

**A stationary Cahn-Hilliard model  
with the  $p$ -Laplacian:  
radial solutions and pattern formation**

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forschung/AngAnalysis/index.html](http://www.math.uni-rostock.de/forschung/AngAnalysis/index.html)

## Time-dependent **Cahn-Hilliard** equation:

$$(1) \quad u_t = \Delta \left[ -\varepsilon^p \nabla \cdot (|\nabla u|^{p-2} \nabla u) + W'(u) \right] \\ \text{for } x \in \Omega \text{ and } t > 0,$$

subject to the **Neumann** (i.e., no-flux) boundary conditions

$$(2) \quad |\nabla u|^{p-2} (\nu \cdot \nabla) u = 0 \quad \text{and} \\ (\nu \cdot \nabla) \left[ -\varepsilon^p \nabla \cdot (|\nabla u|^{p-2} \nabla u) + W'(u) \right] = 0 \\ \text{at } x \in \partial\Omega \text{ for } t > 0,$$

where  $1 < p < \infty$ ,  $\varepsilon > 0$ , and  $W : \mathbb{R} \rightarrow \mathbb{R}$  is a given **potential function** of class  $C^1$  whose first derivative might be only **Hölder-continuous**. Semilinear case  $p = 2$  is well-known; see, e.g., **R. Temam** (monograph by Springer-Verlag).

We abbreviate by  $\Delta_p u \stackrel{\text{def}}{=} \nabla \cdot (|\nabla u|^{p-2} \nabla u)$  the well-known  $p$ -Laplace operator; of course,  $\Delta_2 \equiv \Delta$  is the (linear) Laplace operator.

Material occupies a **bounded domain**  $\Omega \subset \mathbb{R}^N$  with a **sufficiently smooth boundary**  $\partial\Omega$ .

If  $W$  is of class  $C^2$  then

the boundary conditions (2) are equivalent with the Navier boundary conditions

$$(3) \quad \begin{aligned} (\nu \cdot \nabla)u &= (\nu \cdot \nabla)(\Delta_p u) = 0 \\ &\text{at } x \in \partial\Omega \text{ for } t > 0. \end{aligned}$$

Classical choice of  $W$ : double-well potential

$$W(s) = (1 - s^2)^2 \text{ for } s \in \mathbb{R}$$

which attains global minimum at two points,  $s_1 = -1$  and  $s_2 = 1$  (Cahn and Hilliard).

These points of minimum are nondegenerate, with  $W'(\pm 1) = 0$  and  $W''(\pm 1) = 8 > 0$ .

Entirely different behavior of the stationary solutions satisfying

$$(4) \quad -\varepsilon^p \Delta_p u + W'(u) = 0 \quad \text{for } x \in \Omega;$$

$$(5) \quad (\nu \cdot \nabla)u = 0 \quad \text{at } x \in \partial\Omega,$$

for classical linear diffusion ( $p = 2$ ) and degenerate nonlinear diffusion ( $p > 2$ ).

Degenerate nonlinear diffusion ( $p > 2$ ) exhibits a greater variety of stationary solutions.

One can observe the same phenomenon for classical linear diffusion ( $p = 2$ ) if the potential  $W$  is modified to  $W(s) = |1 - s^2|^\alpha$  for  $s \in \mathbb{R}$ , where  $\alpha$  is a constant,  $1 < \alpha < 2$ .

We focus on problem (4), (5) with arbitrary  $p, \alpha > 1$ . More generally, we consider the potential

$$W(s) = |1 - |s|^\beta|^\alpha \quad \text{for } s \in \mathbb{R},$$

where  $p, \alpha, \beta > 1$  are constants.

Asymptotic behavior:

$$W(s) = \beta^\alpha |1 - |s||^\alpha + \mathcal{O}(|1 - |s||^{\alpha+1})$$

for  $s \rightarrow \pm 1$ ,

$$W(s) = 1 - \alpha |s|^\beta + \mathcal{O}(|s|^{2\beta})$$

for  $s \rightarrow 0$ .

The following constant is important for  $N \geq 2$ ,

$$p_{N,\beta} \stackrel{\text{def}}{=} \frac{\beta N}{N - 1 + \beta} \geq 1.$$

One has  $p_{1,\beta} = 1$  and  $p_{N,\beta} > 1$  for  $N \geq 2$ .

## Radially symmetric solutions in a ball

Boundary value problem (4), (5) reduces to

$$(6) \quad -\varepsilon^p r^{-(N-1)} \left( r^{N-1} |u'|^{p-2} u' \right)' + W'(u) = 0, \quad 0 < r < R;$$

$$(7) \quad u'(0) = 0, \quad u'(R) = 0.$$

This problem applies to any choice  $N \geq 1$ .

$$(8) \quad \varepsilon^p (|u'|^{p-2} u')' + \varepsilon^p \frac{N-1}{r} |u'|^{p-2} u' - W'(u) = 0, \quad 0 < r < \infty;$$

$$(9) \quad u(0) = -\theta, \quad u'(0) = 0,$$

where  $u_0 = -\theta \in [-1, 1]$  is given.

### Shooting method – three (3) factors:

- reaction (independent from  $N$ )
- diffusion (for every  $N \geq 1$ )
- damping (only for  $N \geq 2$ )

## Case $N = 1$ .

(Pavel Drábek, U.W.B., Plzeň, Czech Rep.)

Problem (4), (5) is the boundary value problem for all *stationary solutions* of the so-called *bi-stable equation*

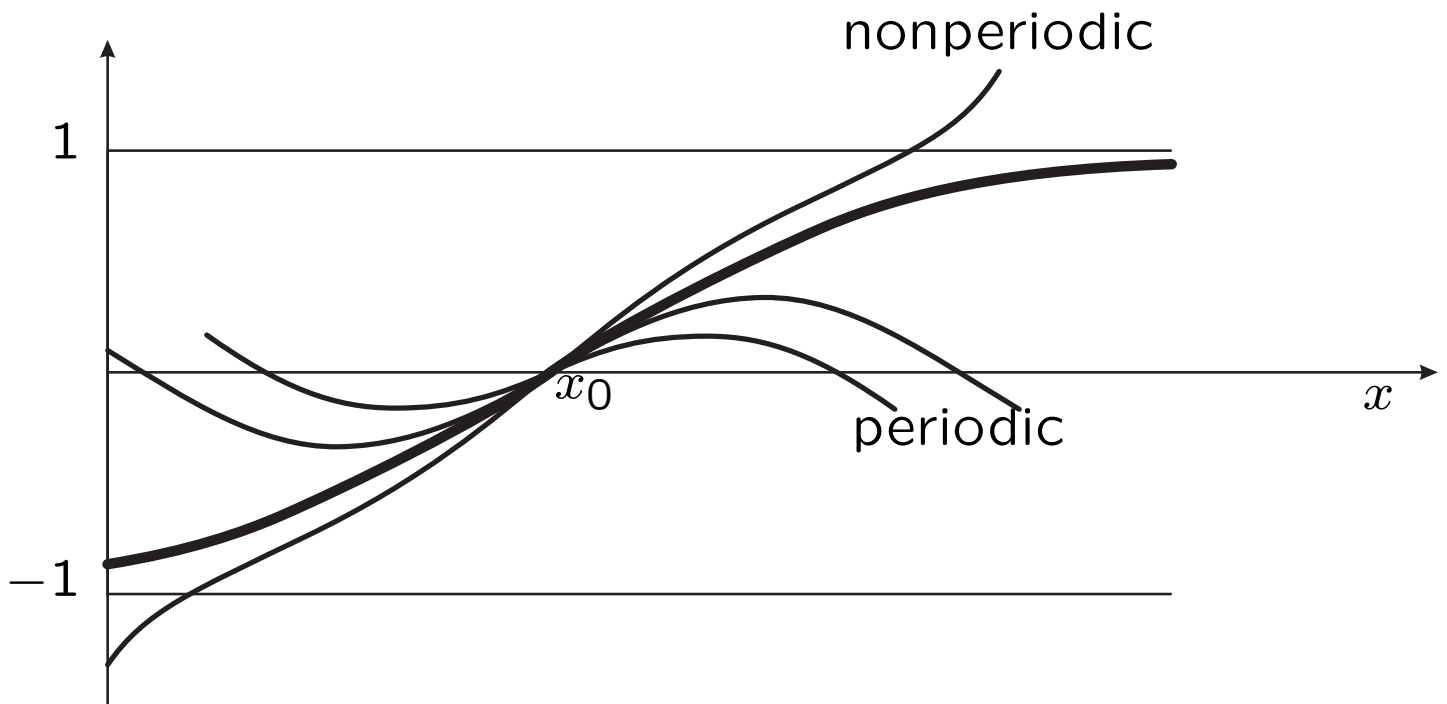
$$(10) \quad \begin{aligned} u_t &= \varepsilon^p \left( |u_x|^{p-2} u_x \right)_x - W'(u) \\ &\text{for } 0 < x < 1 \text{ and } t > 0, \end{aligned}$$

