Multiple positive solutions for quasilinear degenerate equations

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We consider the quasilinear problems of the type:
Let $\Omega \subset \mathbb{R}^N$,

$$\begin{aligned} -\text{div } a(x,u,\nabla u) + b(x,u,\nabla u) &= f(u) \text{ in } \Omega \\ a(x,u,\nabla u)\cdot\nu &= \lambda g(u) \text{ on } \partial\Omega \end{aligned}$$

or

$$u = 0 \text{ on } \partial\Omega$$

with $f(u)$ has critical growth and $g(u)$ is of "sublinear"

We are interested in

- Existence of a positive solution
- Uniform estimates
- Existence of second positive solution

Dirichlet boundary conditions:

$$(PD)_\lambda \begin{cases} -\Delta_p u &= f_\lambda(x,u), \ u > 0 \text{ in } \Omega \\ \qquad u &= 0 \ \text{ on } \ \partial\Omega \end{cases}$$

$\Omega \subset I\!R^N, \ |\Omega| < \infty, \ 1 < p \le N$

$\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$

$t \mapsto f_\lambda(x,t)$ is increasing for $t \ge 0$ and satisfies

$$\lim_{t \to 0,\infty} \frac{f_\lambda(x,t)}{t^{p-1}} = \infty, \quad |f_\lambda(x,t)| \le \begin{cases} C(1 + t^{p^*-1}) & p < N \\ Ct^\alpha e^{|t|^{\frac{N}{N-1}}} & p = N \end{cases}$$

where $p^* = \frac{Np}{N-p}$ and $\alpha > N - 1$

For example: $f_\lambda(x,t) = \lambda t^q + t^\alpha, \ 0 < q < p - 1 < \alpha \le p^* - 1$

Neumann boundary conditions:

$$(PN)_\lambda \begin{cases} -\Delta_p u + u^{p-1} & = f(x,u),\ u > 0 \text{ in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} & = \lambda g(x,u) \ \text{ on } \ \partial\Omega \end{cases}$$

$t \mapsto f(x,t)$ satisfies

$$\lim_{t\to 0}\frac{f(x,t)}{t^{p-1}} < \infty,\ \ \lim_{t\to 0}\frac{g(x,t)}{t^{p-1}} = \infty$$

$f(x,t), g(x,t)$ are increasing in t for $t \geq 0$

$$|f(x,t)| \leq \begin{cases} C(1+t^{p^*-1}) & p < N \\ Ct^{\alpha}e^{t^{\frac{N}{N-1}}} & p = N,\ \alpha > p-1. \end{cases}$$

$|g(x,t)| \leq Ct^q,\ q < p-1$

”Some generalizations”

$$(\Phi N)_\lambda \begin{cases} -\Delta_\phi u + \phi(|u|)u & = f(x,u),\ u>0 \text{ in } \Omega \\ \phi(|\nabla u|)\frac{\partial u}{\partial n} & = \lambda g(x,u) \ \text{ on } \ \partial\Omega \end{cases}$$

where $\Delta_\phi u = \nabla \cdot (\phi(|\nabla u|)\nabla u)$ and $\phi(t)$ is C^1 function satisfying:

$$1 < p_0, p_1 \le N, \qquad \phi(t) \sim \begin{cases} t^{p_0-2} & t \to 0 \\ t^{p_1-2} & t \to \infty \end{cases}.$$

$|g(x,t)| \le Ct^q$, $0 < q < \min\{p_0-1, p_1-1\}$

$f(x,t) \sim t^\alpha,\ \alpha > \max\{p_0-1, p_1-1\}$ near $t=0$ and $f(x,t) \sim t^{p_1^*}$ as $t \to \infty$.

Brief History:

♠ Ambrosetti-Brezis-Cerami, 1994, JFA

$p = 2$ Laplacian case. They proved that $\exists \Lambda > 0$ such that

(a) (P_λ) has two solutions for $\lambda < \Lambda$ and no solution for $\lambda > \Lambda$

(b) At least one solution for $\lambda = \Lambda$.

<u>Outline:</u>

• Minimal solution u_λ

$\underline{u} = \epsilon\phi_1$, (ϕ_1 is the first eigenfunction)

$\overline{u} = Mu$ where $-\Delta u = 1, \Omega, u = 0 \; \partial\Omega$

$\overline{u}$ is a super solution for λ small and $\underline{u} \leq \overline{u}$ for small ϵ.

$\Lambda = \sup\{\lambda : (P_\lambda)$ admits a solution $\}$.

• Ordered minimal solutions

$\lambda_1 < \lambda < \lambda_2 \implies u_{\lambda_1} < u_\lambda < u_{\lambda_2}$ in Ω and

Höpf maximum principle $\implies \frac{\partial u_{\lambda_1}}{\partial \nu} > \frac{\partial u_\lambda}{\partial \nu} > \frac{\partial u_{\lambda_2}}{\partial \nu}$ on $\partial\Omega$.

$\therefore \|u_{\lambda_2} - u_{\lambda_1}\|_{C^1} > 0$

• $C_0^1(\Omega)$ local minimum

Let $\lambda \in (\lambda_1, \lambda_2)$. Then u_λ is a local minimum of J_λ in $C_0^1(\Omega)$.

$$\text{Then } \|u_{\lambda_1} - u_{\lambda_2}\|_{C_0^1(\Omega)} = \delta > 0$$

Define

$$\overline{f}(x,u) = f(u)\big|_{[u_{\lambda_1}, u_{\lambda_2}]}$$

$$\overline{J}(u) = \frac{1}{2}\|u\|^2 - \int_\Omega \overline{F}(x,u)$$

where $\overline{F}(x,u) = \int_0^u \overline{f}(x,s)ds$.
Then $\overline{J}$ admits global minimum $u_\lambda \in [u_{\lambda_1}, u_{\lambda_2}]$.
So if we take $u \in \{v : \|v - u_\lambda\|_{C_0^1(\Omega)} \leq \frac{\delta}{2}\}$
Then Höpf maximum principle, $u_{\lambda_1} < u < u_{\lambda_2}$ in Ω and hence
$\overline{J}(u) = J(u)$

• H^1 Versus C^1 local minimum

Local minimum in H^1 topology is always C^1 local minimum as the topology in C^1 is stronger

But we need converse to apply mountain pass Lemma!

