Bounds on dispersion tensor in periodic media

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Joint work with
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Outline

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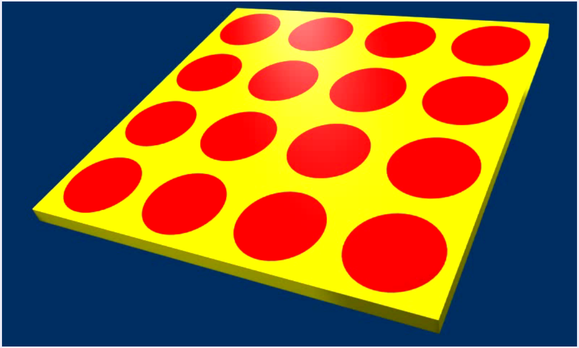
Notations

Equation

$\Omega = \mathbb{R}^N$



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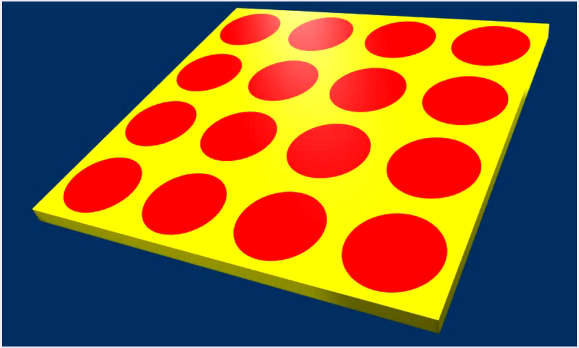
$$A^\varepsilon u^\varepsilon \stackrel{\text{def}}{=} -\frac{\partial}{\partial x_k}\left(\alpha_{k\ell}^\varepsilon(x)\frac{\partial u^\varepsilon}{\partial x_\ell}\right) = f \quad \text{in} \quad \mathbb{R}^N$$



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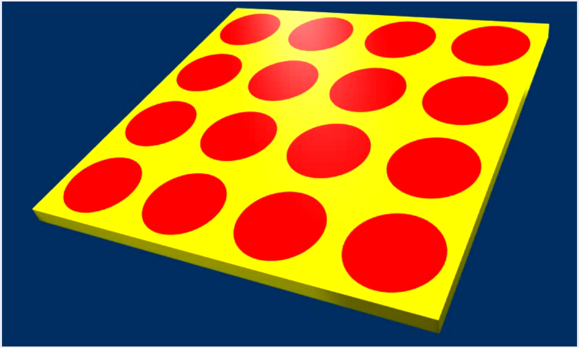

Properties of $(\alpha^\varepsilon_{k\ell})$

εY : the period

$$(\alpha^\varepsilon_{k\ell})(x) \in \{\alpha_0 I, \alpha_1 I\}$$

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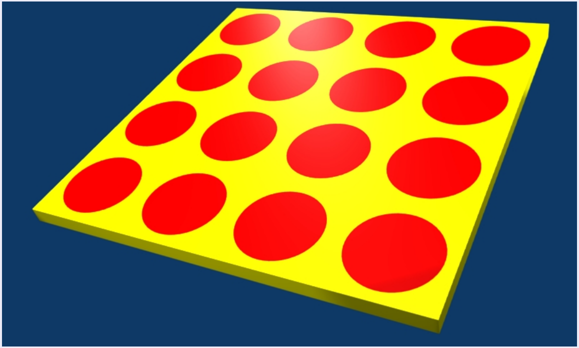

Properties of $(\alpha^\varepsilon_{k\ell})$

εY : the period

$$(\alpha^\varepsilon_{k\ell})(x) \in \{\alpha_0 I, \alpha_1 I\}$$

We are interested in

$$\varepsilon \to 0$$

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The reference cell

$$Y = (0, 2\pi)^N$$

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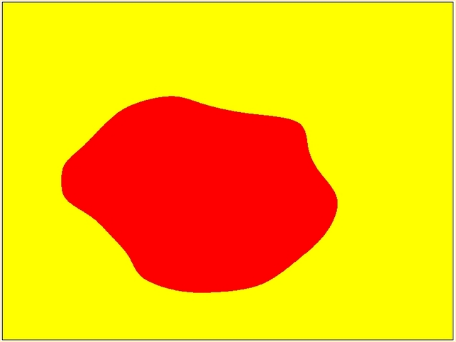
For all $y \in Y$, we consider

$$\alpha_{k\ell}(y) = (\alpha_0 \chi_{T^c}(y) + \alpha_1 \chi_T(y))\delta_{k\ell}$$



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Definition

$$A = -\frac{\partial}{\partial y_k}\left(\alpha_{k\ell}(y)\frac{\partial}{\partial y_\ell}\right)$$

Homogenization problem

Let $\{u^\varepsilon\} \subseteq H^1(\mathbb{R}^N)$ and $f \in L^2(\mathbb{R}^N)$ such that

$$-\frac{\partial}{\partial x_k}\left(\alpha^\varepsilon_{k\ell}(x)\frac{\partial u^\varepsilon}{\partial x_\ell}\right) = f \quad \text{in} \quad \mathbb{R}^N$$

If $u^\varepsilon \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, then u satisfies

$$-\frac{\partial}{\partial x_k}\left(q_{k\ell}\frac{\partial u}{\partial x_\ell}\right) = f \quad \text{in} \quad \mathbb{R}^N$$

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- In dimension $N = 1$: q depends only on γ.
- In dimension $N \geq 2$: q lies in a non-trivial set as T varies.