subject to the boundary conditions

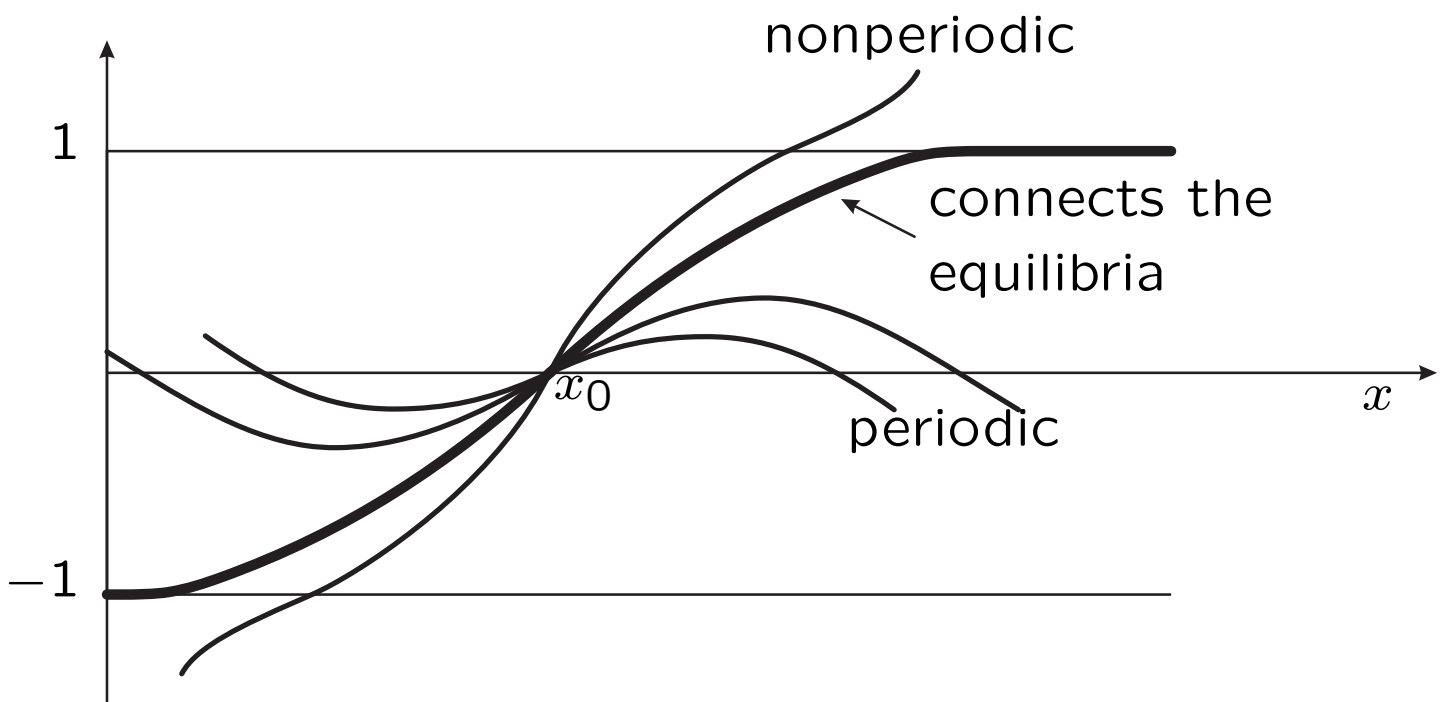
$$(11) \quad u_x = 0 \quad \text{at } x = 0, 1, \quad \text{for } t > 0.$$

Equations (10), (11) describe the **gradient flow** for the **(total free energy)** functional

$$(12) \quad \mathcal{J}_\varepsilon(u) \stackrel{\text{def}}{=} \int_0^1 \left( \frac{\varepsilon^p}{p} |u_x|^p + W(u) \right) dx, \\ u \in W^{1,p}(0, 1).$$



$$1 < p \leq \alpha < \infty, W(s) = |1 - s^2|^\alpha.$$



$$1 < \alpha < p < \infty, W(s) = |1 - s^2|^\alpha.$$

For  $1 < \alpha < p < \infty$  we have **loss of uniqueness** in the initial value problem for the **first integral** of eq. (8),

$$(13) \quad \frac{p-1}{p} \varepsilon^p |u_x(x)|^p - W(u(x)) = \text{const},$$
$$0 \leq x \leq 1.$$

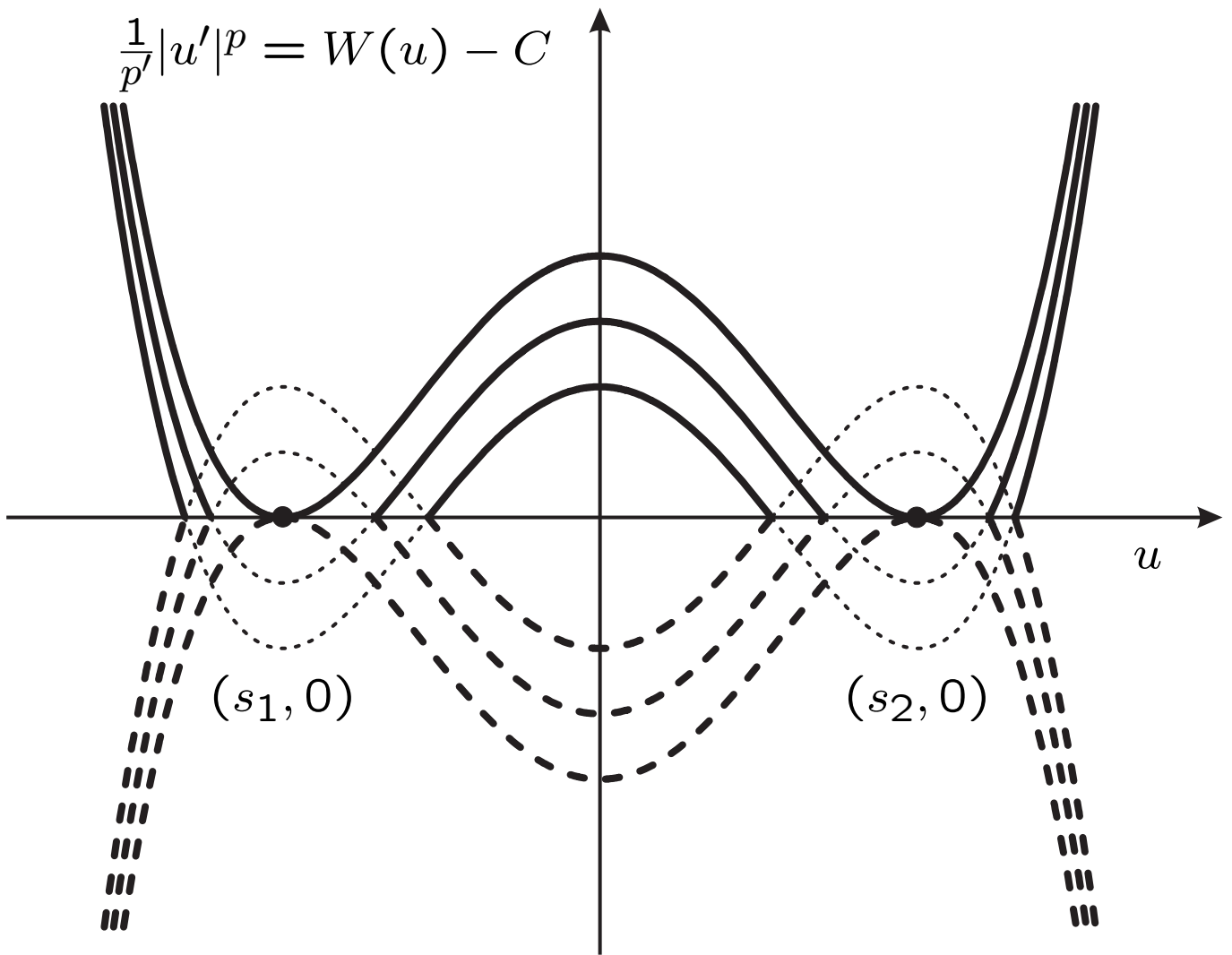
Denote  $p' = p/(p-1)$ . Then eq. (13) reads (we look for a **local** solution  $u$ )