Suppose u_λ is not a local minimum then

By Lagrange multiplier theorem,

$\exists\, u_\epsilon \in \{u \in H_0^1(\Omega) :\ K(u) := \|u_\lambda - u\|_{H^1} \leq \epsilon\}$ and $\mu_\epsilon \leq 0$ such that

$$J_\lambda'(u_\epsilon) = \mu_\epsilon K'(u_\lambda - u_\epsilon)$$

i.e., $-\Delta u_\epsilon - f(u_\epsilon) = \mu_\epsilon(\Delta(u_\lambda - u_\epsilon))$ in Ω, $u_\epsilon = 0$ on $\partial\Omega$

i.e., $-(1-\mu_\epsilon)\Delta u_\epsilon = f(u_\epsilon) - \mu_\epsilon f(u_\lambda)$ in Ω, $u_\epsilon = 0$ on $\partial\Omega$

since $\mu_\epsilon \leq 0$ standard elliptic estimates imply that

$$\|u_\epsilon\|_{C^{1,\alpha}} \leq C.$$

Since $u_\epsilon \to u_\lambda$ pointwise, we get $u_\epsilon \to u_\lambda$ in C^1.

• Mountain-pass around u_λ implies second solution.

♠ Ambrosetti-Garcia-Peral,1996, JFA,
Obtained above result for p-Laplacian radial problem:

$$-\Delta_p u := -(r^{N-1}|u'|^{p-2}u')' = r^{N-1}(f_\lambda(u)),\ u > 0, \quad \text{in} \quad \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

$f_\lambda(u)$ is of sub-critical
• Minimal solution u_λ exists for $\lambda \in (0, \Lambda)$.
• Let $K_\lambda(v) = (\Delta_p)^{-1}(r^{N-1}f(v))$, and $M = \{u; \underline{u} \le u \le \overline{u}\}$, then

$$K_{\lambda_0}(M) \subset M$$

• Höpf Maximum principle $\implies \quad u_\lambda + \epsilon B_1 \subset M$.

- If u_λ, minimal solution is the unique solution, then

$$deg(I - K_\lambda, u_\lambda + \epsilon B_1, 0) = i(K_\lambda, M, M) = 1$$

- a-priori estimates $\implies \quad \exists \rho$ s.t. $\|u_\lambda\|_\infty \leq \rho$

$$deg(I - K_\lambda, \rho B_1, 0) = deg(I - K_\Lambda + \delta, \rho B_1, 0) = 0$$

- By excision property of degree

$$deg(I - K_\lambda, \rho B_1 \backslash \{u_\lambda + \epsilon B_1\}, 0) = -1$$

Hence the second fixed point for K_λ

♠ Critical case in Ball: S.Prashanth-K.S., 2002, ADE

We considered the radial problem in the Ball with $f_\lambda(u)$ critical:

$$-(r^{N-1}|u'|^{p-2}u')' = r^{N-1}(\lambda u^q + u^{p^*-1} \quad \text{in} \quad \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

Use the Transformations:

- $r \to \lambda^{-\frac{\alpha-p+1}{p(\alpha-q)}} r,\ w(r) = \lambda^{-\frac{1}{\alpha-q}} u(\lambda^{-\frac{\alpha-p+1}{p(\alpha-q)}} r),\ r < \lambda^{-\frac{\alpha-p+1}{p(\alpha-q)}}$
- $t = (\frac{\gamma}{r})^\gamma, y(t) = w(r)$. Then $y(t)$ satisfies

$$\begin{cases} -(|y'|^{p-2}y')' & = t^{-k}(y^{p^*-1} + y^q), y > 0, t \in (T, \infty) \\ y(T) = 0, & y'(\infty) = 0 \end{cases}$$

So We study the parametrized problem:

$$(P_\gamma) \quad \begin{cases} -(|y'|^{p-2}y')' & = t^{-k}(y^{p^*-1} + y^q), \; y > 0, \; t \in (T, \infty) \\ y'(\infty) = 0, & \lim_{t\to\infty} y(t) = \gamma. \end{cases}$$

We study the map $\gamma \mapsto T(\gamma)$. We can show that

Theorem: $T(\gamma)$ is continuous and $\lim_{\gamma\to 0,\infty} T(\gamma) = \infty$.

♠ Critical case in bounded domains

Defigueredo-Gossez-Ubilla, JFA 2009
Let $f(s) = \lambda s^q + s^{p^*-1}$, and p, q satisfies:

1. $\frac{2N}{N+2} < p < 3$,
2. $p \geq 3, p^* - \frac{2}{p-1} < q + 1 < p$.

- Minimal solutions and C^1 local minimum follow similarly.
- $W_0^{1,p}(\Omega)$ local minimum If 0 is a local minimizer of $J_\lambda(. + u_{\lambda_2})$ in C_0^1 topology, then u_λ is also local minimizer in $W^{1,N}$ topology.
  Suppose not, $\exists \{v_n\} \subset W_0^{1,p}(\Omega)$ such that

$$\|v_n\| < \frac{1}{n}, \quad J(u_0 + v_n) < J(u_0) \; \forall \; n$$

Hence the minimization problem

$$\min_{u \in B_{\frac{1}{n}}(0)} J_\lambda(u + u_\lambda)$$

admits a minimizer, say u_n. So by Lagrange Multiplier rule, $\exists \mu_\epsilon$ such that

$$\langle J'_\lambda(u_n + u_\lambda), \phi \rangle = \mu_\epsilon \int_\Omega |\nabla u_n|^{p-2} \nabla u_n . \nabla \phi, \ \forall \phi \in W^{1,p}(\Omega)$$

$i.e.,\ \ -\Delta_p(u_n + u_\lambda) + \mu_\epsilon \Delta_p u_n = f(u_n + u_\lambda)$ in Ω $u_n = 0$on $\partial\Omega$.