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- In dimension $N = 1$: q depends only on γ.
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Using Bloch waves,

- we define another parameter d: **the dispersion or Burnett tensor**,
- we look for the set where d lies when T varies preserving the volume proportion γ.

Outline

For $\eta \in \mathbb{R}^N$: find $\lambda(\eta) \in \mathbb{R}$ and $\psi(y;\eta) \neq 0$ such that

$$\begin{cases} A\psi(\cdot;\eta) = \lambda(\eta)\psi(\cdot;\eta) & \text{in } \mathbb{R}^N, \\ \psi(\cdot;\eta) & \text{is } (\eta; Y)\text{-periodic.} \end{cases}$$

(η; Y)-periodicity

$$\psi(y + 2\pi m;\eta) = e^{i\eta\cdot(2\pi m)}\psi(y;\eta) \quad \forall m \in \mathbb{Z}^N,\ y \in \mathbb{R}^N$$

Bloch waves

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By Floquet theory,

$$\psi(y;\eta) = e^{iy\cdot\eta}\phi(y;\eta) \qquad \Rightarrow \qquad \begin{cases} A(\eta)\phi(\cdot;\eta) = \lambda(\eta)\phi(\cdot;\eta) & \text{in } \mathbb{R}^N, \\ \phi(\cdot;\eta) & \text{is } Y\text{-periodic.} \end{cases}$$

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$$A(\eta) = -\left(\frac{\partial}{\partial y_k} + i\eta_k\right)\left(\alpha_{k\ell}(y)\left(\frac{\partial}{\partial y_\ell} + i\eta_\ell\right)\right).$$

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There exists a sequence of finite multiplicity eigenvalues:

$$0 \leq \lambda_1(\eta) \leq \lambda_2(\eta) \leq \cdots \leq \lambda_m(\eta) \leq \cdots \to \infty.$$

$\{\phi_m(\cdot;\eta)\}_{m\geq 1}$ form an orthonormal basis in $L^2_\#(Y)$.

Bloch waves at ε-scale

We introduce Bloch waves at the ε-scale:

$$\lambda_m^\varepsilon(\xi) = \varepsilon^{-2}\lambda_m(\eta), \qquad \phi_m^\varepsilon(x;\xi) = \phi_m(y;\eta),$$

where

$$y = \frac{x}{\varepsilon}, \qquad \eta = \varepsilon\xi.$$

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Taylor expansion for the first Bloch eigenvalue $\lambda_1^\varepsilon(\xi)$:

$$\lambda_1^\varepsilon(\xi) \;=\; \frac{1}{2!}\frac{\partial^2\lambda_1}{\partial\eta_k\partial\eta_\ell}(0)\,\xi_k\xi_\ell \;+\; \frac{1}{4!}\frac{\partial^4\lambda_1}{\partial\eta_k\partial\eta_\ell\partial\eta_m\partial\eta_n}(0)\,\xi_k\xi_\ell\xi_m\xi_n\varepsilon^2 \;+\; \cdots$$

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Remarks

- If $|\xi|^4\varepsilon^2 = o(1)$, then $q_{k\ell}$ is alone important.

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- If $|\xi|^4\varepsilon^2 = \mathcal{O}(1)$, then $d_{k\ell mn}$ need to be considered too.

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- If $|\xi|^4\varepsilon^2 = \mathcal{O}(1)$, then $d_{k\ell mn}$ need to be considered too.

$d = (d_{k\ell mn})$ is called **Burnett** or **dispersion tensor**.

Outline

Characterization of $q(\eta)$

Theorem

The homogenized quantity

$$q(\eta) \stackrel{\text{def}}{=} q_{k\ell}\eta_k\eta_\ell = \frac{1}{2!}\frac{\partial^2\lambda_1}{\partial\eta_k\partial\eta_\ell}(0)\eta_k\eta_\ell$$

is given by

$$q(\eta) = \frac{1}{|Y|}\int_Y \alpha_{k\ell}(y)\left(\frac{\partial X^{(1)}_{(T)}}{\partial y_k} + \eta_k\right)\eta_\ell dy,$$

where the cell function $X^{(1)}_{(T)}$ is defined as solution of

$$\begin{cases} -\dfrac{\partial}{\partial y_k}\left(\alpha_{k\ell}(y)\dfrac{\partial X^{(1)}_{(T)}}{\partial y_\ell}\right) = \dfrac{\partial}{\partial y_k}\left(\alpha_{k\ell}(y)\eta_\ell\right) \quad \text{in} \quad \mathbb{R}^N, \\ X^{(1)}_{(T)} \in H^1_\#(Y), \quad \dfrac{1}{|Y|}\displaystyle\int_Y X^{(1)}_{(T)} dy = 0. \end{cases}$$

