On the three-wave coupling system.

Laser-plasma interaction

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Bengalore, Satelite Conf on PDE and related topics

Advert.

Boyd-Kadmontsev system

$$\begin{aligned} \varepsilon\partial_t u + \partial_x u &= -vw + ... \\ \varepsilon\partial_t v - \partial_x v &= u\overline{w} + ... \\ \partial_t w + \partial_x w &= u\overline{v} + ... \end{aligned}$$

$$u(t,x) \in \mathbb{C}, \qquad v(t,x) \in \mathbb{C}, \qquad w(t,x) \in \mathbb{C}$$

$$\varepsilon > 0$$

Motivations

Simulations of laser propagation in a plasma

(Inertial Confinent Fusion exp.)

$\Rightarrow$ Brillouin backscattering (or Raman backscattering)

3D or 2D numerical simulations : Plasma hydrodynamics + laser propagation

Microscopic level, (wave length 0.35 μm).

Macroscopic level, Geom. Optics ($L_{\text{target}} \sim 2000\ \mu m$)

Mesoscopic level, $L_{\text{simul}} \sim 50 \longrightarrow 500\ \mu m$.

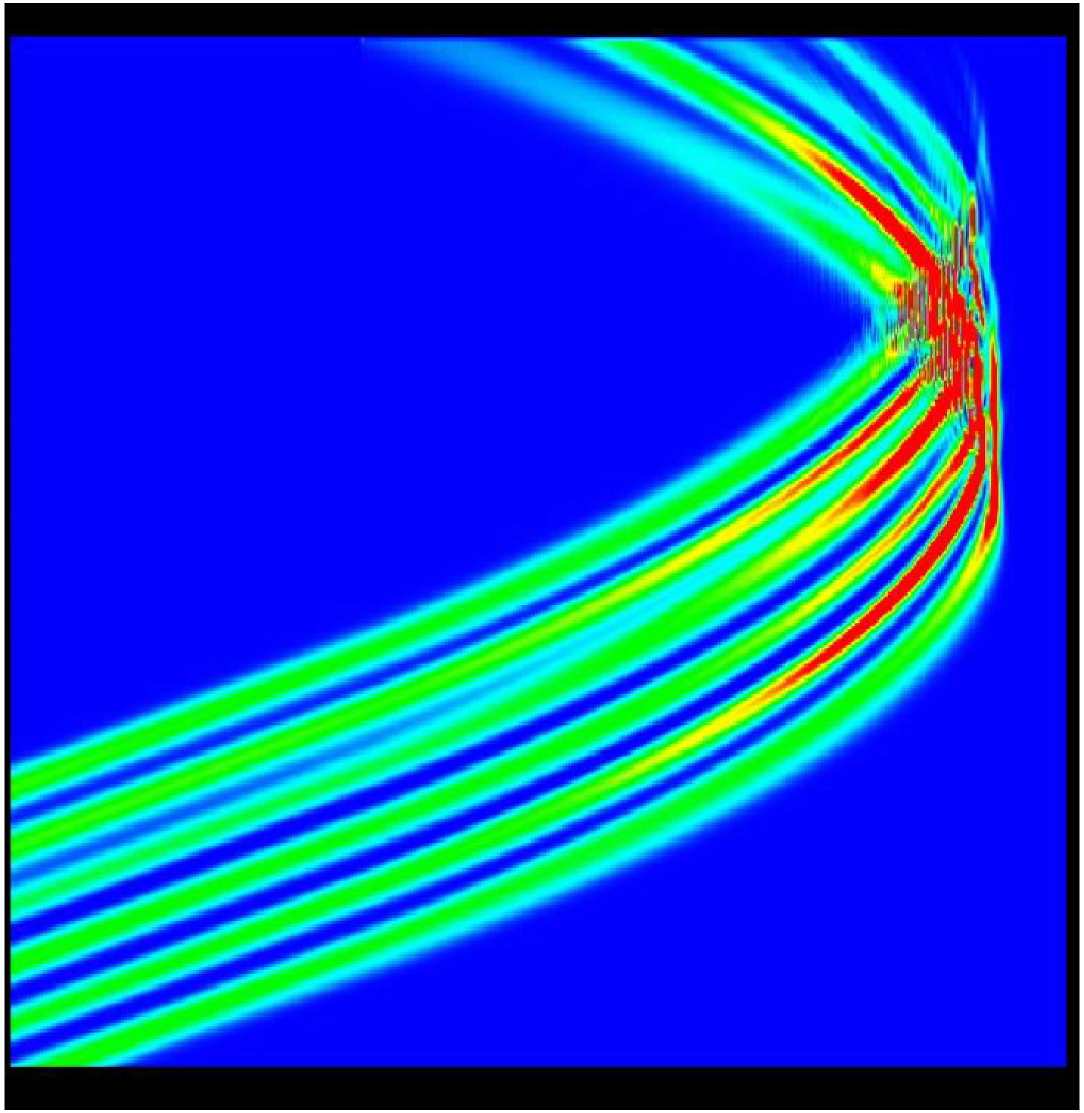

Figure 1: 2D simulation with frequency Maxwell (very expensive). Map of laser intensity