Now if

$$\|u_\lambda + u_n\|_{C^{1,\beta}(\overline{\Omega})} \leq C.$$

then $u_n \to 0$ in C^1 which gives a contradiction.

But This estimate is DIFFICULT !!

Minimize over L^α Balls

$$\text{Let } C_\epsilon = \{u \in W^{1,N} : K(u) = \frac{1}{p^*}\int_\Omega |u|^{p^*} \leq \epsilon\}$$

$\exists u_\epsilon \in C_\epsilon \min_{C_\epsilon} J(u+u_0) = J(u_\epsilon + u_0) < J(u_0)$. $\therefore$ $\{u_\epsilon\}$ is bounded and hence $u_\epsilon + u_0 \rightharpoonup u_0$ in $W^{1,p}(\Omega)$.

By Lagrange Multiplier theorem, $\exists\ \mu_\epsilon \leq 0$ such that

$$J^{'}(u_\epsilon + u_0) = \mu_\epsilon K^{'}(u_\epsilon), \quad \epsilon \in (0,1).$$

$$\begin{aligned} -\Delta_p(u_0+u_\epsilon) &= f(u_0+u_\epsilon) + \mu_\epsilon u_\epsilon^{p^*-1} \text{ in } \Omega \\ u_\epsilon &= 0 \text{ on } \partial\Omega \end{aligned}$$

Then one can show

- $\|u_\epsilon\|_{L^\infty} \leq C$.
- $\|u_\epsilon\|_{C^{1,\alpha}} \leq C$.(By Lieberman's results)

Therefore $u_\epsilon \to 0$ in C^1, contradiction.

On Regularity

G.M.Lieberman, 1988, Nonlinear Analysis: If
$u \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$ satisfies

$$divA(x,u,\nabla u) + B(x,u,\nabla u) = 0 \text{ in } \Omega$$
$$A(x,u,\nabla u).\nu = \phi(x,u) \text{ on } \partial\Omega \; or \; u = 0 \text{ on } \partial\Omega$$

where ϕ is bounded, Hölder continuous and $A(x,u,\nabla u)$ satisfies

$$\lambda(k+|\eta|^{p-2})|\xi|^2 \le a^{ij}(x,z,\eta)\xi_i\xi_j \le \Lambda(k+|p|^{p-2})|\xi|^2, \; \lambda,\Lambda,k>0$$
$$|a^{ij}(x,z,\eta)| \le \Lambda(k+|\eta|^{p-2})$$
$$|A(x,z,\eta)-A(y,z,\eta)| \le \Lambda(1+|\eta|^{p-1})(|x-y|^\alpha)$$
$$|B(x,z,\eta)| \le \Lambda(1+|\eta|)^p.$$

Then there exists $\beta \in (0,1)$ such that

$$\|u\|_{C^{1,\beta}(\overline{\Omega})} \le C(\alpha,\lambda,\Lambda,C_1\|u\|_\infty,N,p,\Omega)$$

Now consider the BVP:

$$(PN)_\lambda \begin{cases} -\Delta_p u + u^{p-1} & = f(u), \ u > 0 \text{ in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} & = \lambda g(u) \ \text{ on } \ \partial\Omega \end{cases}$$

with $f(u) = u^{p^*-1}$ and $g(u) = u^q, 0 < q < p-1$.

• A.Garcia, I.Peral and J.D.Rossi 2004 , JDE
Let $\Omega \subset I\!R^N, N \geq 3$

$$\begin{aligned} -\Delta u + u & = u^{2^*-1}, u > 0 \ \text{ in } \Omega \\ \frac{\partial u}{\partial n} & = \lambda u^q \ \text{ on } \ \partial\Omega, \ \ 0 < q < 1 \end{aligned}$$

• S.Prashanth-K.S- Nonlinear analysis, 2007

$$\begin{aligned} -\Delta u + u & = u^p e^{u^2}, u > 0 \ \ in\ \Omega \\ \frac{\partial u}{\partial n} & = \lambda u^q \ \ on\ \partial\Omega \end{aligned}$$

♠ S.Bhatia, R.Dhanya-K.S. Adv.Nonlinear Studies, 2010

We studied the multiplicity for $(PN)_\lambda$ and obtained the following:

Theorem(Existence): There exists $\Lambda > 0$ such that (P_λ) admits a minimal solution for all $\lambda \in (0, \Lambda)$ and no solution for $\lambda > \Lambda$.

Theorem(Multiplicity): Let $Q(N,p)$ be defined as:

$$Q(N,p) = \begin{cases} \max\left\{2, \frac{(N-1-p)p}{N-p}\right\} & 2 \le p < 3, p^2+p < N \\ 0 & \frac{2N}{N+1} \le p < 3, p^2+p > N \\ \max\left\{\frac{(N-1-p)p}{N-p}, \frac{(N-1)p}{N-p} - \frac{2}{p-1}, \frac{(N-1)(p-1)}{N-p}\right\} & p \ge 3. \end{cases}$$

If p is in one of the ranges in the definition of $Q(N,p)$, and $Q(N,p) < q+1 < p$, then (P_λ) admits a second solution.

- $C^1(\overline{\Omega})$ local minimum

  Let $\lambda \in (\lambda_1, \lambda_2)$. Then u_λ is a local minimum of J_λ in $C^1(\overline{\Omega})$.
  Define $\overline{h}(x,u) = h(u)\big|_{[u_\lambda, u_{\lambda_2}]}$ and $\overline{g}(x,u) = g(u)\big|_{[u_{\lambda_1}, u_{\lambda_2}]}$.

  $$\overline{J}(u) = \frac{1}{p}\|u\|^p - \int_\Omega \overline{H}(x,u) - \int_{\partial\Omega} \overline{G}(x,u)$$

  where $\overline{H}(x,u) = \int_0^u \overline{h}(x,s)ds,\ \overline{G}(x,u) = \int_0^u \overline{g}(x,s)ds$
  Then $\overline{J}$ admits global minimum $u_\lambda \in [u_{\lambda_1}, u_{\lambda_2}]$.