Characterization of $d(\eta)$

Theorem

The dispersion quantity

$$d(\eta) \stackrel{\text{def}}{=} d_{k\ell mn}\eta_k\eta_\ell\eta_m\eta_n = \frac{1}{4!}\frac{\partial^4\lambda_1}{\partial\eta_k\partial\eta_\ell\partial\eta_m\partial\eta_n}(0)\eta_k\eta_\ell\eta_m\eta_n$$

is given by

$$d(\eta) = -\frac{1}{|Y|}\int_Y \alpha_{k\ell}(y)\frac{\partial X^{(2)}_{(T)}}{\partial y_k}\frac{\partial X^{(2)}_{(T)}}{\partial y_\ell}dy,$$

where the cell function $X^{(2)}_{(T)}$ is defined by

$$\begin{cases} -\dfrac{\partial}{\partial y_k}\left(\alpha_{k\ell}(y)\dfrac{\partial X^{(2)}_{(T)}}{\partial y_\ell}\right) = \alpha_{k\ell}(y)\left(\dfrac{\partial X^{(1)}_{(T)}}{\partial y_k} + \eta_k\right)\left(\dfrac{\partial X^{(1)}_{(T)}}{\partial y_\ell} + \eta_\ell\right) - q(\eta) \text{ in } \mathbb{R}^N, \\ X^{(2)}_{(T)} \in H^1_\#(Y), \quad \dfrac{1}{|Y|}\displaystyle\int_Y X^{(2)}_{(T)}dy = 0. \end{cases}$$

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Properties on the dispersion coefficient in low contrast regime.

Outline

Questions on bounds

For a given $\gamma \in [0,1]$ and $\eta \in \left(-\frac{1}{2}, \frac{1}{2}\right)^N$, we consider **all measurable sets** $T \subseteq Y$ such that

$$\frac{|T|}{|Y|} = \gamma.$$

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For all T, we consider the map

$$T \longmapsto d(\eta) = -m\left(\alpha(y)\nabla X^{(2)}_{(T)} \cdot \nabla X^{(2)}_{(T)}\right)$$

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Questions

- Is $\{d(\eta) : |T| = \gamma |Y|\}$ bounded?
- There exists $\max \{d(\eta) : |T| = \gamma |Y|\}$?
- There exists $\min \{d(\eta) : |T| = \gamma |Y|\}$?

Outline

Laminated structures

We consider

$$\alpha(y) \quad = \quad \alpha(y_1)$$

and then

$$X_{(T)}^{(1)}(y) = X_{(T)}^{(1)}(y_1) \qquad \text{and} \qquad X_{(T)}^{(2)}(y) = X_{(T)}^{(2)}(y_1).$$

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and then

$$X^{(1)}_{(T)}(y) = X^{(1)}_{(T)}(y_1) \qquad \text{and} \qquad X^{(2)}_{(T)}(y) = X^{(2)}_{(T)}(y_1).$$

The equations of $X^{(1)}_{(T)}$ and $X^{(2)}_{(T)}$ become

$$\begin{cases} -\dfrac{\partial}{\partial y_1}\left(\alpha(y_1)\dfrac{\partial X^{(1)}_{(T)}}{\partial y_1}\right) = \dfrac{\partial}{\partial y_1}(\alpha(y_1)\eta_1) \quad \text{in} \quad \mathbb{R}, \\ X^{(1)}_{(T)} \in H^1_{\#}(0, 2\pi), \; \mathcal{m}(X^{(1)}_{(T)}) = 0 \end{cases}$$

Laminated structures

We consider

$$\alpha(y) = \alpha(y_1)$$

and then

$$X^{(1)}_{(T)}(y) = X^{(1)}_{(T)}(y_1) \qquad \text{and} \qquad X^{(2)}_{(T)}(y) = X^{(2)}_{(T)}(y_1).$$

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and

$$\begin{cases} -\dfrac{\partial}{\partial y_1}\left(\alpha(y_1)\dfrac{\partial X^{(2)}_{(T)}}{\partial y_1}\right) = \alpha(y_1)\left|\dfrac{\partial X^{(1)}_{(T)}}{\partial y_1}\mathbf{e}_1 + \eta\right|^2 - q(\eta) \quad \text{in} \quad \mathbb{R}, \\ X^{(2)}_{(T)} \in H^1_{\#}(0, 2\pi), \; m(X^{(2)}_{(T)}) = 0. \end{cases}$$

Characterization of $q(\eta)$ in laminates

By integrating equation of $X^{(1)}_{(T)}$, we get

$$-\alpha(y_1)\left(\frac{\partial X^{(1)}_{(T)}}{\partial y_1} + \eta_1\right) = C_1, \qquad \text{where} \quad C_1 = -\frac{\eta_1}{m(1/\alpha)}.$$

Characterization of $q(\eta)$ in laminates

By integrating equation of $X_{(T)}^{(1)}$, we get

$$-\alpha(y_1)\left(\frac{\partial X_{(T)}^{(1)}}{\partial y_1}+\eta_1\right)=C_1, \qquad \text{where} \quad C_1=-\frac{\eta_1}{m(1/\alpha)}.$$

Therefore, $X_{(T)}^{(1)}$ satisfies

$$\frac{\partial X_{(T)}^{(1)}}{\partial y_1}+\eta_1=\frac{\eta_1}{m(1/\alpha)\alpha(y_1)}.$$

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We obtain the homogenized quantity can be written as

$$\begin{aligned} q(\eta) &= m\left(\alpha(y_1)\left|\frac{\partial X^{(1)}_{(T)}}{\partial y_1}\mathbf{e}_1+\eta\right|^2\right) \\ &= \frac{\eta_1^2}{m(1/\alpha)}+m(\alpha)|\tilde{\eta}|^2 \end{aligned}$$

where we have denoted $\tilde{\eta}=(0,\eta_2,\ldots,\eta_N)^T$.