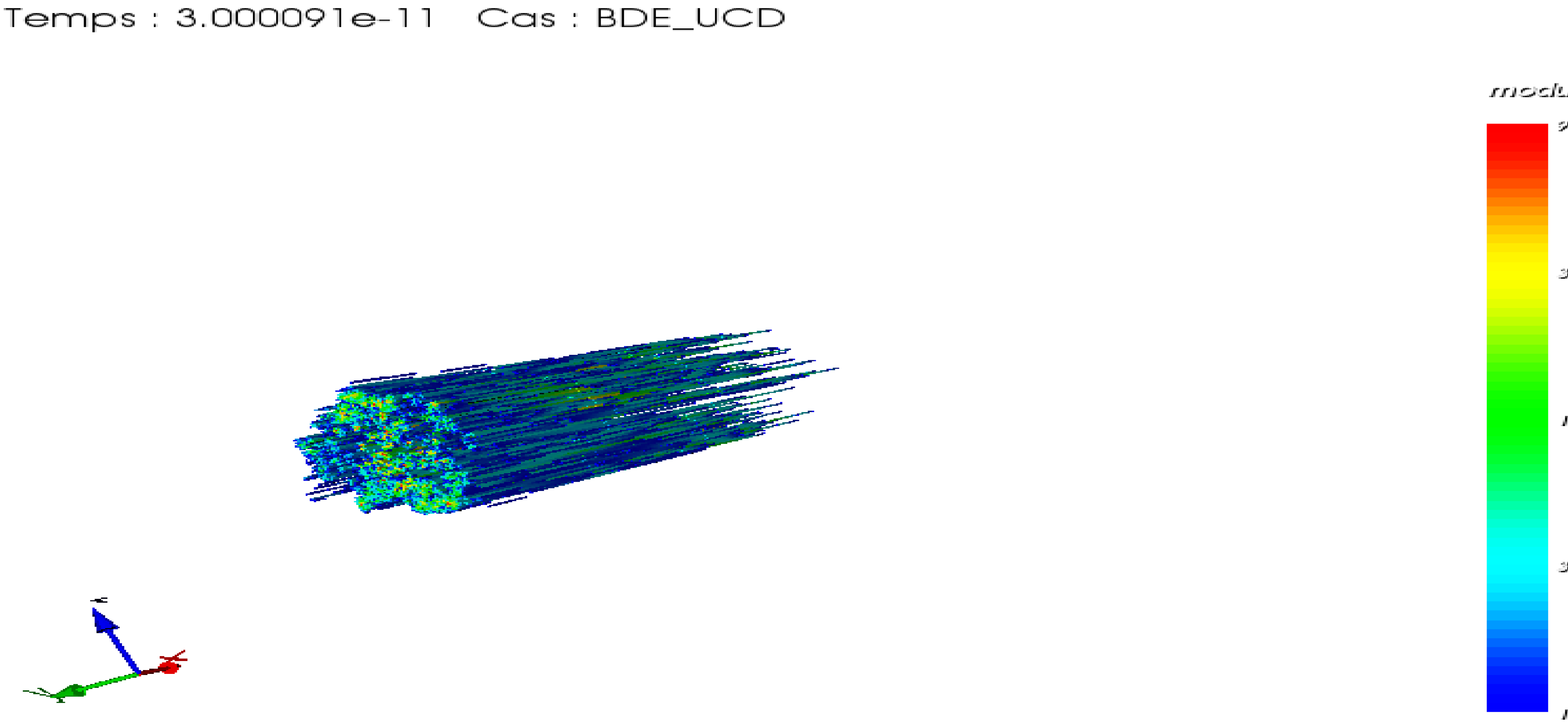

Figure 2: 3D simulation box $\sim 500 \times 500 \times 700\ \mu m$, using paraxial appr. (a lot of speckels: width $\sim 3\ \mu m$)

Outlines

1. Derivation of the model: coupling an acoustic wave with a laser beam

2. Analysis of the three-wave coupling system

3. Numerical issues

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1. Derivation of the model

■ Initial model for Brillouin backscattering.

Laser field (electric field) $\Rightarrow \Psi(t, \mathbf{x})$

♦ Plasma dynamics

Assumption : Density is close to N_{ref} .

Pertubation of denstity denoted by nN_{ref}, so $n \ll 1$. Velocity $\mathcal{Q}$.

$$\begin{aligned} \frac{\partial}{\partial t} n + \nabla \mathcal{Q} &= 0 \\ \frac{\partial}{\partial t} \mathcal{Q} + c_s^2 \nabla n + 2\eta \mathcal{Q} &= -\gamma_{\text{p}} \nabla |\Psi|^2. \end{aligned}$$

c_s the sound speed, η the Landau damping coef.

At r.h.s., ponderomotive force from the laser onto the plasma (γ_{p} constant)

♦ Paraxial approximation of the laser field

$$\frac{\partial^2}{\partial t^2}\Psi - c^2\Delta\Psi + \omega_0^2 N_{\text{ref}}(1+n)\Psi = 0$$