  <u>Claim :-</u> $u_{\lambda_1} < u_\lambda < u_{\lambda_2}$ in $\overline{\Omega}$.
  Let $\mathcal{C} = \{x \in \Omega : \ u_{\lambda_1}(x) = u_\lambda(x)\ \}$, **Then** $\mathcal{C} \subset\subset \Omega$.
  Choose $\Omega' \subset\subset \Omega$ such that $u_\lambda > u_{\lambda_1}$ on $\partial\Omega'$. Now by Strong comparison we get $\mathcal{C} = \emptyset$. Let
  $\delta = \min\limits_{\overline{\Omega}}\{|u_{\lambda_2} - u_{\lambda_1}| + |\nabla u_{\lambda_1} - \nabla u_{\lambda_2}|\}$
  Then $\delta > 0$. So if $v \in B_{\delta/2} = \{u \in C^1(\overline{\Omega}) : \|u - u_\lambda\|_{C^1(\Omega)} < \frac{\delta}{2}\}$ then $u_{\lambda_1} < u < u_{\lambda_2}$ for $x \in \Omega$ and $\forall\ u \in B_\delta$.

- Uniform L^∞ estimates

Let $g_\epsilon, h_\epsilon : \Omega \times I\!\!R \to [0,\infty), \epsilon \in [0,\epsilon_0)$ for some $\epsilon_0 > 0$ be a one parameter family of Holder continuous maps satisfying:
$(H1)$ $|g_\epsilon(x,s)| \leq C_1(1+|s|^{p^*-1})$
$(H2)$ $h_\epsilon(x,s)| \leq C_2(1+|s|^q), q < p-1.$

$$(Q_\epsilon) \quad \begin{cases} -\Delta_p u + u^{p-1} &= \quad g_\epsilon(x,u) \;\; in \; \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} &= \quad h_\epsilon(x,u) \; on \; \partial\Omega \end{cases}$$

Theorem: Let $\{u_\epsilon\}$ be a sequence of solutions to (Q_ϵ) bounded in $W^{1,p}(\Omega)$ such that $u_\epsilon \to u_0$ in $L^{p^*}(\Omega)$. Then for $\beta \in (1, \frac{p^*}{p})$,

$$\|u_\epsilon\|_{L^{\beta p^*}(\Omega)} \leq C(1 + \|u_\epsilon\|_{L^{p^*}(\Omega)}).$$

Theorem: Let $\{u_\epsilon\}$ be a sequence of solutions to (Q_ϵ) in $W^{1,p}(\Omega)$ such that $\|u_\epsilon\|_{L^{p*+\delta}(\Omega)} < C$. Then $\|u_\epsilon\|_{L^\infty(\Omega)} < M$.

Define the cut off functions for $k > 0$, let

$$T_k(s) = \begin{cases} s+k & s \in (\infty, -k] \\ 0 & s \in [-k, k] \\ s-k & s \in [k, \infty) \end{cases}$$

and $\Omega_k^\epsilon = \{x \in \Omega; |u_\epsilon| \geq k\}, \partial\Omega_k^\epsilon = \{x \in \partial\Omega; |u_\epsilon| \geq k\}$. Taking $T_k(u_\epsilon)$ as test function in (Q_ϵ), to get,

$$\int_\Omega |\nabla u_\epsilon|^{p-2}\nabla u_\epsilon \cdot \nabla T_k(u_\epsilon) + |u_\epsilon|^{p-2}u_\epsilon T_k(u_\epsilon) = \int_\Omega g_\epsilon(u_\epsilon)T_k(u_\epsilon) + \int_{\partial\Omega} h_\epsilon(u_\epsilon)T_k(u_\epsilon)d\sigma$$

$$
\begin{aligned}
R.H.S. &\le C\int_{\Omega_k^\epsilon}|u_\epsilon|^{p^*}+\int_{\partial\Omega_k^\epsilon}|u_\epsilon|^{q+1}d\sigma \\
&\le C(\int_{\Omega_k^\epsilon}|u_\epsilon|^\alpha)^{\frac{p^*}{\alpha}}+(\int_{\partial\Omega}|u_\epsilon|^{\frac{(q+1)\alpha}{p^*}})^{\frac{p^*}{\alpha}}(|\Omega_k^\epsilon|+|\partial\Omega_k^\epsilon|)^{1-\frac{p^*}{\alpha}} \\
&\le C(|\Omega_k^\epsilon|+|\partial\Omega_k^\epsilon|)^{1-\frac{p^*}{\alpha}}
\end{aligned}
$$

$$
\begin{aligned}
L.H.S. &\ge C(\int_\Omega|\nabla T_k(u_\epsilon)|^p+\int_\Omega|T_k(u_\epsilon)|^p) \\
&\ge C(\int_\Omega|T_k(u_\epsilon)|^{p_*}+\int_{\partial\Omega}|T_k(u_\epsilon)|^{p_*})^{\frac{p}{p_*}} \\
&\ge C((h-k)^{p_*}|\Omega_h^\epsilon|+(h-k)^{p_*}|\partial\Omega_h^\epsilon|)^{\frac{p}{p_*}}
\end{aligned}
$$

Combining upper and lower bounds, we get for any $h>k$,

$$
(h-k)^p(|\Omega_h^\epsilon|+|\partial\Omega_h^\epsilon|)\le C(|\Omega_h^\epsilon|+|\partial\Omega_h^\epsilon|)^{(1-\frac{p^*}{\alpha})\frac{p_*}{p}}
$$

We obtain the estimate by the following Lemma of Stampacchia, 1966

Lemma: Let $\Phi : [0, \infty) \to I\!R^{+}$ be a nonincreasing function such that

$$\Phi(h) \leq (\frac{C}{(h-k)})^{\alpha} \Phi(k)^{\beta},\ h > k$$

for some $\beta > 1$, then $\Phi(d) = 0$ for $d = C\Phi(0)^{\frac{\beta-1}{\alpha}} 2^{\frac{\beta}{\beta-1}}$.