We recall

$$\begin{cases} -\dfrac{\partial}{\partial y_1}\left(\alpha(y_1)\dfrac{\partial X_{(T)}^{(2)}}{\partial y_1}\right) = \underbrace{\alpha(y_1)\left|\dfrac{\partial X_{(T)}^{(1)}}{\partial y_1}\mathbf{e}_1 + \eta\right|^2}_{RHS} - q(\eta) \quad \text{in} \quad \mathbb{R}, \\ X_{(T)}^{(2)} \in H^1_{\#}(0, 2\pi),\ m(X_{(T)}^{(2)}) = 0. \end{cases}$$

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The first term in the right hand side could be written

$$\begin{aligned} RHS &= \alpha(y_1)\left|\frac{\partial X^{(1)}_{(T)}}{\partial y_1} + \eta_1\right|^2 + \alpha(y_1)|\tilde{\eta}|^2 \\ &= \frac{\eta_1^2}{\mathcal{M}^2(1/\alpha)\alpha(y_1)} + \alpha(y_1)|\tilde{\eta}|^2. \end{aligned}$$

Characterization of $d(\eta)$ in laminates

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$$\begin{cases} -\dfrac{\partial}{\partial y_1}\left(\alpha(y_1)\dfrac{\partial X^{(2)}_{(T)}}{\partial y_1}\right) = \underbrace{\alpha(y_1)\left|\dfrac{\partial X^{(1)}_{(T)}}{\partial y_1}\mathbf{e}_1 + \eta\right|^2}_{RHS} - q(\eta) \quad \text{in} \quad \mathbb{R}, \\ X^{(2)}_{(T)} \in H^1_\#(0,2\pi),\ m(X^{(2)}_{(T)}) = 0. \end{cases}$$

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Therefore, the first equation of $X^{(2)}_{(T)}$ yields

$$-\frac{\partial}{\partial y_1}\left(\alpha(y_1)\frac{\partial X^{(2)}_{(T)}}{\partial y_1}\right) = \frac{\eta_1^2}{m^2(1/\alpha)\alpha(y_1)} + \alpha(y_1)|\tilde{\eta}|^2 - \frac{\eta_1^2}{m(1/\alpha)} - m(\alpha)|\tilde{\eta}|^2,$$

Characterization of $d(\eta)$ in laminates

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By integration, one could obtain

$$-\alpha(y_1)\frac{\partial X^{(2)}_{(T)}}{\partial y_1} = C_2 + \frac{\eta_1^2}{m^2(1/\alpha)}\int_0^{y_1}\left(\frac{1}{\alpha(s)} - m(1/\alpha)\right)ds + |\tilde{\eta}|^2\int_0^{y_1}(\alpha(s) - m(\alpha))ds.$$

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Characterization of $d(\eta)$ in laminates

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Characterization of $d(\eta)$ in laminates

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Let us observe that for $k \in \{-1, 1\}$,

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Characterization of $d(\eta)$ in laminates

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Let us observe that for $k \in \{-1, 1\}$,

$$\begin{aligned} \alpha^k(s) &= \alpha_1^k \chi_T(s) + \alpha_0^k(1-\chi_T(s)), \\ m(\alpha^k) &= \alpha_1^k\gamma + \alpha_0^k(1-\gamma). \end{aligned}$$

By integrating we then get

$$\int_0^{y_1}(\alpha^k(s) - m(\alpha^k))ds = (\alpha_1^k - \alpha_0^k)\underbrace{\int_0^{y_1}(\chi_T(s)-\gamma)ds}_{a_T(y_1)}.$$

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$$-\alpha(y_1)\frac{\partial X^{(2)}_{(T)}}{\partial y_1} = C_2 + \frac{\eta_1^2}{\mathcal{M}^2(1/\alpha)}\int_0^{y_1}\left(\frac{1}{\alpha(s)} - \mathcal{M}(1/\alpha)\right)ds + |\tilde{\eta}|^2\int_0^{y_1}(\alpha(s) - \mathcal{M}(\alpha))ds.$$

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The constant C_2 is computed by taking the average on $(0,2\pi)$:

$$C_2\,\mathcal{M}(1/\alpha) = -\left[\frac{\eta_1^2}{\mathcal{M}^2(1/\alpha)}\left(\frac{1}{\alpha_1}-\frac{1}{\alpha_0}\right)+|\tilde{\eta}|^2(\alpha_1-\alpha_0)\right]\mathcal{M}(a_T/\alpha).$$

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Thus,

$$-\alpha(y_1)\frac{\partial X^{(2)}_{(T)}}{\partial y_1} = \left[\frac{\eta_1^2}{m^2(1/\alpha)}\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0}\right) + |\tilde{\eta}|^2(\alpha_1 - \alpha_0)\right]\left[a_T(y_1) - \frac{m(a_T/\alpha)}{m(1/\alpha)}\right].$$