$\Rightarrow$ WKB approximation ($x = \mathbf{x}_1$ propagation direction)

laser freq. $\omega_0 \gg T_{\text{char}}^{-1}$. wave number $k_p = \frac{\omega_0}{c}(1 - N_{\text{ref}})^{\frac{1}{2}} \simeq \frac{2\pi}{0.3}\ \mu m^{-1}$

Field $\Psi \Rightarrow$ 2 waves. $\Psi(t,\mathbf{x}) = \left[E(t,\mathbf{x})e^{ik_px - i\omega_0 t} + c.c\right] + \left[V(t,\mathbf{x})e^{-ik_px - i\omega_0 t} + c.c.\right]$

$$\begin{aligned}(\frac{\partial}{\partial t} + cg\frac{\partial}{\partial x})E - \frac{ic^2}{2\omega_0}\Delta_\perp E &= -i\frac{\omega_0 N_{\text{ref}}}{2} n e^{-i2k_px} V \\ (\frac{\partial}{\partial t} - cg\frac{\partial}{\partial x})V - \frac{ic^2}{2\omega_0}\Delta_\perp V &= -i\frac{\omega_0 N_{\text{ref}}}{2} n e^{i2k_px} E\end{aligned} \tag{1}$$

$c_g = c(1 - N_{\text{ref}})^{\frac{1}{2}}$ group velocity. Ponderomotive force reads as

$$\begin{aligned}\partial_x|\Psi|^2 &= \partial_x\left[e^{i2k_px}E\overline{V} + c.c.\right] + \nabla_\perp(|E|^2) + \ldots \\ &\simeq i2k_pE\overline{V}e^{i2k_px} + c.c.\end{aligned}$$

■ From a 4-wave model to a 3-wave model.

1D $\Rightarrow$ Generation of a wave in the propagation direction $k_S = 2k_P$.