- $W^{1,p}(\Omega)$ local minimum

u_λ is a local minimum in $W^{1,p}(\Omega)$ topology.

We take the constraint set

$$\mathcal{C}_\epsilon = \{u \in W^{1,p}(\Omega) : \frac{1}{p^*} \|u\|_{L^{p^*}}^{p^*} + \frac{1}{q+1} \|u\|_{L^{q+1}}^{q+1} \leq \epsilon\}$$

- Compactness and energy levels The functional defined as

$$\tilde{J}_\lambda(w) = \frac{1}{p}\int_\Omega |\nabla(u_\lambda+w)|^p+|u_\lambda+w|^p - \int_\Omega G_\lambda(x,w) - \lambda \int_{\partial\Omega} H_\lambda(x,w)\,d\sigma$$

$$\tilde{J}_\lambda(w) \geq \tilde{J}_\lambda(0) \text{ for } \|w\|_{1,p} \leq \epsilon$$

Theorem: Assume that 0 is the only critical point of $\tilde{J}_\lambda$. Let $\{w_k\}$ be a $(P.S.)_c$ for $\tilde{J}_\lambda$ with

$$c < c_0 := J_\lambda(u_\lambda) + \frac{S^{N/p}}{2N}.$$

Then, $c = \frac{1}{p}\|u_\lambda\|_{1,p}^p$ and $w_k \to 0$ in $W^{1,p}(\Omega)$.

Lemma: Let c_0 be as above. Then

(a) $\tilde{J}_\lambda(tu_\epsilon) \to -\infty$ as $t \to \infty$ for any $\epsilon > 0$,

(b) if $1 < q+1 < p$ are such that p is in one of the ranges in the definition of $Q(N,p)$ and $Q(N,p) < q+1$, then we have $\sup_{t\geq 0} \tilde{J}_\lambda(tu_\epsilon) < c_0$ for all $\epsilon > 0$ small.

Limiting case $p = N$

Moser-Trudinger embedding:

1. For any $u \in W_0^{1,N}(\Omega)$, we have $\int_\Omega e^{\alpha|u|^{N/N-1}} < \infty$. Moreover
$$\sup_{\|u\|_{1,N} \le 1} \int_\Omega e^{\alpha|u|^{N/N-1}} < \infty, \quad \text{for all } \alpha \le \alpha_N := N w_{N-1}^{1/N-1}$$
2. For any $u \in W^{1,N}(\Omega)$, we have $\int_\Omega e^{\alpha|u|^{N/N-1}} < \infty$. Moreover
$$\sup_{\|u\|_{1,N} \le 1} \int_\Omega e^{\alpha|u|^{N/N-1}} < \infty, \quad \text{for all } \alpha \le \frac{\alpha_N}{2}$$
where $w_{N-1} = |S^{N-1}|$
3. The embedding is not compact in the limiting case.

♠ Radial case

Joint work with S.Prashanth, DIE,2004

We considered the problem (P_λ) in the Ball $B_1(0)$

$$-\Delta_N u = \lambda f(u),\ u > 0,\ \text{in } B_1(0),\ u = 0 \text{ on } \partial B_1(0).$$

The Radial version of this problem is

$$(P_\lambda^r) \begin{cases} -\text{div}(r^{N-1}|u'|^{N-2}u')' & = \lambda r^{N-1} f(u),\ u > 0 \text{ in } r \in (0,1), \\ u'(0) = u(1) & = 0. \end{cases}$$

Changing the variable, as $\lambda = R^N, , s = Rr$ and taking $v(s) = u(r)$, then v solves:

$$\begin{aligned} -\text{div}(s^{N-1}|v'|^{N-2}v')' & = s^{N-1} f(v),\ v > 0 \text{ in } s \in (0,R), \\ v'(0) = v(R) & = 0. \end{aligned}$$

We study this problem for varying values of R via the parametrized problem:

$$\begin{aligned} -\mathrm{div}(s^{N-1}|w'|^{N-2}w')' &= s^{N-1}f(w),\ w>0 \text{ in } s\in(0,R), \\ w^{'}(0)=0 &\quad w(0)=\gamma. \end{aligned}$$

Again defining the Emden-Fowler transformation:

$$r = Ne^{-t/N},\ y(t) = w(r)$$

Then y satisfies

$$-(|y'|^{N-2}y')' = e^{-t}f(y),\ y^{'}(\infty)=0,\ y(\infty)=\gamma.$$

Let $T_0(\gamma)$ be the first zero of $y_\gamma(t)$ as t decreases from ∞.

For multiplicity it is enough to show that

$$\lim_{\gamma \to 0,\infty} T_0(\gamma) = \infty.$$

We obtain the results for nonlinearities that look like

1. $f(s) = s^m e^s, m < 0$
2. $f(s) = s^m e^{s^\alpha \pm s^\beta}, 1 \leq b < \alpha < \frac{N}{N-1}$
3. $f(s) = s^m e^{s^{\frac{N}{N-1}}}, 1 \leq \beta < \frac{N}{N-1}, m \in I\!R.$