Properties of function a_T

Lemma

For any $k \in \mathbb{Z}$ *and any* $\ell \in \mathbb{N}$*, we have*

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We obtain

$$\left[\alpha^k - \mathfrak{m}(\alpha^k)\right] a_T^\ell = (\alpha_1^k - \alpha_0^k) a_T^\ell \cdot a_T' = \frac{(\alpha_1^k - \alpha_0^k)}{\ell + 1}(a_T^{\ell+1})'$$

and then

$$\mathfrak{m}\Big(\alpha^k a_T^\ell - \mathfrak{m}(\alpha^k) a_T^\ell\Big) = 0.$$

Characterization of $d(\eta)$ in laminates

Using Lemma with $k = -1$ and $\ell = 1$, the equation

$$-\alpha(y_1)\frac{\partial X^{(2)}_{(T)}}{\partial y_1} = \left[\frac{\eta_1^2}{\mathcal{M}^2(1/\alpha)}\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0}\right) + |\tilde{\eta}|^2(\alpha_1 - \alpha_0)\right]\left[a_T(y_1) - \frac{\mathcal{M}(a_T/\alpha)}{\mathcal{M}(1/\alpha)}\right]$$

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Since

$$d(\eta)=-\mathcal{M}\left(\alpha(y_1)\frac{\partial X^{(2)}_{(T)}}{\partial y_1}\cdot\frac{\partial X^{(2)}_{(T)}}{\partial y_1}\right),$$

we obtain

$$-d(\eta)=\left[\frac{\eta_1^2}{\mathcal{M}^{3/2}(1/\alpha)}\left(\frac{1}{\alpha_1}-\frac{1}{\alpha_0}\right)+|\tilde{\eta}|^2\mathcal{M}^{1/2}(1/\alpha)(\alpha_1-\alpha_0)\right]^2\left[\mathcal{M}(a_T^2)-\mathcal{M}^2(a_T)\right]$$

Relation between laminates and 1D

We have

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Theorem

The homogenized and the dispersion coefficients in 1D are given by

$$\frac{1}{q_1} = m(1/\alpha) = \frac{\gamma}{\alpha_1} + \frac{1-\gamma}{\alpha_0}, \qquad d_1 = -q_1\, m\Big((X_{(T)})^2\Big),$$

with the cell function $X_{(T)}$ defined by

$$\begin{cases} -\dfrac{dX_{(T)}}{dy} = 1 - q_1\left(\dfrac{\chi_T}{\alpha_1} + \dfrac{1-\chi_T}{\alpha_0}\right) \quad \text{in} \quad \mathbb{R}, \\ X_{(T)} \in H^1_{\#}(Y),\ m(X_{(T)}) = 0. \end{cases}$$

Characterizations of q_1 and d_1 in 1D

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d_1 depends on the microstructure: $\chi_T \rightarrow X_{(T)} \rightarrow d_1.$

Outline

Notations

$$\mathrm{Char}(Y) = \{x : Y \to \{0,1\}\}, \qquad T(x) = \{y \in Y : x(y) = 1\},$$

$$\mathcal{C}_\gamma \;=\; \{x \in \mathrm{Char}(Y) \;:\; |T(x)| = \gamma|Y|\}.$$

$$\mathrm{Char}(Y) = \{\chi : Y \to \{0,1\}\}, \qquad T(\chi) = \{y \in Y : \chi(y) = 1\},$$

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We define

$$\begin{aligned} J_0 : \mathrm{Char}(Y) \; &\longrightarrow \; \mathbb{R} \\ \chi \; &\longmapsto \; J_0(\chi) \stackrel{\text{def}}{=} m((X_{(T(\chi))})^2), \end{aligned}$$

then the dispersion coefficient can be rewritten as follows

$$d_1 = -q_1 \, J_0(\chi_T).$$

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Moreover,

$$-q_1 \sup_{\chi \in \mathcal{C}_\gamma} J_0(\chi) \;\le\; d_1 \;\le\; -q_1 \inf_{\chi \in \mathcal{C}_\gamma} J_0(\chi) \qquad \forall \chi_T \in \mathcal{C}_\gamma.$$

Relaxation

We define

$$\mathcal{D}_\gamma = \{\theta \in L^\infty_\#(Y;[0,1]) \ : \ m(\theta) = \gamma\}$$

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X_θ is the solution of

$$\begin{cases} -\dfrac{dX_\theta}{dy} = 1 - q_1(m(\theta))\left(\dfrac{\theta}{\alpha_1} + \dfrac{1-\theta}{\alpha_0}\right) & \text{in} \quad \mathbb{R}, \\ X_\theta \in H^1_\#(Y),\ m(X_\theta) = 0, \end{cases}$$

where

$$\frac{1}{q_1(\tau)} = \frac{\tau}{\alpha_1} + \frac{1-\tau}{\alpha_0}.$$

Outline

First result on bounds in 1D

Theorem

For any $\gamma \in (0,1)$*, we have that*

$$\inf_{\chi\in\mathcal{C}_\gamma} J_0(\chi) = \min_{\theta\in\mathcal{D}_\gamma} J(\theta), \qquad \sup_{\chi\in\mathcal{C}_\gamma} J_0(\chi) = \max_{\theta\in\mathcal{D}_\gamma} J(\theta),$$

$$\min_{\theta\in\mathcal{D}_\gamma} J(\theta) = 0,$$

$$\max_{\theta\in\mathcal{D}_\gamma} J(\theta) = \frac{1}{12} q_1^2 \gamma^2 (1-\gamma)^2 |2\pi|^2 \left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0}\right)^2.$$

Moreover, $\exists \chi^*_{max} \in \mathcal{C}_\gamma$ *such that*

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C. Conca, J. San Martín, L. Smaranda and M. Vanninathan,
Optimal bounds on dispersion coefficient in one-dimensional periodic media,
Math. Models Methods Appl. Sci., 19 (2009).