$$(14) \quad u' = \text{sgn } u'_0 \cdot [p'(W(u) - C)]^{1/p};$$

$$(15) \quad u(x_0) = u_0.$$

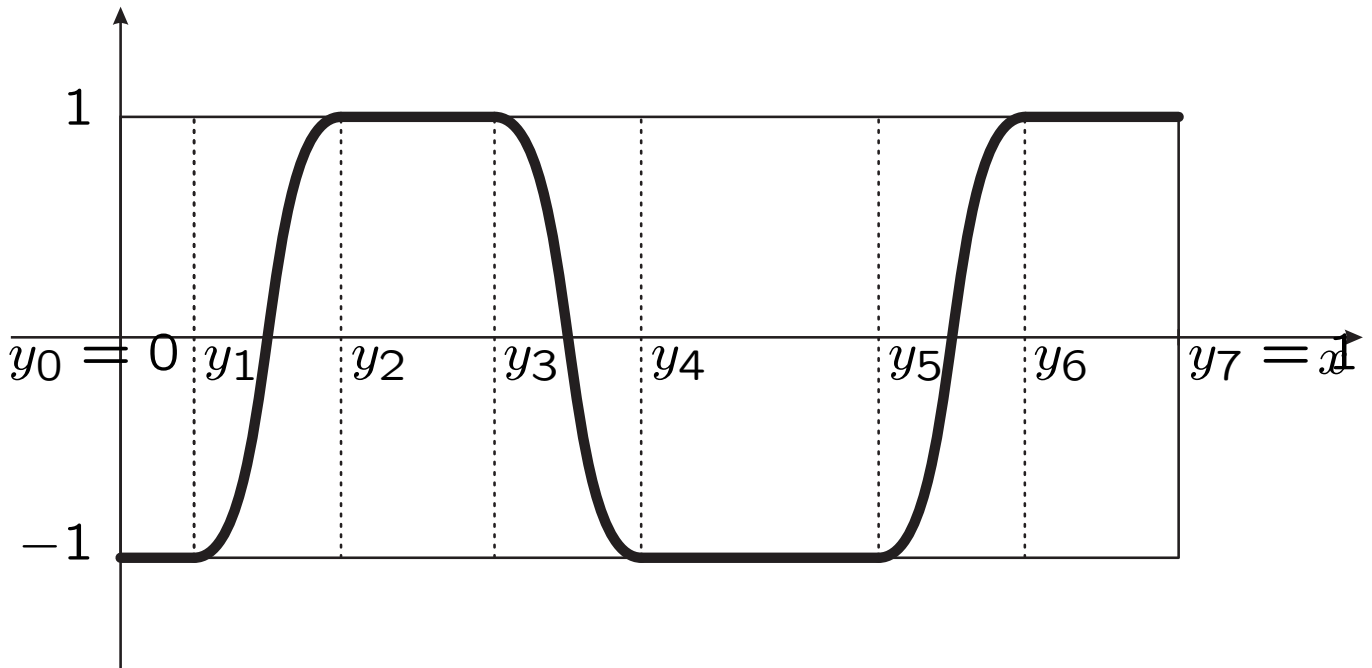
Here, we take  $u'_0 = u'(x_0)$  if  $u'(x_0) \neq 0$  (hence,  $W(u_0) - C > 0$ ); otherwise  $C = W(u_0)$  and we may take any number  $u'_0 \in \mathbb{R}$ .

The right-hand side is locally Lipschitz if  $u'(x_0) \neq 0$ ; **local existence and uniqueness**.



Shifts of the potential  $W$  by a constant  $C$ .

A stationary solution for  $1 < \alpha < p < \infty$  which gives a nonperiodic pattern formation:



Function  $u_\varepsilon$  for  $m = 3$ .

A partition of the interval  $[0, 1]$  with the points

$$(16) \quad \begin{aligned} 0 = y_0 \leq y_1 < y_2 \leq y_3 < \cdots < \\ y_{2k} \leq y_{2k+1} < \cdots < y_{2m} \leq y_{2m+1} = 1, \end{aligned}$$

where  $k = 0, 1, 2, \dots, m$  ( $m \geq 0$  – an integer),

$$(17) \quad \begin{aligned} y_{2k} - y_{2k-1} &= 2\vartheta_\varepsilon = 2\varepsilon\vartheta_1 \quad (> 0), \\ y_{2k+1} - y_{2k} &= d_k \geq 0. \end{aligned}$$

## Case $N \geq 2$ .

(Electr. J.D.E., Conf. **17** (2009), 227–254)

The first integral of eq. (8) gets a damping term  $Z(r)$ ,

$$(18) \quad \frac{\varepsilon^p}{p'} |u'(r)|^p - W(u(r)) + \varepsilon^p Z(r) = 0, \\ 0 \leq r < \infty,$$

where the damping term  $Z(r)$  satisfies

$$(19) \quad Z'(r) = \frac{N-1}{r} |u'(r)|^p \geq 0 \quad \text{for } r > 0.$$

**Rescaling:** We replace  $r$  by  $\varepsilon^{-1}r$  to get  $\varepsilon = 1$ .

If  $u(0) = s_0 = \pm 1$  (a local minimizer for  $W$ ) and  $u'(0) = 0$ , then the functions

$$r \mapsto r^{-1} |u'(r)|^{p-1} \quad \text{and} \\ r \mapsto r^{-p'} (Z(r) - Z(0)) : (0, \delta) \rightarrow \mathbb{R}_+$$

are monotone increasing, for some  $\delta > 0$  small.  
( $W$  is convex in an open interval containing  $s_0$ .)

From these facts we derive the inequalities

$$(20) \quad \begin{aligned} (0 \leq r) \quad & \frac{1}{N} [W(u(r)) - W(u(0))] \\ & \leq W(u(r)) - Z(r) \\ & \leq W(u(r)) - W(u(0)) \end{aligned}$$

for all  $r \in [0, \delta)$ , where

$$Z(0) = W(u(0)) = W(s_0).$$

Applying (20) to (18), we arrive at

$$(21) \quad \begin{aligned} & \frac{p'}{N} [W(u(r)) - W(u(0))] \\ & \leq |u'(r)|^p \leq p' [W(u(r)) - W(u(0))] \end{aligned}$$

for all  $r \in [0, \delta)$ .

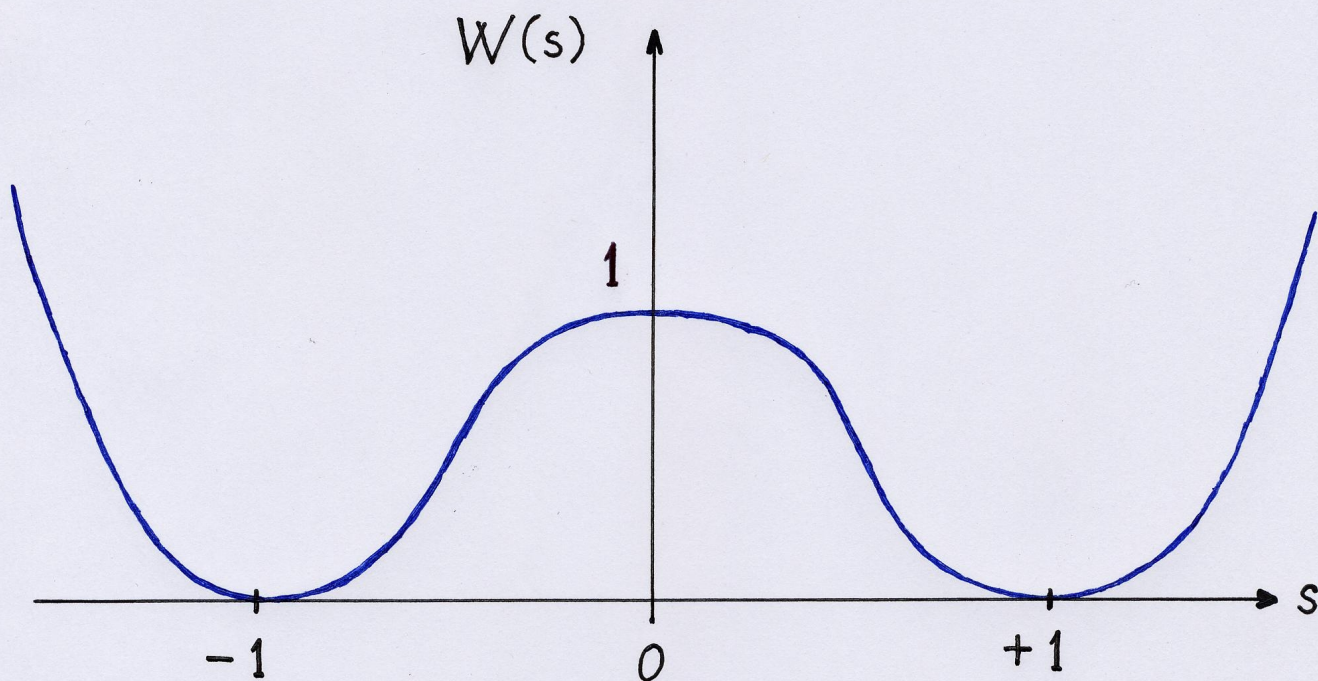
This means the *uniqueness* or *nonuniqueness* of a local solution  $u$  to equation (8) with the initial conditions  $u(0) = s_0$  (where  $s_0$  is a local minimizer for  $W$ ) and  $u'(0) = 0$  at  $r = 0$ , depending on

$$(22) \quad \int_{s_0}^{s_0+\zeta} |W(s) - W(s_0)|^{-1/p} ds$$

being *infinite* (forcing uniqueness)

or *finite* (forcing nonuniqueness), respectively:

$p \leq \alpha$  (infinite) or  $p > \alpha$  (finite).



$$W(s) = |1 - |s|^\beta|^\alpha \quad \text{for } s \in \mathbb{R},$$

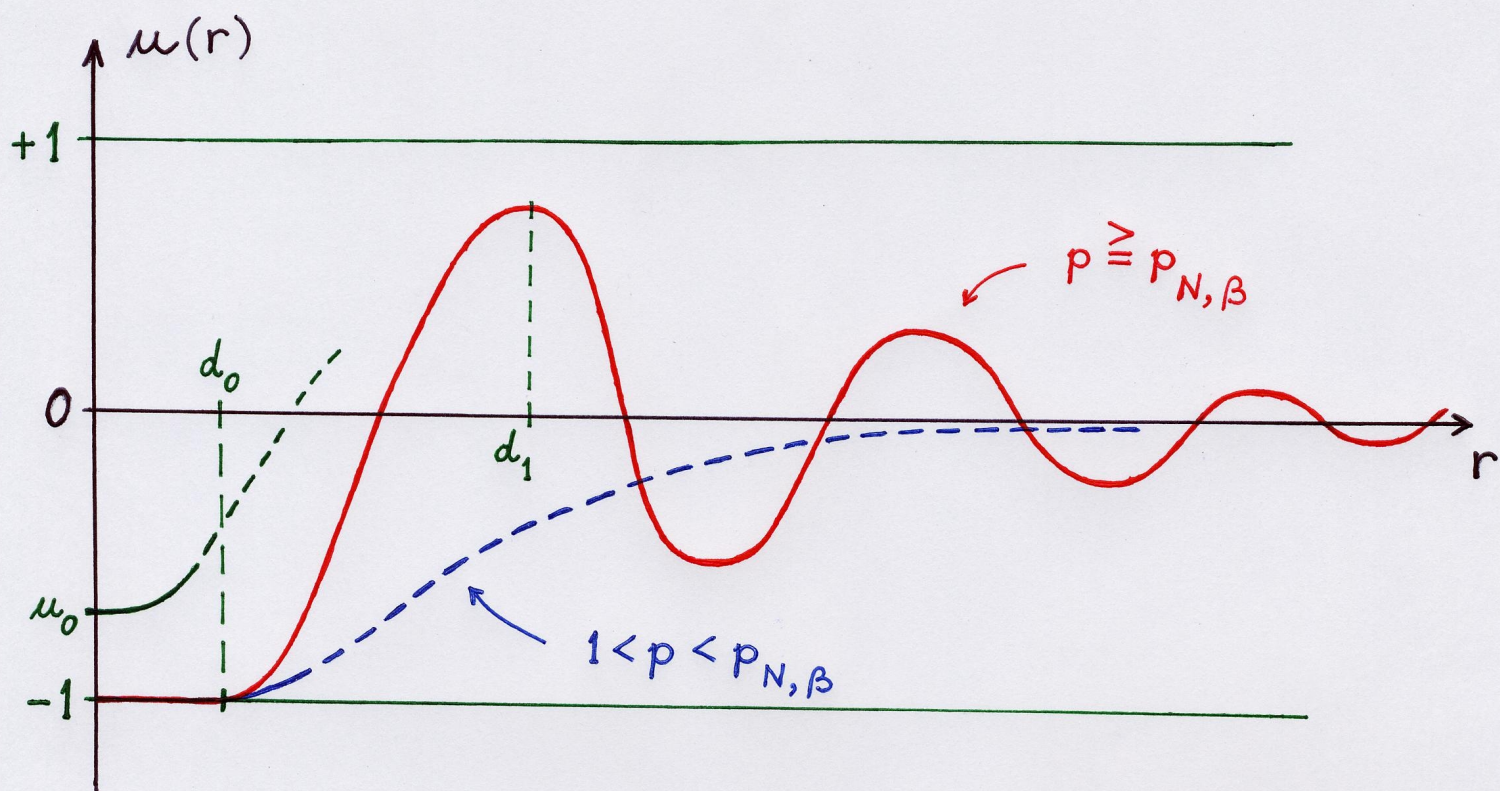
$\alpha, \beta > 1$  are constants.

- $W(s) = \beta^\alpha \cdot |1 - |s||^\alpha + \mathcal{O}(|1 - |s||^{\alpha+1})$   
as  $|s| \rightarrow 1$ ;

- $W(s) = 1 - \alpha \cdot |s|^\beta + \mathcal{O}(|s|^{2\beta})$   
as  $|s| \rightarrow 0$ .

$$p_{N,\beta} \stackrel{\text{def}}{=} \frac{\beta N}{N-1+\beta} \implies p_{1,\beta} = 1 < p$$

$N \geq 2$ :  $u(x) \equiv u(r)$  with  $r = |x|$



$d_0 = 0$  if  $1 < p \leq \alpha$  (strong diffusion)

$d_0 \geq 0$  if  $p > \alpha$  (weak diffusion)

$d_0 = 0$  if  $|u(0)| < 1, u(0) \neq 0$

3 Factors:

reaction

diffusion ( $N \geq 1$ )

damping ( $N \geq 2$ )

$$\frac{1}{p_{N, \beta}} = \frac{1}{N} + \frac{1}{\beta} - \frac{1}{\beta N} \leq 1$$

## Isolated pure state regions ( $N \geq 2$ )

(Yavdat Sh. Il'yasov, Ufa, Russia)

(Russian Academy of Sciences)

**Problem.**  $\Omega, \Omega_1 \subset \mathbb{R}^N$  bounded domains with  $C^2$ -boundaries,  $\overline{\Omega_1} \subset \Omega$  and the open set  $\Omega'_1 = \Omega \setminus \overline{\Omega_1}$  is **connected**.  $u : \overline{\Omega'_1} \rightarrow \mathbb{R}$  is called a phase transition solution of

$$(23) \quad -\operatorname{div}(|\nabla u|^{p-2} \nabla u) + W'(u) = 0 \quad \text{in } \Omega'_1$$