♠ **Bounded domains**
with J.Giacomoni and S.Prashanth, JDE, 2007

We consider the problem in $\Omega \subset I\!R^N, |\Omega| < \infty$:

$$\begin{aligned} -\nabla \cdot (|\nabla u|^{N-2} \nabla u) &= \lambda f(u),\ u > 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

where $f(t) = h(t) e^{|t|^{\frac{N}{N-1}}}$ satisfies

(H1) $h \in C^1(0, \infty); h(t) = 0, t \leq 0, h(t) > 0, t > 0.$
(H2) The map $t \to f(t)$ is nondecreasing for $t \in (0, t_*) \cup (\frac{1}{t_*}, \infty)$
(H3) The map $t \to t^{1-N} f(t)$ is non increasing in $(0, t_*)$
(H4) for all $\epsilon > 0$
$\liminf_{t\to\infty} h(t) e^{\epsilon t^{\frac{N}{N-1}}} = \infty;\ \limsup_{t\to\infty} h(t) e^{-\epsilon t^{\frac{N}{N-1}}} = 0.$
(H5) $\liminf_{t\to\infty} h(t) t e^{\epsilon t^{\frac{1}{N-1}}} = \infty, \forall \epsilon > 0$

The nonlinearities that satisfy $(H1) - (H5)$ may be classified as follows based on their "strength"

$$I = \{h(t) = t^{\alpha}(1+t)^{m}e^{-t^{\beta}},\ \alpha \in (0, N-1), m \geq 0, 0 < \beta < \frac{1}{N-1}\}$$

$$II = \{h(t) = t^{\alpha}(1+t)^{m}e^{t^{\beta}}, \alpha \in (0, N-1), m \geq 0, 0 < \beta < \frac{N}{N-1}\}$$

Theorem: There exists two positive solutions for $\lambda \in (0, \Lambda)$, one solution for $\lambda = \Lambda$ and no solution for $\lambda > \Lambda$.

We also obtained the uniqueness in the radial case for $h(t)$ that look like

$$h(t) = t^{\alpha}(1+t)^{m}e^{-t^{\beta}}, \alpha \in (0, N-1), m \geq 0, \frac{1}{N-1} < \beta < \frac{N}{N-1}$$

The borderline between uniqueness and multiplicity is described by the condition $(H5)$.

♠ J.Giacomoni, S.Prashanth and K.S.- C.R.Acad, 2009, DIE,2010:

We consider the problem in $\Omega \subset I\!R^N, |\Omega| < \infty$:

$$\begin{aligned} -\nabla \cdot (|\nabla u|^{N-2} \nabla u) + |u|^{N-2} u &= \lambda f(u),\ u > 0 \text{ in } \Omega \\ |\nabla u|^{p-2} \frac{\partial u}{\partial n} &= \lambda u^q \text{ on } \partial\Omega \end{aligned}$$

with $f(t) = h(t) e^{|t|^{\frac{N}{N-1}}}$ and $0 \leq q < N - 1$.

- Existence, Non-existence and $C^1(\Omega)$ local minimum are is similar
- $W_0^{1,N}(\Omega)$ local minimum
  We take the constraint set

  Let $C_\epsilon = \{u \in W^{1,N} : K(u) = \|G(u)\|_{L^1} + \|u\|^{q+1}_{L^q(\partial\Omega)} \leq \epsilon\}$

where $G(s) = |s|^{N+1} e^{2s^{\frac{N}{N-1}}}$

$$\|v_n\| \to 0 \Longrightarrow K(v_n) \to 0$$

$\therefore\ \exists u_\epsilon \in C_\epsilon$ such that $u_\epsilon \neq 0$ and

$$\min_{C_\epsilon} J(u + u_0) = J(u_\epsilon + u_0) < J(u_0).$$

$\therefore\ \{u_\epsilon\}$ is bounded and hence $u_\epsilon + u_0 \rightharpoonup u_0$ in $W^{1,N}(\Omega)$.
By Lagrange Multiplier theorem, $\exists\ \mu_\epsilon \leq 0(!)$ such that

$$J^{'}(u_\epsilon + u_0) = \mu_\epsilon K^{'}(u_\epsilon), \quad \epsilon \in (0,1).$$

Then we can show

- $\|u_\epsilon\|_{W^{1,N}} \to 0$ as $\epsilon \to 0$
- $\|u_\epsilon\|_{L^\infty} \leq C$.
- $\|u_\epsilon\|_{C^{1,\alpha}} \leq C$.(By Lieberman's results)

- Lion's Lemma for P.S. sequences:

  If u_n satisfies

  - $\|u_n\| = 1$
  - $u_n \rightharpoonup u, u \not\equiv 0.$
  - $-\Delta_N u_n = f_n + g_n, f_n \to f$ in $L^1(\Omega)$ and $g_n \to 0$ in $W^{1,N'}(\Omega)$.

  Then for $1 < p < (1 - \|u\|)^{\frac{-1}{N-1}}$

$$\sup \int_\Omega e^{p \frac{\alpha_N}{2} |u|^{\frac{N}{N-1}}} < \infty.$$

- $(P.S)_\rho$ sequence

$$\sup_{t>0} J_\lambda(u_\lambda + t\phi_n) < J_\lambda(u_\lambda) + \frac{(\alpha_N)^{N-1}}{2N} := \rho_0$$

Theorem: Suppose $\{u_n\}$ be a $(P.S)_\rho$, $(\rho < \rho_0)$ sequence. Then $u_n \rightharpoonup v_\lambda$, the second solution of (P_λ).