Corollary

For any $\gamma \in (0,1)$, *the following inclusion holds:*

$$\left\{ d_1 \;:\; |T| = \gamma |Y| \right\} \subseteq \left[-\frac{1}{12} q_1^3 \gamma^2 (1-\gamma)^2 |2\pi|^2 \left(\tfrac{1}{\alpha_1} - \tfrac{1}{\alpha_0}\right)^2, 0 \right).$$

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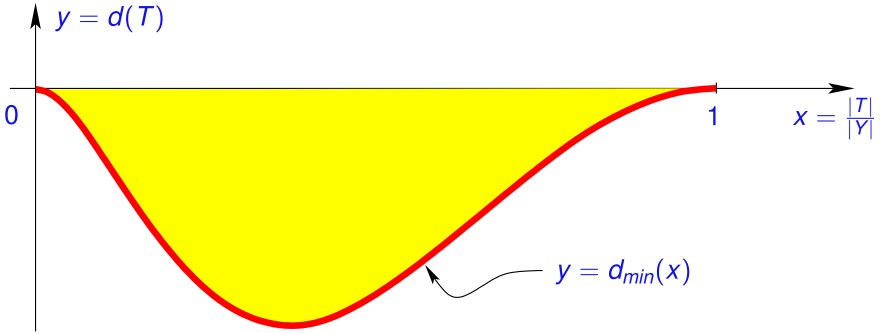

Consequence of the first result in 1D

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Second result in 1D

Theorem (d_1 fills up the interval)

For any $\gamma \in (0,1)$, the following converse inclusion holds:

$$\left\{ d_1 \ : \ |T| = \gamma |Y| \right\} \supseteq \left[-\frac{1}{12} q_1^3 \gamma^2 (1-\gamma)^2 |2\pi|^2 \left(\tfrac{1}{\alpha_1} - \tfrac{1}{\alpha_0}\right)^2, 0 \right).$$

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Since

$$\left\{ d_1 \ : \ |T| = \gamma |Y| \right\} = \left[-\frac{1}{12} q_1^3 \gamma^2 (1-\gamma)^2 |2\pi|^2 \left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0} \right)^2, 0 \right),$$

and

$$d_1 = -\left[\frac{1}{m^{3/2}(1/\alpha)} \left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0} \right) \right]^2 \left(m(a_T^2) - m^2(a_T) \right),$$

one can prove that

Theorem (Result for laminates)

For any $\gamma \in (0,1)$ and for any $\eta \in Y'$, the following equality holds:

$$\left\{ d(\eta) \ : \ |T| = \gamma |Y| \right\} =$$

$$\left[-\frac{1}{12} \gamma^2 (1-\gamma)^2 |2\pi|^2 \left[\frac{\eta_1^2}{m^{3/2}(1/\alpha)} \left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0} \right) + |\tilde{\eta}|^2 m^{1/2}(1/\alpha)(\alpha_1 - \alpha_0) \right]^2, 0 \right).$$

Outline

Minimization problem

We have

$$J(\theta) \geq 0 \qquad \forall \theta \in \mathcal{D}_\gamma.$$

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$$q_1 \left(\frac{\theta^*_{min}(y)}{\alpha_1} + \frac{1 - \theta^*_{min}(y)}{\alpha_0} \right) = 1,$$

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Remark

All minimizers are equal to θ^*_{min}.

Outline

Step 1: Lagrange multipliers

Since $\mathcal{D}_\gamma$ is defined with the constraint $m(\theta) = \gamma$, we introduce the Lagrangian

$$L(\theta, \lambda) = J(\theta) + \lambda(m(\theta) - \gamma)$$

for all $\theta \in L^\infty_\#(Y; [0, 1])$ and $\lambda \in \mathbb{R}$.

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Proposition

For each $\theta^* \in \mathcal{D}_\gamma$ with $J(\theta^*) = \max_{\theta \in \mathcal{D}_\gamma} J(\theta)$, there exists $\lambda^* \in \mathbb{R}$ such that

$$D_\theta L(\theta^*, \lambda^*)(\theta - \theta^*) \quad \leq \quad 0$$

for all $\theta \in L^\infty_\#(Y; [0,1])$.