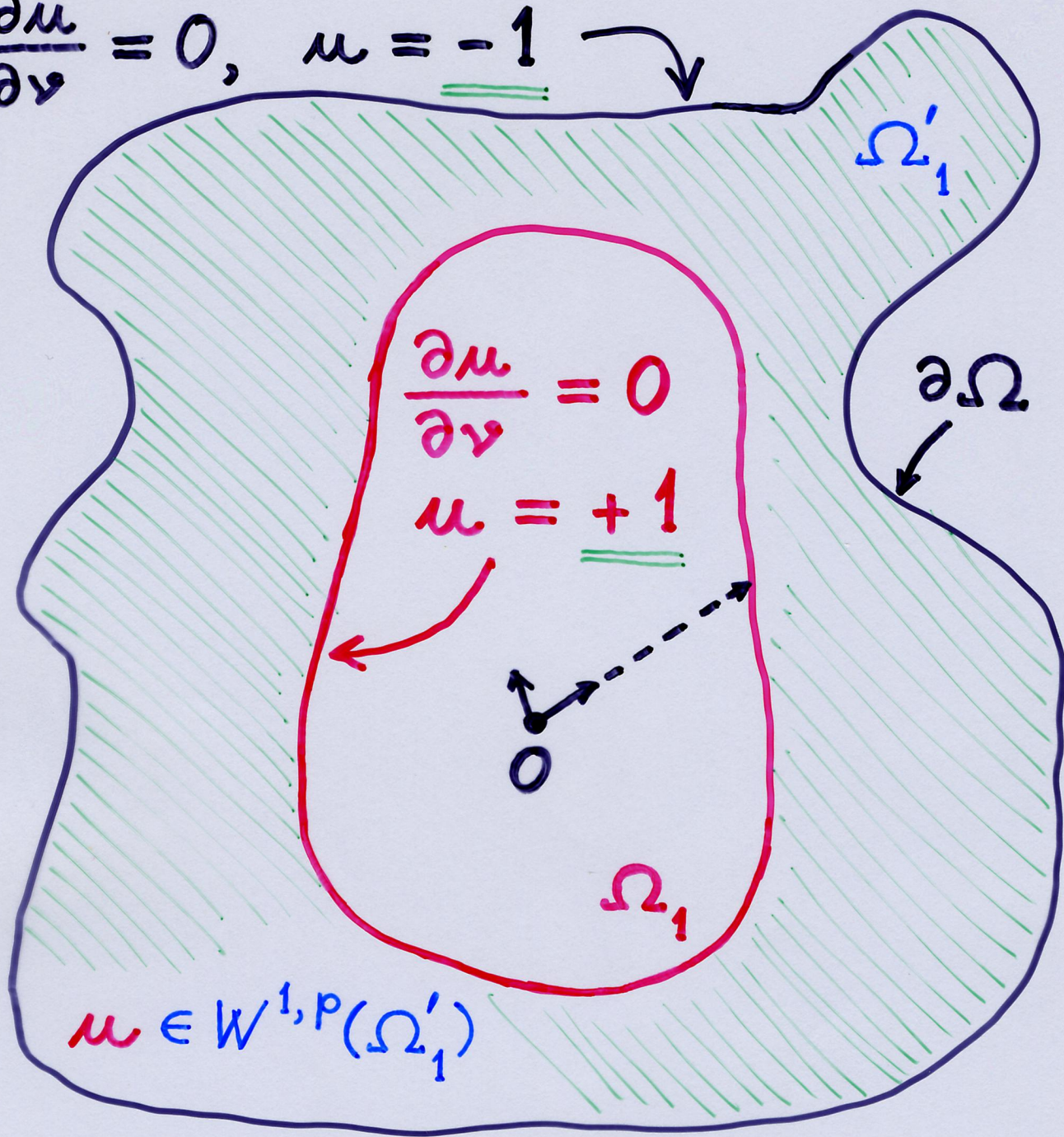
if  $u \in W^{1,p}(\Omega'_1)$  **verifies** eq. (23) above (in the weak sense) with the Neumann boundary conditions

$$(24) \quad \partial u / \partial \nu = 0 \quad \text{on } \partial \Omega'_1 = \partial \Omega \cup \partial \Omega_1$$

and the “**phase transition**” property

$$(25) \quad u = -1 \quad \text{on } \partial \Omega \quad \text{and} \quad u = 1 \quad \text{on } \partial \Omega_1.$$

$$\frac{\partial \mu}{\partial \nu} = 0, \quad \mu = -1$$



$$\begin{cases} -\Delta_p \mu + W'(\mu) = 0 & \text{in } \Omega'_1; \\ \frac{\partial \mu}{\partial \nu} = 0 & \text{on } \partial\Omega'_1 = \partial\Omega \cup \partial\Omega_1. \end{cases}$$

## Theorem.

(Nonexistence of a phase transition solution)

$\Omega$  and  $\Omega_1$  as above and  $1 < p \leq N$ . Furthermore, let  $\Omega_1$  be star-shaped with respect to the origin  $\mathbf{0} \in \mathbb{R}^N$ . Then *any* weak solution  $u \in W^{1,p}(\Omega'_1)$  of eq. (23) with the Neumann boundary conditions (24), such that  $|u| = 1$  on  $\partial\Omega$ , must be a *constant*  $= \pm 1$ .

**Proof** is based on

### Proposition 1. (Pohozaev's identity)

Let  $1 < p < \infty$ . Assume  $f \in L^1(\Omega)$  possesses derivatives  $\partial f / \partial x_i \in L^1_{\text{loc}}(\Omega)$ ;  $i = 1, 2, \dots, N$ .

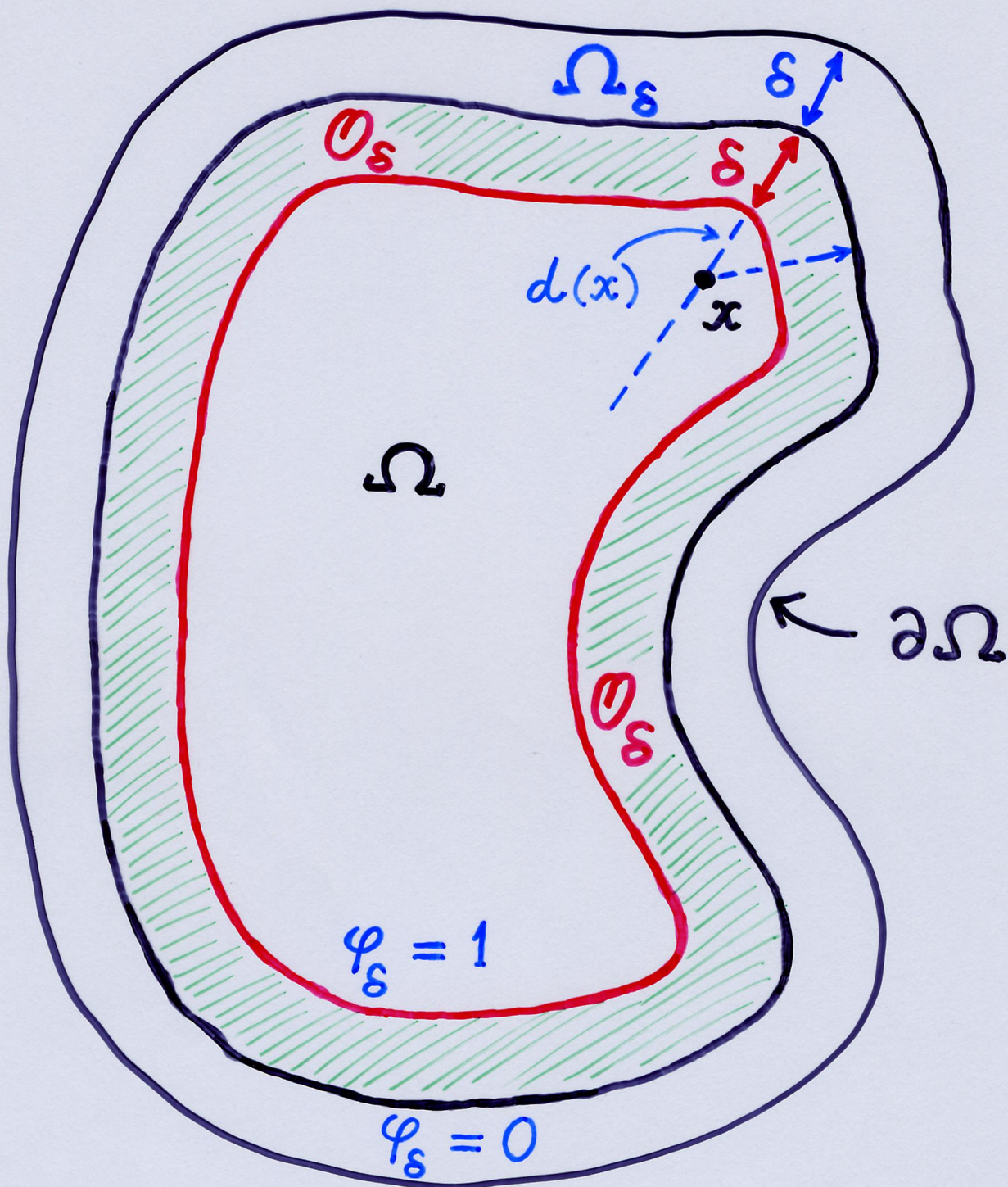
Let  $u \in C^1(\overline{\Omega})$  satisfy

$$(26) \quad -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = f(x) \quad \text{in } \Omega$$

in the sense of distributions in  $\Omega$ . Then

$$\begin{aligned} & \frac{N-p}{p} \int_{\Omega} |\nabla u|^p \, dx + \int_{\Omega} f(x) (x \cdot \nabla u) \, dx \\ &= \frac{1}{p} \int_{\partial\Omega} |\nabla u|^p (x \cdot \nu(x)) \, d\sigma(x) \\ & - \int_{\partial\Omega} |\nabla u|^{p-2} (x \cdot \nabla u) (\nu(x) \cdot \nabla u) \, d\sigma(x). \end{aligned}$$