Dimensionless form

$$c_S \mapsto 1, \qquad \varepsilon = c_S/c_G$$

=(acoustic velocity) / (speed of light) $\simeq .002$

$$E \mapsto iu, \qquad V \mapsto v. \qquad \mathcal{Q} \mapsto q.$$

Simplified model with n, q, u, v. For the plasma:

$$\begin{aligned} \frac{\partial}{\partial t} n + \frac{\partial}{\partial x} q = 0 \\ \frac{\partial}{\partial t} q + \frac{\partial}{\partial x} n + 2\eta q = u\overline{v}e^{ik_s x} + c.c. \end{aligned} \tag{2}$$

coupled with (1).

Conjecture: it is a well-posed problem.

$$\text{Set : } \frac{n+q}{2} = we^{-ik_s x} + c.c, \qquad \frac{n-q}{2} = se^{-ik_s x} + c.c$$

Assume for a while that u is fixed. We get the linear system.

$$\begin{array}{rcl} \frac{\partial}{\partial t}w + (ik_S + \frac{\partial}{\partial x})w + \eta(w - s) & = & u\overline{v} \\ \frac{\partial}{\partial t}s - (ik_S + \frac{\partial}{\partial x})s + \eta(s - w) & = & -u\overline{v}. \\ \frac{\partial}{\partial t}v - \frac{1}{\varepsilon}\frac{\partial}{\partial x}v & = & \frac{1}{\varepsilon}(\overline{w} + \overline{s})u \end{array} \tag{3}$$

Stability analysis $\Rightarrow$ Dispersion relation for (Ω, ζ) (here $\eta = 0$)

$$\begin{vmatrix} \Omega - (\zeta + k_S) & 0 & -iu \\ 0 & \Omega + (\zeta + k_S) & iu \\ -i\overline{u}/\varepsilon & -i\overline{u}/\varepsilon & \Omega + \zeta/\varepsilon \end{vmatrix} = 0$$

Proposition. *The largest value of* Im(Ω) *is* >0 *and is obtained when* $\zeta = -k_S\varepsilon/(1+\varepsilon)$. *For system without s-wave, we get* Im$(\Omega) = 0$.

$\Longrightarrow$*In linear system (3), the* s*–wave don't grow; we can neglect it:* $(\overline{w} + \overline{s})u \rightarrowtail \overline{w}u$.

Introducing afresh the quantity u, the system to be addressed is.

$$(\varepsilon\partial_t + \partial_x)u = -vw$$

$$(\varepsilon\partial_t - \partial_x)v = u\overline{w}$$

$$(\partial_t + \partial_x)w + (\eta + ik_s)w = u\overline{v}$$

■ Remark. Model in large parallel 3D codes ($\mathbb{HERA}$, PF3D)

hydrodynamics macroscopic level ($\Rightarrow \Gamma$ plasma density) + system

$$
\begin{aligned}
\varepsilon \frac{\partial}{\partial t} u + \frac{\partial}{\partial x} u + i\alpha \Delta_{\perp} u &= -\Gamma w v + .. \\
\varepsilon \frac{\partial}{\partial t} v - \frac{\partial}{\partial x} v + i\alpha \Delta_{\perp} v &= \Gamma \overline{w} u + .. \\
\frac{\partial}{\partial t} w + \frac{\partial}{\partial x} w + (\eta + + i k_s) w &= \Gamma u \overline{v} + ..
\end{aligned}
$$

Numerical difficulties :

diffraction $i\alpha\Delta_{\perp}$

u in one way with velocity $1/\varepsilon$

v in the opposite with velocity $1/\varepsilon$.

Some 10^8 grid points

Up to now, in these large codes, one uses

$\delta t = \delta x / \varepsilon^{-1} \ll$ characteristic time of instability growth which is $\sim \varepsilon^{1/2}$ (sub-cycling...)

2. Analysis of the three-wave coupling system

The 1D system is addressed in plasma physics since:

Kadomtsev, 1964

Boyd-Turner, 1972

Novikov-Zakharov, 1984

Eliseev et al. PoP (1995) ; Mounaix et al. Phys Rev. (1997); Huller, (2000), (2006)...; Loiseau et al. (2008)

Berger-Still et al. PoP (1998), .. ColinM, Colin T, 2006 (for Raman), ..