Maximization problem

$$D_\theta L(\theta^*, \lambda^*)(\theta - \theta^*) =$$

$$q_1 \left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0}\right) m\left(\left[P_{\theta^*} + \frac{\lambda^*}{q_1 \left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0}\right)} - q_1 \left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0}\right) m(P_{\theta^*} \cdot \theta^*) \right] \cdot (\theta - \theta^*) \right),$$

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where $P_{\theta*}$ is solution of the adjoint state equation

$$\begin{cases} -\dfrac{dP_{\theta*}}{dy} = 2X_{\theta*} \quad \text{in} \quad \mathbb{R}, \\ P_{\theta*} \in H^1_\#(Y), \ m(P_{\theta*}) = 0. \end{cases}$$

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The condition $D_\theta L(\theta^*, \lambda^*)(\theta - \theta^*) \leq 0$ becomes:

$$\int_Y \left(P_{\theta*}(y) - p^*\right)\left(\theta(y) - \theta^*(y)\right) dy \leq 0 \qquad \forall \theta \in L^\infty_\#(Y; [0, 1]).$$

Maximization problem

Step 2: Optimality condition

We define

$$\mathscr{A}(\theta^*, p^*) = \{y \in \mathbb{R} : P_{\theta^*}(y) = p^*\},$$

$$\mathscr{B}(\theta^*, p^*) = \{y \in \mathbb{R} : P_{\theta^*}(y) > p^*\}, \qquad \mathscr{C}(\theta^*, p^*) = \{y \in \mathbb{R} : P_{\theta^*}(y) < p^*\}.$$

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Proposition

For each $\theta^* \in \mathcal{D}_\gamma$ *with* $J(\theta^*) = \max_{\theta \in \mathcal{D}_\gamma} J(\theta)$, *there exists* $p^* \in \mathbb{R}$ *such that the optimality condition (O.C.) holds:*

$$\begin{cases} \theta^* = 1 & \text{a.e. in } \mathscr{B}(\theta^*, p^*), \\ \theta^* = 0 & \text{a.e. in } \mathscr{C}(\theta^*, p^*). \end{cases}$$

Maximization problem

Step 3: New expression of $J(\theta^*)$

We denote

$$\Theta_\gamma = \left\{\theta^* \in \mathcal{D}_\gamma : \exists p^* \in \mathbb{R} \text{ such that O.C. holds}\right\}.$$

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Lemma

For any $(\theta^*, p^*) \in \Theta_\gamma \times \mathbb{R}$ *satisfying O.C., there exists* $y_{\mathscr{A}} \in \mathscr{A}(\theta^*, p^*)$ *such that:*

$$\mathscr{B}(\theta^*, p^*) \cap (y_{\mathscr{A}} + Y) = \bigcup_{i=1}^{N_{\mathscr{B}}} (a_i, b_i), \qquad \mathscr{C}(\theta^*, p^*) \cap (y_{\mathscr{A}} + Y) = \bigcup_{j=1}^{N_{\mathscr{C}}} (c_j, d_j),$$

where $N_{\mathscr{B}}, N_{\mathscr{C}} \in \mathbb{N} \cup \{+\infty\}$ *and* $a_i, b_i, c_j, d_j \in \mathscr{A}(\theta^*, p^*)$, $\forall i \in \{1, \ldots, N_{\mathscr{B}}\}$, $j \in \{1, \ldots, N_{\mathscr{C}}\}$.

Maximization problem

By integrating X_{θ^*} and P_{θ^*} in each (a_i, b_i) and (c_j, d_j), we prove

Proposition

For any $(\theta^*, p^*) \in \Theta_\gamma \times \mathbb{R}$ *satisfying O.C. and* $y_{\mathscr{A}} \in \mathscr{A}(\theta^*, p^*)$*:*

$$J(\theta^*) = \frac{q_1^2}{12}\gamma^2(1-\gamma)^2|2\pi|^2\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0}\right)^2 \left[\gamma \sum_{i=1}^{N_{\mathscr{B}}} x_i^3 + (1-\gamma)\sum_{j=1}^{N_{\mathscr{C}}} z_j^3\right],$$

where $x_i = \dfrac{b_i - a_i}{\gamma|Y|}$ *and* $z_j = \dfrac{d_j - c_j}{(1-\gamma)|Y|}$.

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Maximization problem

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1. $J(\theta^*) \leq \frac{q_1^2}{12}\gamma^2(1-\gamma)^2|2\pi|^2\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_0}\right)^2.$

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$$\theta^*_{max}(y) = \begin{cases} 1 & \text{if} \quad y \in [0, \gamma|Y|], \\ 0 & \text{if} \quad y \in (\gamma|Y|, |Y|), \end{cases}$$

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3. All maximizers of J are characteristic functions and are obtained for $N_{\mathscr{B}} = N_{\mathscr{C}} = 1$ and $x_1 = z_1 = 1$.

Outline

Proof of second result

For any $n \in \mathbb{N}^*$ and $\delta \in (0, 1)$ we define $\theta^*_{n,\delta}$ in $[0, 2\pi]$ as follows:

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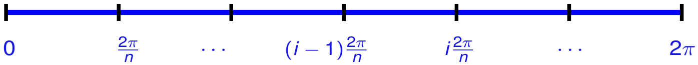

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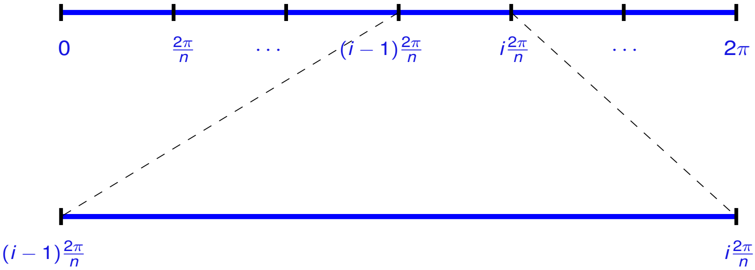