♠ S.Bhatia, I.Schindler and K.S.-

Consider the following problem:

$$(\Phi N)_\lambda \quad \begin{cases} \left.\begin{aligned} -div(\phi(|\nabla u|)\nabla u) + \phi(|u|)u &= u^p e^{u^\alpha} \\ u &> 0 \end{aligned}\right\} \text{in } \Omega, \\ \phi(|\nabla u|)\dfrac{\partial u}{\partial n} = \lambda \psi u^q \text{ on } \partial\Omega, \end{cases}$$

- Let Ω be a bounded domain in $I\!R^N$ with smooth boundary $\partial\Omega, N \geq 3$.
- $\alpha \in (0, \frac{N}{N-1}], \lambda > 0, p_0 > 1, q \in (0, \min\{p_0 - 1, N - 1\})$, $p > \max\{p_0 - 1, N - 1\}$
- $\phi(t)$ satisfies the following properties:
  - $\phi(t) \in C^1(0, \infty),\ t\phi(t) \in C(I\!R)$
  - $\phi(t) > 0,\ (t\phi(t))' > 0$ for $t > 0$
  - There exists $p_0 > 1$ and $\alpha_0, \alpha_1 > 0$ such that $\phi(t) = \alpha_0 t^{p_0-2} + o(t^{p_0-2})$ as $t \to 0$ and $\phi(t) = \alpha_1 t^{N-2} + o(t^{N-2})$ as $t \to \infty$
  - There exists $a_0, a_1 > 0$ and $0 < T_0 < T_1$ such that $a_0 t^{p_0-2} \leq (t\phi(t))' \leq a_1 t^{p_0-2}$ for $0 < t < T_0$ and $a_0 t^{N-2} \leq (t\phi(t))' \leq a_1 t^{N-2}$ for $t > T_1$

✠ Let $\Phi(t) = \int\limits_0^t s\phi(s)ds,\ f(t) = t^p e^{t^\alpha},\ F(t) = \int\limits_0^t f(s)ds.$

Associated to $(\Phi N)_\lambda$, we have the functional $J_\lambda^2 : W^{1,N}(\Omega) \to I\!R$ defined as follows:

$$J_\lambda^2(u) = \int_\Omega \Phi(|\nabla u|) + \int_\Omega \Phi(|u|) - \int_\Omega F(u) - \frac{\lambda}{q+1}\int_{\partial\Omega} \psi u^{q+1}$$

N. Fukagai and K. Narukawa, Comm. in Contemporary Mathematics, (2003)

Authors considered the following problem:

$$(Q_\phi) \qquad \begin{cases} -div(\phi(|\nabla u|)\nabla u) &= \lambda f(x,u), \;\; u > 0 \quad \text{in } \Omega \\ u &= 0 \;\text{ on }\; \partial\Omega, \end{cases}$$

satisfying the asymptotic conditions on ϕ and f such as
$\phi(t) \sim t^{p_0-1}, f(x,t) \sim t^{q_0-1}$ as $t \to 0^+$ and
$\phi(t) \sim t^{p_1-2}, f(x,t) \sim t^{q_1-1}$ as $t \to \infty$ where $p_\leq N$ and $q_1 < \frac{Np_1}{N-p_1}$.
They proved the following theorem:

Theorem: *Suppose* $q_0 \in (1, p_0), q_1 \in (p_1, p_1^*)$. *Then for* $\lambda \in (0, \Lambda)$, (Q_ϕ) *admits two solutions.*

✠ Existence and multiplicity for the above problem was shown only in the subcritical case.

We obtain the following results

✠ Existence of solution for the critical case.

✠ Non-existence for λ large ($\Lambda < \infty$)

✠ Existence of solution for $\lambda = \Lambda$.

✠ Existence of two solutions for exponential subcritical nonlinearity.

✠ Existence of two solutions with critical nonlinearity i.e., $\alpha = \frac{N}{N-1}$ in the special case $\Phi(t) = \sqrt{1 + |t|^{2N}} - 1$.

✠ First positive solution and $C^1(\Omega)$ local minimum are similar.

✠ **On Uniform estimates**

Let $g_\epsilon, h_\epsilon : \Omega \times I\!R \to [0, \infty), \epsilon \in [0, \epsilon_0)$ for some $\epsilon_0 > 0$ be a one parameter family of Holder continuous maps satisfying:
$(H1)$ $|g_\epsilon(x,s)| \leq C_1(1 + |s|^{p^*-1})$
$(H2)$ $h_\epsilon(x,s)| \leq C_2(1 + |s|^q), q < p - 1.$

$$(\Phi_\epsilon) \quad \begin{cases} -\nabla \cdot (\phi(|\nabla u|\nabla u) + \phi(|u|)u & = & g_\epsilon(x,s) \ \ in\ \Omega \\ \phi(|\nabla u|)\frac{\partial u}{\partial n} & = & h_\epsilon(x,u)\ on\ \partial\Omega \end{cases}$$

In case of $p_1 < N$:

Theorem: Let $q \leq p_1 - 1$ and let $\{u_\epsilon\}$ be a sequence of solutions to (Φ_ϵ) in $W^{1,p_1}(\Omega)$ such that $u_\epsilon \to u_0$ in $L^{p_1^*}(\Omega)$. Fix any $\beta \in (1, \frac{p_1^*}{p_1})$. Then, there exists a constant $C_1 > 0$ such that $\|u_\epsilon\|_{L^{\beta p_1^*}(\Omega)} \leq C_1(1 + \|u_\epsilon\|_{L^{p_1^*}(\Omega)})$.

Theorem: Let $\{u_\epsilon\}$ be a sequence of solutions to (Φ_ϵ) bounded in $W^{1,p_1}(\Omega)$ and $L^{p_1^*+\delta}$ for some $\delta > 0$. Then there exists a constant $M > 0$ independent of ϵ such that

$$\|u_\epsilon\|_{L^\infty(\Omega)} \leq M(\Omega, p_1, N, \|u_\epsilon\|_{L^{p_1^*+\delta}}).$$

In case $p_1 = N$, we have the following theorem:

Theorem: Let $u \in W^{1,N}(\Omega)$ be a solution to (Φ_ϵ) such that

$$\int_\Omega u^{(p+1-N)(N+1)} e^{(N+1)|u|^{\frac{N}{N-1}}} \mathrm{d}x < C_3.$$

Then there exists $M = M(C_1, C_2, C_3, p, q, \Omega) > 0$ such that $\|u\|_{L^\infty(\Omega)} \leq M$.