The 3-wave coupling system is related also to vibrating string models and plasma turbulence...

$$
\begin{aligned}
(\varepsilon\partial_t + \partial_x)u &= -vw \\
(\varepsilon\partial_t - \partial_x)v &= u\overline{w} \\
(\partial_t + \partial_x)w + \eta w &= u\overline{v}
\end{aligned}
$$

On the interval $[0, L]$, where the main difficulties occur

$$
\begin{array}{lll}
u(t,0) = u^{in}, & v(t,L) = 0, & w(t,0) = 0 \\
u|_{t=0} = u_0, & v|_{t=0} = 0, & w|_{t=0} = w_0,
\end{array}
$$

Balance Relations

$$\begin{aligned} \varepsilon\partial_t(|u|^2+|u|^2)+\partial_x(|u|^2-|v|^2) &= 0; \\ \partial_t(|w|^2+\varepsilon|u|^2)+2\eta|w|^2+\partial_x(|w|^2+|u|^2) &= 0. \end{aligned}$$

$|u|^2+|v|^2 =$ total laser energy ; $|w|^2+\varepsilon|u|^2$ related to conservation of momentum (acoustic+ electrom.).

Theorem 1. *If u_0, w_0 are in L^2_x, there exists an unique solution* $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ in $L^2_{t,x}$

Proof based on Compensated Int. (cf. Murat-Tartar)

There is C_0, such that if u, v are $u|_{\partial\mathcal{D}}$, $v|_{\partial\mathcal{D}}$ are in L^2 and $(\varepsilon\partial_t+\partial_x)u, (\varepsilon\partial_t-\partial_x)v$ in $L^2_{t,x}$,

$$\|uv\|_{L^2_{t,x}} \le C_0\left[\|u|_{\partial\mathcal{D}}\|^2_{L^2}+\|(\varepsilon\partial_t+\partial_x)u\|^2_{L^2_{t,x}}\right]\left[\|v|_{\partial\mathcal{D}}\|^2_{L^2}+\|(\varepsilon\partial_t-\partial_x)v\|^2_{L^2_{t,x}}\right]$$

Asymptotic Analysis

This limiting system ($\varepsilon = 0$) reads as

$$\partial_x u_* = -v_* w_*, \qquad u_*(t,0) = 0$$

$$-\partial_x v_* = \overline{w_*} u_*, \qquad v_*(t,L) = 0$$

$$(\partial_t + \partial_x) w_* + \eta w_* = u_* \overline{v_*}, \qquad w_*(t,0) = 0$$

with initial condition only : $\qquad w_*(t = 0, \cdot) = w_0(\cdot).$

Assume w_0 *bounded, there exists a unique solution* (u_*, v_*, w_*). *Moreover*

$$|v_*(0)| < |U^{in}|.$$

Theorem 2. (G. Metivier, R. S.) *If* u_0, w_0 *are smooth*,

$$(u^\varepsilon, v^\varepsilon, w^\varepsilon) \to (u_*, v_*, w_*), \qquad \text{in } [L^2_{t,x}]^3$$

Proof. Difficulties:Initial layer; non-linear system.

[Comm. Math. Physics, 2010.]

3. Numerical issues.

Previous result justifies that: Time derivative = Correction term.

NOT OBVIOUS.

Time step δt such that $\delta t/\sqrt{\varepsilon} \sim \delta x$. (If explicit $\Rightarrow \delta t/\varepsilon \sim \delta x$).

Each time step

1) solve equation for w by a classical explicit scheme

2) solve the system of 2 coupled ODEs

$$\begin{aligned} \partial_x u + \Gamma v W + \frac{\varepsilon}{\delta t} u &= \frac{\varepsilon}{\delta t} u^{(n)}, && \text{bound. c. } u(0) = U^{in} \\ -\partial_x v - \Gamma u \overline{W} + \frac{\varepsilon}{\delta t} v &= \frac{\varepsilon}{\delta t} v^{(n)}, && \text{bound. c. } v(L) = 0. \end{aligned}$$

Energy balance : $|u(0)|^2 - |v(0)|^2 = 1 - |v(L)|^2 + O(\varepsilon)$, $\Rightarrow$ stability.