$\delta > 0$  small,  $1 < p < 2$



$$\int_{\partial_\delta} |\nabla u|^{-(2-p)} dx < \infty$$

## Proposition 2.

(Optimal regularity in weighted  $W_{\text{loc}}^{2,2}$ )

Assume  $1 < p < \infty$ ; if  $p < 2$  then assume also

**Hypothesis.**  $\nabla u \neq 0$  a.e. in  $\mathcal{O}_\delta$ , for  $\delta > 0$  small enough, where

$$\mathcal{O}_\delta = \{x \in \Omega'_\delta : d_\delta(x) < \delta\},$$

$$\Omega'_\delta = \Omega \setminus \overline{\Omega}_\delta, \quad \Omega_\delta = \{x \in \Omega : d(x) < \delta\},$$

$$d_\delta(x) = \begin{cases} 0 & \text{if } x \in \Omega_\delta; \\ \text{dist}(x, \Omega_\delta) & \text{if } x \in \Omega \setminus \Omega_\delta. \end{cases}$$

and

$$(27) \quad \int_{\mathcal{O}_\delta} |\nabla u|^{-(2-p)} dx < \infty.$$

Let us construct  $\varphi_\delta : \overline{\Omega} \rightarrow [0, 1]$  as follows:

$$\varphi_\delta(x) = \begin{cases} (\delta^{-1} d_\delta(x))^2 & \text{if } x \in \overline{\Omega}_\delta \cup \mathcal{O}_\delta; \\ 1 & \text{if } x \in \Omega \setminus (\overline{\Omega}_\delta \cup \mathcal{O}_\delta). \end{cases}$$

(a) Then the vector field  $|\nabla u|^{(p-2)/2} \nabla u$  is in the local Sobolev space  $W_{\text{loc}}^{1,2}(\Omega'_\delta)$  and

$$\int_{\Omega'_\delta} \left| \nabla \left( |\nabla u|^{(p-2)/2} \nabla u \right) \right|^2 \varphi_\delta dx < \infty.$$

(b) Moreover,  $u \in W_{\text{loc}}^{2,2}(U)$  where

$$U = \Omega'_\delta \cap \{x \in \Omega : \nabla u(x) \neq \mathbf{0}\}$$

and one has

$$\int_U |\nabla u|^{p-2} |\nabla(\nabla u)|^2 \varphi_\delta \, dx < \infty.$$

(c) Finally, if  $p < 2$  then  $u \in W_{\text{loc}}^{2,2}(\Omega'_\delta)$  and  $\nabla(\nabla u)(x) = \mathbf{0} \in \mathbb{R}^{N \times N}$  holds for almost every

$$x \in U' = \Omega'_\delta \cap \{x \in \Omega : \nabla u(x) = \mathbf{0}\}.$$

Consequently, we may write

$$\int_{\Omega'_\delta} |\nabla u|^{p-2} |\nabla(\nabla u)|^2 \varphi_\delta \, dx < \infty.$$

### Remark 3. (The $p$ -Laplacian)

For  $1 < p < \infty$  we have  $\Delta_p u$

$$\begin{aligned} &= |\nabla u|^{p-2} \left( \Delta u + \frac{p-2}{|\nabla u|^2} \sum_{i,j=1}^N \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} \right) \\ &= |\nabla u|^{p-2} \sum_{i,j=1}^N \left( \delta_{ij} + \frac{p-2}{|\nabla u|^2} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) \frac{\partial^2 u}{\partial x_i \partial x_j} \end{aligned}$$

with  $\left( \delta_{ij} + \frac{p-2}{|\nabla u|^2} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right)_{i,j=1}^N$  uniformly elliptic.

Notice that the entries  $A_{ij} = \partial a_i / \partial \eta_j$  of the Jacobian matrix  $\mathbf{A} = (A_{ij})_{i,j=1}^N$  of the mapping  $\boldsymbol{\eta} \mapsto \mathbf{a}(\boldsymbol{\eta}) \stackrel{\text{def}}{=} |\boldsymbol{\eta}|^{p-2} \boldsymbol{\eta} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ ,

$$\mathbf{A}(\boldsymbol{\eta}) = |\boldsymbol{\eta}|^{p-2} \left( \mathbf{I} + (p-2) \frac{\boldsymbol{\eta} \otimes \boldsymbol{\eta}}{|\boldsymbol{\eta}|^2} \right), \quad \boldsymbol{\eta} \in \mathbb{R}^N \setminus \{\mathbf{0}\},$$

satisfy the ellipticity and growth inequalities, with some constants  $0 < \gamma \leq \Gamma < \infty$ ,

$$\begin{aligned} \gamma |\boldsymbol{\eta}|^{p-2} |\boldsymbol{\xi}|^2 &\leq \langle \mathbf{A}(\boldsymbol{\eta}) \boldsymbol{\xi}, \boldsymbol{\xi} \rangle = \sum_{i,j=1}^N \frac{\partial a_i}{\partial \eta_j}(\boldsymbol{\eta}) \cdot \xi_i \xi_j \\ &\leq \Gamma |\boldsymbol{\eta}|^{p-2} |\boldsymbol{\xi}|^2 \end{aligned}$$

for all  $\boldsymbol{\eta} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$  and for all  $\boldsymbol{\xi} \in \mathbb{R}^N$ .

## Proof of Proposition 2.

Optimal regularity  $|\nabla u|^{(p-2)/2} \nabla u \in W_{\text{loc}}^{1,2}(\Omega'_\delta)$

Take  $h \in \mathbb{R}^N$  with  $0 < |h| < \delta$ .

Let  $\varphi \in C_0^1(\Omega)$ ,  $\varphi \geq 0$ , be supported in

$$\Omega'_\delta = \Omega \setminus \overline{\Omega_\delta} = \{x \in \Omega : d(x) > \delta\},$$

i.e.,  $\text{supp } \varphi \subset \Omega'_\delta$ .

Hence, the difference quotient

$$(31) \quad \delta_h \varphi(x) \stackrel{\text{def}}{=} \frac{\varphi(x+h) - \varphi(x)}{|h|}, \quad x \in \Omega,$$

satisfies  $\delta_h \varphi \in C_0^1(\Omega)$ . Multiply eq. (26) by  $\delta_h \varphi$  and integrate over  $\Omega$ ,

$$\int_{\Omega} \mathbf{a}(\nabla u) \cdot \nabla(\delta_h \varphi) \, dx = \int_{\Omega} f(\delta_h \varphi) \, dx$$

with the notation  $\mathbf{a}(\nabla u) = |\nabla u|^{p-2} \nabla u$ .

A simple substitution above yields

$$(32) \quad \int_{\Omega} \frac{\mathbf{a}(\nabla u(x+h)) - \mathbf{a}(\nabla u(x))}{|h|} \cdot \nabla \varphi(x) \, dx \\ = \int_{\Omega} \frac{f(x+h) - f(x)}{|h|} \varphi \, dx.$$

Now we use the Taylor formula

$$(33) \quad \mathbf{a}(\nabla u(x+h)) - \mathbf{a}(\nabla u(x)) \\ = \tilde{\mathbf{A}}(x; h) (\nabla u(x+h) - \nabla u(x))$$

with the abbreviation

$$\tilde{\mathbf{A}}(x; h) \stackrel{\text{def}}{=} \int_0^1 \mathbf{A}((1-s)\nabla u(x+h) + s\nabla u(x)) \, ds \in \mathbb{R}^{N \times N}$$

and replace the function  $\varphi$  by  $(\delta_h u) \varphi$  in (32),

$$\begin{aligned} & \int_{\Omega} \langle \tilde{\mathbf{A}}(x; h) \delta_h(\nabla u), \delta_h(\nabla u) \rangle \varphi \, dx \\ (34) \quad & + \int_{\Omega} \langle \tilde{\mathbf{A}}(x; h) \delta_h(\nabla u), \nabla \varphi \rangle (\delta_h u) \, dx \\ & = \int_{\Omega} (\delta_h f) (\delta_h u) \varphi \, dx. \end{aligned}$$