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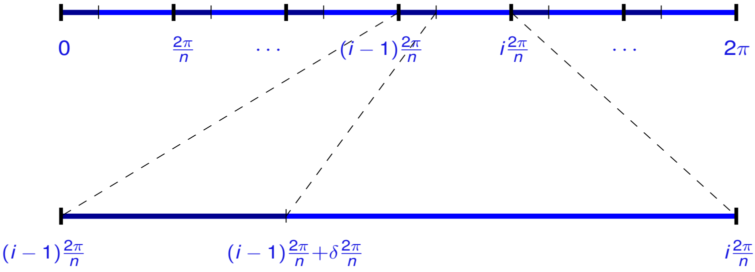

$$2n \text{ intervals:} \begin{cases} n \text{ of length } \delta\frac{2\pi}{n} \\ n \text{ of length } (1-\delta)\frac{2\pi}{n} \end{cases}$$

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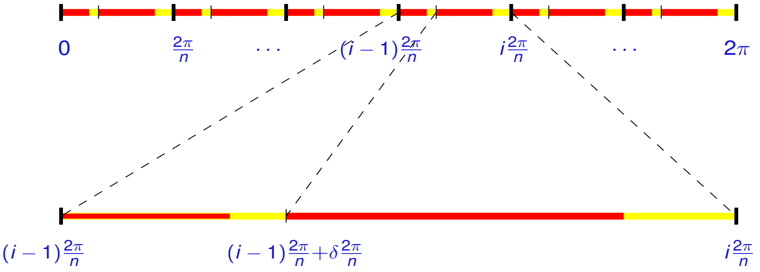

$$a_{2i-1} = (i-1)\frac{2\pi}{n} \qquad b_{2i-1} = c_{2i-1} = (i-1+\gamma\delta)\frac{2\pi}{n}$$
$$d_{2i-1} = a_{2i} = (i-1+\delta)\frac{2\pi}{n} \qquad b_{2i} = c_{2i} = (i-(1-\gamma)(1-\delta))\frac{2\pi}{n} \qquad d_{2i} = i\frac{2\pi}{n}$$

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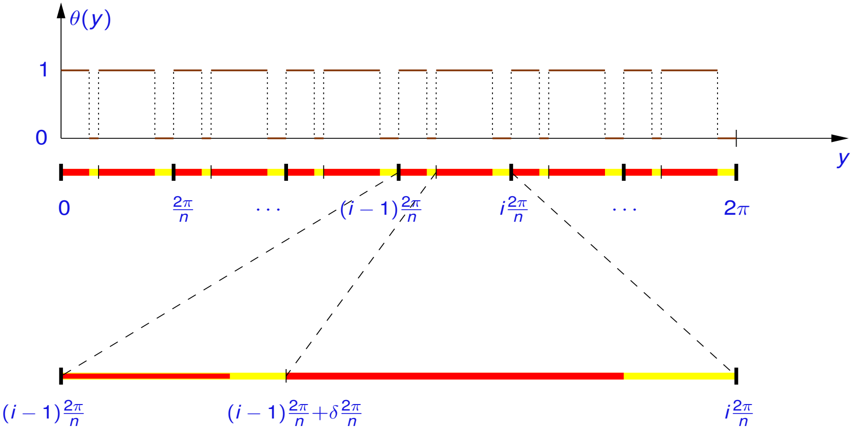

$$\theta^*_{n,\delta}(y) = \begin{cases} 1 & \text{if } y \in \bigcup_{i=1}^{2n} [a_i, b_i], \\ 0 & \text{if } y \in \bigcup_{j=1}^{2n} (c_j, d_j). \end{cases}$$

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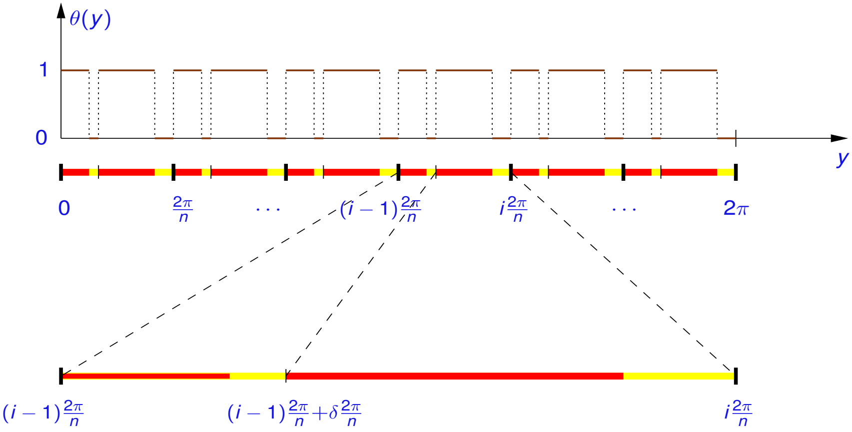

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We have

$$J(\theta^*_{n,\delta}) = \frac{1 - 3\delta + 3\delta^2}{n^2} \max_{\theta \in \mathcal{D}_\gamma} J(\theta).$$

Outline

Perspectives

- The behavior of the dispersion tensor for other structures with some kind of symmetry.

  Interesting particular cases are the **spherical Hashin structures**.

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- Properties on the dispersion for materials with ellipsoidal geometries.

- Find bounds in higher dimensions for general materials.

Thank you!

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