Centered discretization for the coupling terms,

$$\begin{aligned} u_{j+1} - u_j + \phi_{j+1/2} v_j + \phi_{j+1/2} v_{j+1} + \frac{\varepsilon}{\delta t} u_j &= \frac{\varepsilon}{\delta t} u_j^{(n)}, \\ -v_{j+1} + v_j - \overline{\phi_{j+1/2}} u_j - \overline{\phi_{j+1/2}} u_{j+1} + \frac{\varepsilon}{\delta t} v_{j+1} &= \frac{\varepsilon}{\delta t} v_{j+1}^{(n)}. \end{aligned}$$

$$\phi_{j+1/2} = \frac{\delta x}{2} \Gamma_{j+1/2} \widetilde{W_{j+1/2}}$$

Iterative method. Space marching $\Rightarrow$ u forward, v backward

Numerical results.

Profiles in 1D of $|u|^2$, $|v|^2$ $|w|^2$

Interesting quantity : Backscattering $r(t) = |v(t,0)|^2$.

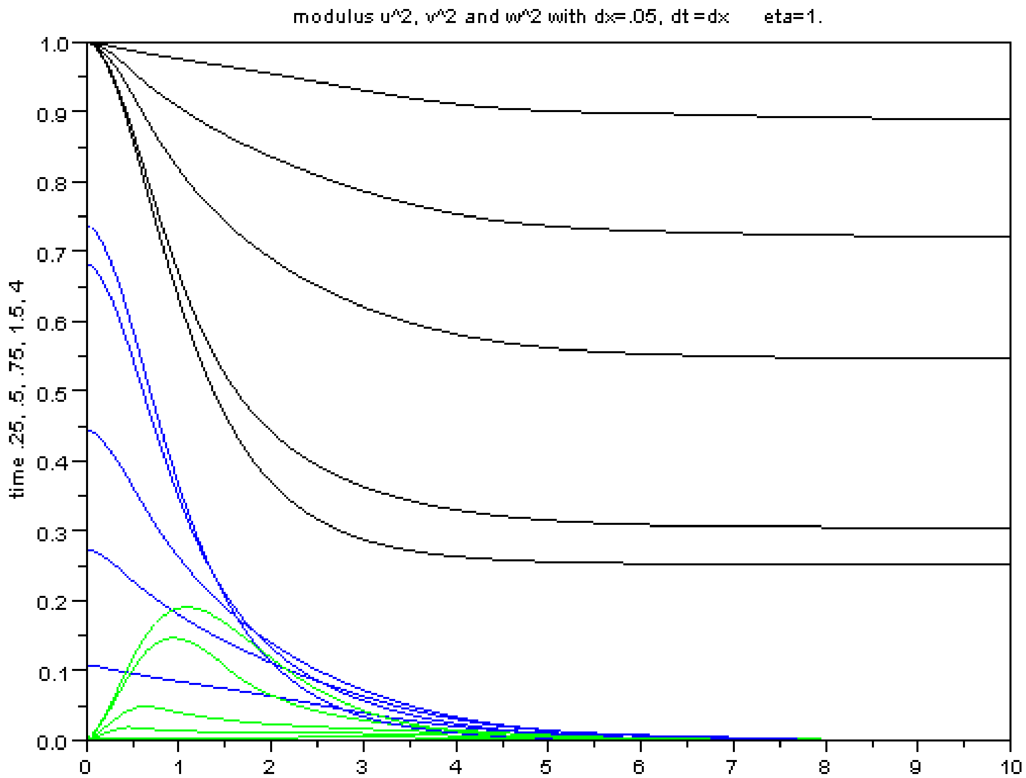

Figure 3: Profiles of $|u|^2$, $|v|^2$, $|w|^2$ versus $x-$variable for $\eta = 1$ at different time values