Idea of Proof:

Let $\psi_k = \max\{u^+ - k, 0\}$ and A^k =supp(ψ_k) Taking ψ_k as test function, we get

$$\int_\Omega \phi(|\nabla u|)\nabla u.\nabla \psi_k \mathrm{d}x + \int_\Omega (\phi|u|)u\psi_k \mathrm{d}x = \int_\Omega f(u)\psi_k \mathrm{d}x + \lambda \int_{\partial\Omega} g(u)\psi_k \mathrm{d}\sigma. \tag{0.1}$$

$$\int_{A_k} |u|^l \mathrm{d}x = \int_{A_k} |u-k+k|^l \mathrm{d}x \le 2^{l-1}\left(\int_{A_k}(u-k)^l \mathrm{d}x + k^l|A_k|\right).$$

Setting $l = N+1$ and using (A1), the Hölder inequality we get

$$\begin{aligned} \int_\Omega f(u)\psi_k \mathrm{d}x &\le C_1 \int_{A_k} u^{p+1} e^{|u|^{N/N-1}} \mathrm{d}x \\ &\le C\left(\left(\int_{A_k}(u-k)^l \mathrm{d}x\right)^{N/l} + k^N|A_k|^{\frac{N}{l}}\right). \end{aligned}$$

The boundary integral may be estimated using:

$$\int_{\partial\Omega} |w| \mathrm{d}\sigma \leq \int_{\Omega} |\nabla w| \mathrm{d}x + K(\Omega) \int_{\Omega} |w| \mathrm{d}x, \ \forall w \in W^{1,1}(\Omega).$$

Applying this to the boundary integral with $w = u^q(u^+ - k)^+$ and using (A2) with Young's inequality,

$$\int_{\partial\Omega} g(u)\psi_k \mathrm{d}\sigma \leq C \left(\frac{(q+1)\delta}{N} \int_{A_k} |\nabla u|^N \mathrm{d}x + \frac{(q+1)N}{\delta^{\frac{1}{(N-1)}}(N-1)} \int_{A_k} u^N \mathrm{d}x \right.$$
$$\left. + K \int_{\Omega} |u|^q (u-k)^+ \mathrm{d}x \right).$$

Also,

$$\int_{A_k} u^q (u-k)^+ \leq \int_{A_k} |u|^N \leq A_k^{1/l} \left(\int_{A_k} (u-k)^l \right)^{N/l} + k^N |A_k|.$$

Hence we obtain

$$\int_{\partial\Omega} g(u)\psi_k d\sigma \leq C_2 \frac{(q+1)\delta}{N} \int_{A_k} |\nabla u|^N \mathrm{d}x + C|A_k|^{\frac{1}{l}} (\int_{A_k} (u-k)^l \mathrm{d}x)^{\frac{N}{l}}$$
$$+ Ck^N |A_k|.$$

The first term on the L.H.S. of (0.1) may be estimated as:

$$\int_{\Omega} \phi(|\nabla u|)\nabla u.\nabla\psi_k \mathrm{d}x \geq (\alpha_1 - \epsilon) \int_{A_k} |\nabla u|^N \mathrm{d}x - C|A_k|.$$

Similarly, we can estimate the second term on the L.H.S.
Substituting these estimates in (0.1) and choosing $\delta < \frac{(\alpha_1-\epsilon)N}{2C_2(q+1)}$,

$$\frac{1}{2}\int_{A_k} |\nabla u|^N \mathrm{d}x \leq C_4 \left[\left(\int_{A_k} (u-k)^l \mathrm{d}x \right)^{N/l} |A_k|^{\frac{1}{l}} \right.$$
$$\left. + k^N |A_k|^{\frac{N}{l}} + k^N |A_k| + |A_k| \right],$$

Also,

$$k^N|A_k| \le \|u\|^N_{L^N(\Omega)}.$$

Now using the Sobolev inequality

$$(\int_{A_k}(u-k)^m \mathrm{d}x)^{1/m} \le C(\int_{A_k}|\nabla u|^N \mathrm{d}x)^{1/N}|A_k|^{1/m},$$

we obtain

$$\begin{aligned}(\int_{A_k}(u-k)^l)^{N/l}|A_k|^{\frac{1}{l}} &\le C(\int_{A_k}|\nabla u|^N \mathrm{d}x)|A_k| \\ &\le \frac{C\|u\|^N_{L^N}}{k^N}\int_{A_k}|\nabla u|^N \mathrm{d}x.\end{aligned}$$

Therefore for $k > k_0$ large,

$$\begin{aligned}\int_{A_k}|\nabla u|^N \mathrm{d}x &\le C(k^N|A_k|^N + +k^N|A_k|^{\frac{N}{l}} + |A_k|) \\ &\le 2Ck^N|A_k|^{N/l}.\end{aligned}$$

Now the result follows from:

Lemma:(Ladyzhenskaya): Let $0 \leq u \in W^{1,N}(\Omega)$. Suppose there exists $l > N, k_0 > 1$ and $C_0 > 0$ such that

$$\int_{A_k} |\nabla u|^N \leq C_0 k^N |A_k|^{\frac{N}{l}},\ k > k_0$$

where $A_k = \{x \in \Omega,\ u \geq k\}$. Then there exists $M = M(\Omega, C_0, l, k_0) > 0$ such that $\|u\|_{L^\infty} \leq M$.

Let v_λ solves

$$\begin{aligned} -\nabla \cdot (\phi(|\nabla u|)\nabla u) + \phi(|u|)u &= 0,\ u > 0 \text{ in } \Omega, \\ \phi(|\nabla u|)\tfrac{\partial u}{\partial n} &= \lambda u^q \end{aligned}$$

Theorem: There exists a sequence $\{\lambda_n\}$ such that $\lambda_n \to \infty$ and $\|v_{\lambda_n}\|_{L^\infty} \to 0$ as $n \to \infty$.

Multiplicity: Assume that $f(u)$ is of sub-critical at ∞. Then There exists $\Lambda > 0$ such that $(\Phi N)_\lambda$ admits two solutions for $\lambda \in (0, \Lambda)$.

Thank you!