We used  $\nabla[(\delta_h u)\varphi] = \varphi \cdot \delta_h(\nabla u) + (\delta_h u) \cdot \nabla \varphi$ . We estimate the integrals on the left-hand side in (34) by the inequalities (30), combined with Cauchy's inequality, and abbreviate

$$(35) \quad \tilde{a}(x; h) \stackrel{\text{def}}{=} \int_0^1 |(1-s)\nabla u(x+h) + s\nabla u(x)|^{p-2} \, ds,$$

in order to get, with  $\varphi = \varphi_{\delta}$  from (28),

$$\begin{aligned} & \gamma \int_{\Omega} \tilde{a}(x; h) |\delta_h(\nabla u)|^2 \varphi_{\delta} \, dx \\ (36) \quad & \leq \Gamma \int_{\mathcal{O}_{\delta}} \tilde{a}(x; h) |\delta_h(\nabla u)| |\nabla \varphi_{\delta}| |\delta_h u| \, dx \\ & + \int_{\Omega} |\delta_h f| |\delta_h u| \varphi_{\delta} \, dx. \end{aligned}$$

To estimate the first integral on the right-hand side in ineq. (36), we use

$$(37) \quad |\nabla \varphi_{\delta}(x)|^2 \leq C\delta^{-2} \varphi_{\delta}(x), \quad x \in \Omega \setminus \partial \mathcal{O}_{\delta},$$

then apply Cauchy's inequality, thus arriving at

$$\begin{aligned}
& \gamma \int_{\Omega} \tilde{a}(x; h) |\delta_h(\nabla u)|^2 \varphi_{\delta} \, dx && \text{(by (37))} \\
& \leq \Gamma C^{1/2} \delta^{-1} \int_{\mathcal{O}_{\delta}} \tilde{a}(x; h) |\delta_h(\nabla u)| \cdot \varphi_{\delta}^{1/2} \cdot |\delta_h u| \, dx \\
& + \int_{\Omega} |\delta_h f| |\delta_h u| \varphi_{\delta} \, dx \\
& \leq \Gamma C^{1/2} \delta^{-1} \left( \int_{\mathcal{O}_{\delta}} \tilde{a}(x; h) |\delta_h(\nabla u)|^2 \varphi_{\delta} \, dx \right)^{1/2} \\
& \times \left( \int_{\mathcal{O}_{\delta}} \tilde{a}(x; h) |\delta_h u|^2 \, dx \right)^{1/2} \\
& + \int_{\Omega} |\delta_h f| |\delta_h u| \varphi_{\delta} \, dx \\
& \leq \frac{\gamma}{2} \int_{\mathcal{O}_{\delta}} \tilde{a}(x; h) |\delta_h(\nabla u)|^2 \varphi_{\delta} \, dx \\
& + \frac{1}{2} \gamma^{-1} \Gamma^2 C \delta^{-2} \int_{\mathcal{O}_{\delta}} \tilde{a}(x; h) |\delta_h u|^2 \, dx \\
& + \int_{\Omega} |\delta_h f| |\delta_h u| \varphi_{\delta} \, dx
\end{aligned}$$

which yields

$$\begin{aligned}
& \int_{\Omega} \tilde{a}(x; h) |\delta_h(\nabla u)|^2 \varphi_{\delta} \, dx \leq \\
& C (\Gamma / (\gamma \delta))^2 \int_{\mathcal{O}_{\delta}} \tilde{a}(x; h) |\delta_h u|^2 \, dx \\
& + (2/\gamma) \int_{\Omega} |\delta_h f| |\delta_h u| \varphi_{\delta} \, dx .
\end{aligned}$$

We state a few geometric inequalities

(Takáč, IUMJ 2002, Lemma A.1, p. 233).

Let  $1 < p < \infty$  and  $p \neq 2$ . Assume that  $\Theta \in L^\infty(0, 1)$  satisfies  $\Theta \geq 0$  in  $(0, 1)$  and  $T = \int_0^1 \Theta(s) ds > 0$ . Then there is a constant  $c_p \equiv c_p(\Theta) > 0$  such that, for all  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$ :

If  $p > 2$  then

(38)

$$\begin{aligned} c_p(\Theta)^{p-2} \left( \max_{0 \leq s \leq 1} |\mathbf{a} + s\mathbf{b}| \right)^{p-2} &\leq \int_0^1 |\mathbf{a} + s\mathbf{b}|^{p-2} \Theta(s) ds \\ &\leq T \cdot \left( \max_{0 \leq s \leq 1} |\mathbf{a} + s\mathbf{b}| \right)^{p-2}, \end{aligned}$$

and if  $1 < p < 2$  and  $|\mathbf{a}| + |\mathbf{b}| > 0$  then

(39)

$$\begin{aligned} T \cdot \left( \max_{0 \leq s \leq 1} |\mathbf{a} + s\mathbf{b}| \right)^{p-2} &\leq \int_0^1 |\mathbf{a} + s\mathbf{b}|^{p-2} \Theta(s) ds \\ &\leq c_p(\Theta)^{p-2} \left( \max_{0 \leq s \leq 1} |\mathbf{a} + s\mathbf{b}| \right)^{p-2}. \end{aligned}$$

Equivalently, in both cases ( $p \neq 2$ ), the ratio

$$\int_0^1 |\mathbf{a} + s\mathbf{b}|^{p-2} \Theta(s) dx \Big/ \left( \max_{0 \leq s \leq 1} |\mathbf{a} + s\mathbf{b}| \right)^{p-2}$$

is bounded below and above by positive constants, whenever  $|\mathbf{a}| + |\mathbf{b}| > 0$ .

Now we apply inequalities (38) and (39) to the expression  $\tilde{a}(x; h)$  (defined in (35)) in order to conclude that there exist constants  $C'_1, C'_2 > 0$  such that

(40)

$$\int_{\Omega} \tilde{a}(x; h) |\delta_h(\nabla u)|^2 \varphi_{\delta} dx \leq C'_1 \int_{\mathcal{O}_{\delta}} \tilde{a}(x; h) |\delta_h u|^2 dx + C'_2 \int_{\Omega} |\delta_h f| |\delta_h u| \varphi_{\delta} dx,$$

where we have abbreviated

(41)

$$\tilde{a}(x; h) \stackrel{\text{def}}{=} \left( \max_{0 \leq s \leq 1} |(1-s)\nabla u(x+h) + s\nabla u(x)| \right)^{p-2}.$$

Now we recall our hypothesis  $\int_{\Omega} |\nabla f| dx < \infty$  and the regularity result  $|\nabla u| \leq \text{const} < \infty$  in  $\Omega$ , together with hypothesis (27) if  $1 < p < 2$ . Applying these inequalities to the right-hand side of (40), we arrive at

$$(42) \quad \int_{\Omega} \tilde{a}(x; h) |\delta_h(\nabla u)|^2 \varphi_{\delta} dx \leq C' < \infty$$

where the constant  $C' > 0$  is independent from  $h \in \mathbb{R}^N$  with  $0 < |h| < \delta$ .

Finally, we deduce from (29) that the Jacobian matrix  $\mathbf{B} = (B_{ij})_{i,j=1}^N$  of the mapping

$\boldsymbol{\eta} \mapsto \mathbf{b}(\boldsymbol{\eta}) = |\boldsymbol{\eta}|^{(p-2)/2} \boldsymbol{\eta} : \mathbb{R}^N \rightarrow \mathbb{R}^N$  equals  
(43)

$$\mathbf{B}(\boldsymbol{\eta}) = |\boldsymbol{\eta}|^{(p-2)/2} \left( \mathbf{I} + \frac{p-2}{2} \frac{\boldsymbol{\eta} \otimes \boldsymbol{\eta}}{|\boldsymbol{\eta}|^2} \right), \quad \boldsymbol{\eta} \in \mathbb{R}^N \setminus \{\mathbf{0}\}.$$

In analogy with the Taylor formula in (33),

$$(44) \quad \begin{aligned} & \mathbf{b}(\nabla u(x+h)) - \mathbf{b}(\nabla u(x)) \\ & = \tilde{\mathbf{B}}(x; h) (\nabla u(x+h) - \nabla u(x)) \end{aligned}$$

with the abbreviation

$$\tilde{\mathbf{B}}(x; h) \stackrel{\text{def}}{=} \int_0^1 \mathbf{B}((1-s)\nabla u(x+h) + s\nabla u(x)) \, ds \in \mathbb{R}^{N \times N}.$$