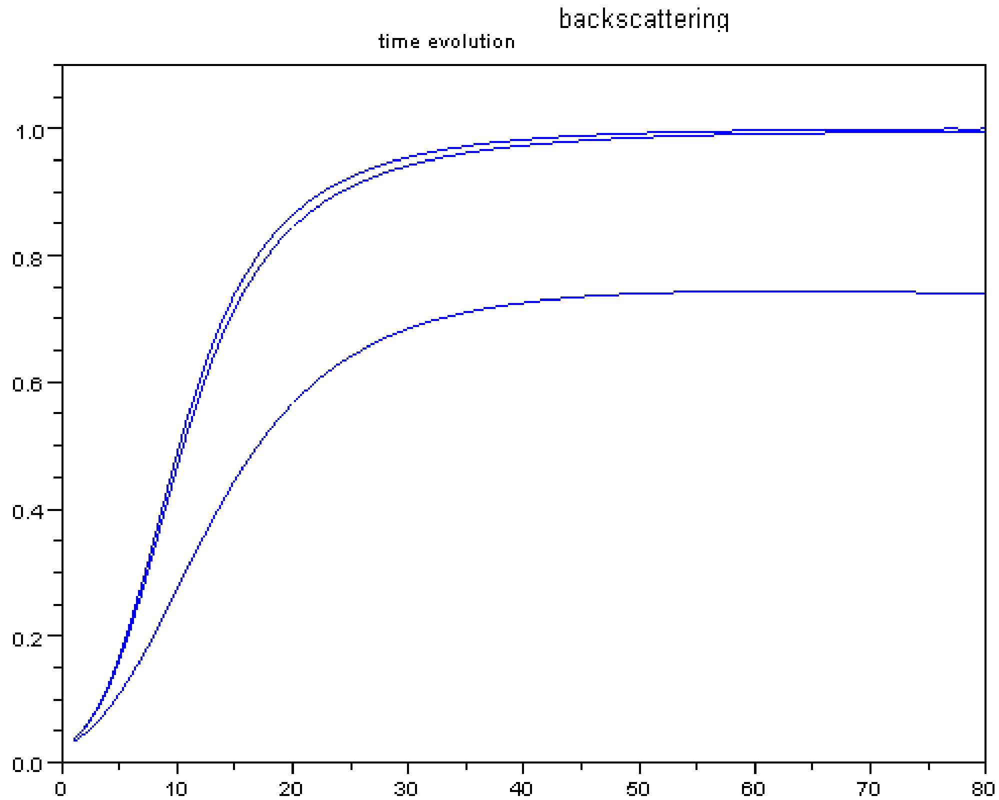

Figure 4: Time evolution of the backscattering $r(t)$ for different values of η (from bottom to top $\eta = 1, 0.1$ and 0.01)

