L^p(\mathbb{R}^n_x, L^2(\mathbb{R}_t))}$, and $|||D_x|e^{it(-\Delta)^{3/2}}\varphi||_{L^p(\mathbb{R}^n_x, L^2(\mathbb{R}_t))}$ for propagators of the wave, Schrödinger, and KdV type equations are equivalent.

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Minkowski's inequality $\Longrightarrow$ for $p_1 \le 2 \le p_2$

$$||f||_{L^2(\mathbb{R}_t, L^{p_1}(\mathbb{R}^n_x))} \le C||f||_{L^{p_1}(\mathbb{R}^n_x, L^2(\mathbb{R}_t))}, \quad ||f||_{L^{p_2}(\mathbb{R}^n_x, L^2(\mathbb{R}_t))} \le C||f||_{L^2(\mathbb{R}_t, L^{p_2}(\mathbb{R}^n_x))},$$

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Note: $L^2(\mathbb{R}_t)$–norm in time is often critical;
also, Strichartz estimates with $p = \infty$ may fail, so the smaller $L^\infty(\mathbb{R}^n_x, L^2(\mathbb{R}_t))$–norms may be a good substitute.

We can not only relate estimates, but also get new ones for new & old equations.

Let $u(t,x)$ be the solution of $\boxed{(i\partial_t + a(D_x))u(t,x) = 0; u(0,x) = \varphi}$ Then

(H), $n \geq 1, m > 0, s > 1/2$, or
(L), $n \geq 1, m \geq 1, s > 1/2$, imply

$$\boxed{\left\| \langle x \rangle^{-s} |D_x|^{(m-1)/2} u(t,x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C \|\varphi\|_{L^2(\mathbb{R}^n_x)}}$$

previously: Ben-Artzi–Klainerman: $a(\xi) = |\xi|^2$, $n \geq 3$; Chihara: (H) and $m > 1$;
(but $m = 1$ is hyperbolic case, which is now also included)

(H), $m > 1, n > m + 1$; or $n > m > 1$ for elliptic $a$,

$$\boxed{\left\| \langle x \rangle^{-m/2} \langle D_x \rangle^{(m-1)/2} u(t,x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C \|\varphi\|_{L^2(\mathbb{R}^n_x)}}$$

(L), $m > 0$, $s > 1/2$, then

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previously: Kato–Yajima: $a(\xi) = |\xi|^2$, $n \geq 3$; Walther: $a(\xi) = |\xi|^m$, $n > m > 1$.

# Further related things

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- Consequently, we get the critical case of the Kato–Yajima's smoothing estimate; we also get critical case $s = \frac{1}{2}$ of restriction/trace theorems for $||\widehat{f}|_{\rho\Sigma}||_{L^2(\rho\Sigma)} \leq C(\rho)||f||_{H^s}$, with corresponding formulae for $C(\rho)$.

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• We can apply some of these results to nonlinear problems. For example, we improve the global time well-posedness for derivative nonlinear Schrödinger equations with small data (for $f$ polynomial of order $N$):

$$(i\partial_t + \triangle_x)u(t,x) = f(\nabla u(t,x)), \quad u(0,x) = \varphi(x), \quad t \in \mathbb{R}, x \in \mathbb{R}^n.$$

Chihara 1996: $N \geq 3$, $\varphi$ is smooth, rapid decay, and small.
Hayashi–Miao–Naumkin 1999: $N \geq 2$, $\varphi \in H^{[n/2]+5}$, rapid decay, and small.
R.–Sugimoto 2006: $N \geq 2$, null–form structure, $\langle x\rangle\varphi \in H^{\frac{n+3}{2}+\epsilon}$, small.