Now we treat the  $L^2$ -norm

$$(45) \quad \begin{aligned} & \int_{\Omega} |\delta_h(\mathbf{b}(\nabla u))|^2 \varphi_{\delta} \, dx \\ & = \int_{\Omega} \langle \tilde{\mathbf{B}}(x; h) \delta_h(\nabla u), \tilde{\mathbf{B}}(x; h) \delta_h(\nabla u) \rangle \varphi_{\delta} \, dx \\ & = \int_{\Omega} \langle \tilde{\mathbf{B}}(x; h)^2 \delta_h(\nabla u), \delta_h(\nabla u) \rangle \varphi_{\delta} \, dx \end{aligned}$$

where, using the abbreviation

$$\boldsymbol{\eta}(s) = (1-s)\nabla u(x+h) + s\nabla u(x) \in \mathbb{R}^N, \quad 0 \leq s \leq 1,$$

we have

$$(46) \quad \begin{aligned} \tilde{\mathbf{B}}(x; h)^2 & = \int_0^1 \int_0^1 \mathbf{B}(\boldsymbol{\eta}(s)) \mathbf{B}(\boldsymbol{\eta}(t)) \, ds \, dt = \\ & \int_0^1 \int_0^1 |\boldsymbol{\eta}(s)|^{(p-2)/2} |\boldsymbol{\eta}(t)|^{(p-2)/2} \mathbf{C}(\boldsymbol{\eta}(s), \boldsymbol{\eta}(t)) \, ds \, dt \end{aligned}$$

with the  $(N \times N)$ -matrix

$$\mathbf{C}(\mathbf{a}, \mathbf{b}) = \left( \mathbf{I} + \frac{p-2}{2} \frac{\mathbf{a} \otimes \mathbf{a}}{|\mathbf{a}|^2} \right) \left( \mathbf{I} + \frac{p-2}{2} \frac{\mathbf{b} \otimes \mathbf{b}}{|\mathbf{b}|^2} \right) \in \mathbb{R}^{N \times N}$$

being uniformly bounded for  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N \setminus \{0\}$ , cf. (43). Furthermore,  $\mathbf{C}(\mathbf{a}, \mathbf{b}) = \mathbf{C}(\mathbf{a}) \mathbf{C}(\mathbf{b})$  where

$$\mathbf{C}(\mathbf{a}) \stackrel{\text{def}}{=} \left( \mathbf{I} + \frac{p-2}{2} \frac{\mathbf{a} \otimes \mathbf{a}}{|\mathbf{a}|^2} \right) \in \mathbb{R}^{N \times N}$$

is a symmetric matrix with the eigenvalues 1 and  $p/2$  (if  $N \geq 2$ ). Consequently, there is a constant  $\Gamma' > 0$  such that the kernel  $\mathbf{B}(\boldsymbol{\eta}(s)) \mathbf{B}(\boldsymbol{\eta}(t))$  of the quadratic form contained in the integrand on the right-hand side of eq. (45) satisfies

$$\begin{aligned} \langle \mathbf{B}(\boldsymbol{\eta}(s)) \mathbf{B}(\boldsymbol{\eta}(t)) \boldsymbol{\xi}, \boldsymbol{\xi} \rangle &= \langle \mathbf{B}(\boldsymbol{\eta}(t)) \boldsymbol{\xi}, \mathbf{B}(\boldsymbol{\eta}(s)) \boldsymbol{\xi} \rangle \\ &\leq |\mathbf{B}(\boldsymbol{\eta}(t)) \boldsymbol{\xi}| \cdot |\mathbf{B}(\boldsymbol{\eta}(s)) \boldsymbol{\xi}| \\ &\leq |\mathbf{B}(\boldsymbol{\eta}(t))| \cdot |\mathbf{B}(\boldsymbol{\eta}(s))| \cdot |\boldsymbol{\xi}|^2 \\ &\leq \Gamma' |\boldsymbol{\eta}(t)|^{(p-2)/2} |\boldsymbol{\eta}(s)|^{(p-2)/2} |\boldsymbol{\xi}|^2 \end{aligned}$$

for  $\boldsymbol{\xi} \in \mathbb{R}^N$  and  $s, t \in [0, 1]$  with  $\boldsymbol{\eta}(s), \boldsymbol{\eta}(t) \neq 0$ . We first integrate this inequality with respect to  $s, t \in [0, 1]$ , then apply it to (46) to get

$$(47) \quad \langle \tilde{\mathbf{B}}(x; h)^2 \boldsymbol{\xi}, \boldsymbol{\xi} \rangle \leq \Gamma' \tilde{b}(x; h)^2 |\boldsymbol{\xi}|^2 \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^N,$$

with the abbreviation

$$\begin{aligned}\tilde{b}(x; h) &\stackrel{\text{def}}{=} \int_0^1 |\boldsymbol{\eta}(s)|^{(p-2)/2} \, ds \\ &= \int_0^1 |(1-s)\nabla u(x+h) + s\nabla u(x)|^{(p-2)/2} \, ds,\end{aligned}$$

in analogy with (35). This integral is estimated from above by inequalities (38) and (39),

$$(48) \quad \tilde{b}(x; h) \leq C_p \hat{b}(x; h),$$

where  $C_p > 0$  is a numerical constant depending only on  $p$  ( $1 < p < \infty$ ) and

$$\hat{b}(x; h) \stackrel{\text{def}}{=} \left( \max_{0 \leq s \leq 1} |(1-s)\nabla u(x+h) + s\nabla u(x)| \right)^{(p-2)/2}.$$

We combine inequalities (47) and (48) and apply them to the quadratic form contained in the integrand on the right-hand side of eq. (45),

$$(49) \quad \begin{aligned}\int_{\Omega} |\delta_h(\mathbf{b}(\nabla u))|^2 \varphi_{\delta} \, dx &\leq \Gamma' \int_{\Omega} \tilde{b}(x; h)^2 |\delta_h(\nabla u)|^2 \varphi_{\delta} \, dx, \\ &\leq \Gamma'' \int_{\Omega} \hat{b}(x; h)^2 |\delta_h(\nabla u)|^2 \varphi_{\delta} \, dx,\end{aligned}$$

with the constant  $\Gamma'' = \Gamma' C_p^2 > 0$ . As

$\hat{a}(x; h) = \hat{b}(x; h)^2$  by (41), ineq. (42) implies

$$(50) \quad \int_{\Omega} |\delta_h(\mathbf{b}(\nabla u))|^2 \varphi_{\delta} \, dx \leq C'' < \infty$$

where the constant  $C'' > 0$  is independent from  $h \in \mathbb{R}^N$  with  $0 < |h| < \delta$ . We are now ready to derive all conclusions of our proposition from this estimate. ■

## Proof of **Proposition 1**. (Pohozaev's id.)

$$\begin{aligned}
 & \frac{N-p}{p} \int_{\Omega} |\nabla u|^p \, dx + \int_{\Omega} f(x) (x \cdot \nabla u) \, dx \\
 &= \frac{1}{p} \int_{\partial\Omega} |\nabla u|^p (x \cdot \nu(x)) \, d\sigma(x) \\
 & \quad - \int_{\partial\Omega} |\nabla u|^{p-2} (x \cdot \nabla u) (\nu(x) \cdot \nabla u) \, d\sigma(x).
 \end{aligned}$$

We begin with the formal calculations

$$\begin{aligned}
 & \operatorname{div} \left( (x \cdot \nabla u) |\nabla u|^{p-2} \nabla u \right) \\
 &= (x \cdot \nabla u) \Delta_p u \\
 (51) \quad & + \nabla(x \cdot \nabla u) \cdot |\nabla u|^{p-2} \nabla u \\
 &= (x \cdot \nabla u) \Delta_p u + |\nabla u|^p + \frac{1}{p} x \cdot \nabla |\nabla u|^p,
 \end{aligned}$$

$$(52) \quad \operatorname{div} (x |\nabla u|^p) = N |\nabla u|^p + x \cdot \nabla |\nabla u|^p$$

which are valid pointwise in  $\Omega$  if  $u \in C^2(\Omega)$ .

From eq. (51) we subtract  $(1/p)$ -multiple of eq. (52) and observe that the difference

$$\begin{aligned}
 (53) \quad & \operatorname{div} \mathbf{v}(x) = \operatorname{div} \left( (x \cdot \nabla u) |\nabla u|^{p-2} \nabla u \right) - \frac{1}{p} \operatorname{div} (x |\nabla u|^p) \\
 &= (x \cdot \nabla u) \Delta_p u + \left( 1 - \frac{N}{p} \right) |\nabla u|^p \\
 &= -f(x) (x \cdot \nabla u) - \frac{N-p}{p} |\nabla u|^p
 \end{aligned}$$

belongs to  $L^1(\Omega)$ , with the vector field

$$\mathbf{v}(x) \stackrel{\text{def}}{=} (x \cdot \nabla u) |\nabla u|^{p-2} \nabla u - \frac{1}{p} x |\nabla u|^p$$

being continuous in  $\bar{\Omega}$ , i.e.,  $\mathbf{v} \in [C(\bar{\Omega})]^N$ . We complete the proof of Pohozaev's identity by applying the divergence theorem to eq. (53).