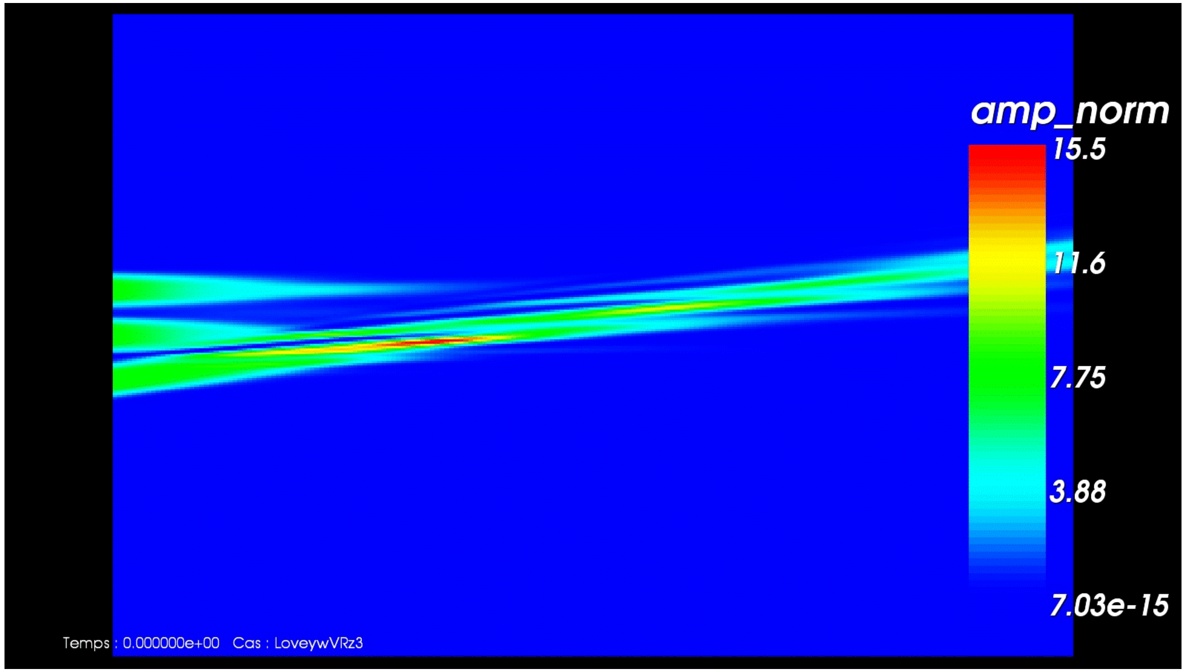

Figure 5: intensity of main laser wave (2D computation, begining of simulation)

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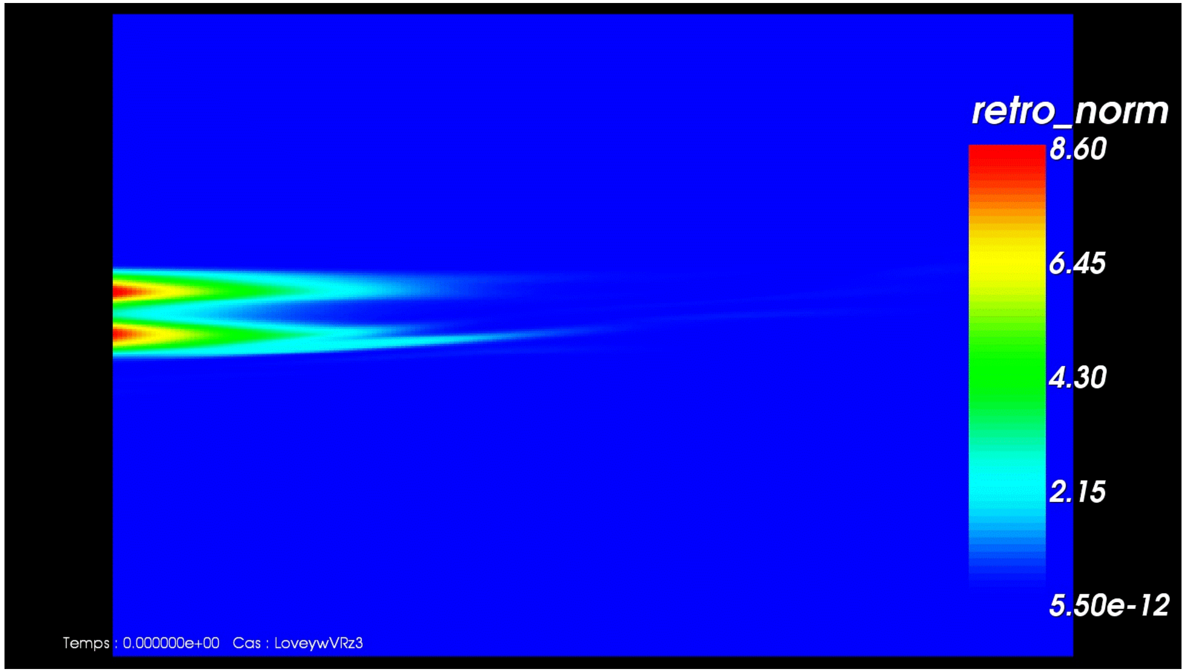

Figure 6: intensity of backscettered wave (idem)

Conclusions.

Maths

The solution Boyd-Kadomstev system with finite speed of light is close to a non-hyperbolic one.

Study stochastic initial/boundary conditions.

Numerics.

* For the eq. for (u,v), the time derivative is a perturbation.

* Extension to 3D Brillouin simulations (account for diffraction) with a δt not constrainted by the speed of light (in progress).

*Extension to Raman instability (+ supplementary terms, ε larger)