# Restriction/trace theorems and restriction norms

For any operator $A = A(X, D_x)$ acting on the variable $x$, smoothing estimate

$$\boxed{\left\| A e^{ita(D_x)} \varphi \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C \, \| \varphi \|_{L^2(\mathbb{R}^n_x)}}$$

is equivalent to the restriction estimate

$$\boxed{\left\| \widehat{A^* f}_{|\rho\Sigma_a} \right\|_{L^2(\rho\Sigma_a \, ; \, \rho^{n-1} d\omega / |\nabla a|)} \leq C \sqrt{\rho} \, \| f \|_{L^2(\mathbb{R}^n_x)}}$$

where $\rho > 0$, $\rho\Sigma_a = \{\rho\omega : \omega \in \Sigma_a\}$, $\Sigma_a = \{\xi \in \mathbb{R}^n : a(\xi) = 1\}$.

Let $a(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$ be real-valued and satisfy $a(\xi) > 0$ and $a(\lambda\xi) = \lambda^2 a(\xi)$ for $\lambda > 0$ and $\xi \neq 0$. Then (using different weights in smoothing estimates)

$$\left\| f_{|\rho\Sigma_a} \right\|_{L^2(\rho\Sigma_a, \rho^{n-1} d\omega)} \leq C \| f \|_{H^s(\mathbb{R}^n)} \qquad (s > 1/2),$$

$$\left\| f_{|\rho\Sigma_a} \right\|_{L^2(\rho\Sigma_a, \rho^{n-1} d\omega)} \leq C \rho^{s-1/2} \| f \|_{\dot{H}^s(\mathbb{R}^n)} \qquad (n/2 > s > 1/2).$$

For $\rho = 1$ this gives a trace theorem, and for all $\rho$,

global smoothing $\Longrightarrow$ growth order of norms, Thus, we have (copied from the last page)

$$\left\| f_{|\rho\Sigma_a} \right\|_{L^2(\rho\Sigma_a, \rho^{n-1} d\omega)} \le C \|f\|_{H^s(\mathbb{R}^n)} \qquad (s > 1/2),$$

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If we in addition assume that the Gaussian curvature of $\Sigma_a$ is non-vanishing, then we have also the the critical cases:

$$\left\| \left( \frac{\nabla a(x)}{|\nabla a(x)|} \wedge \frac{D_x}{|D_x|} \right) f_{|\rho\Sigma_a} \right\|_{L^2(\rho\Sigma_a, \rho^{n-1} d\omega)} \le C \|f\|_{\dot{H}^{1/2}(\mathbb{R}^n)},$$

$$\left\| \left( \frac{x}{|x|} \wedge \frac{\nabla a^*(D_x)}{|\nabla a^*(D_x)|} \right) f_{|\rho\Sigma_a} \right\|_{L^2(\rho\Sigma_a, \rho^{n-1} d\omega)} \le C \|f\|_{\dot{H}^{1/2}(\mathbb{R}^n)}.$$

Here the dual function $a^*(x)$ is determined by the relation

$$\Sigma_{a^*} = \Sigma_a^*$$

where the *dual hypersurface* is defined by

$$\Sigma_a^* = \{\nabla a(\xi) : \xi \in \Sigma_a\}.$$

Let us consider equations of the form

$$\begin{cases} (i\partial_t + a(t, D_x))\, u(t,x) = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}^n_x, \\ u(0,x) = \varphi(x) & \text{in } \mathbb{R}^n_x. \end{cases}$$

We can conjecture the following estimate (e.g. for type [1])

$$\boxed{\left\| \langle x \rangle^{-s} |\nabla a(t, D_x)|^{1/2} e^{i \int_0^t a(t,D_x)dt} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C \|\varphi\|_{L^2(\mathbb{R}^n_x)} \quad (s > 1/2)}$$

• in the dispersive case it is equivalent to the usual type [1] estimate;

• it does continue to hold for a variety of non-dispersive equations, where $\nabla a(\xi)$ may become zero on some set and when the usual estimate fails;

• it does take into account zeros of the gradient $\nabla a(\xi)$, which is also responsible for the interface between dispersive and non-dispersive zone (e.g. how quickly the gradient vanishes);

• it is invariant under canonical transforms of the equation;

• the proposed estimates are scaling invariant or "almost" invariant;

• the estimates are also "sharp".

## Conclusions

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- Comparison principles allow us to compare estimates by comparing certain expressions for symbols appearing in the estimates.

# Thank you