On some optimal partition problems

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Outline

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Partitions associated with strongly competing systems

We consider systems of k nonlinear elliptic equations, of the following type

$$\begin{cases} -\Delta u_i = f_i(x, u_i) - \beta g_i(u_1, \dots, u_k) & \text{in } \Omega \\ u_i \geq 0 & \text{in } \Omega \\ u_i = \gamma_i & \text{on } \partial\Omega. \end{cases}$$

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In the main applications g_i are either:

$$g_i(u_1, \dots, u_k) = \underbrace{u_i \sum_{j \neq i} \beta_{ij} u_j^2}_{\text{gradient iff symmetric}} \quad , \quad g_i(u_1, \dots, u_k) = \underbrace{u_i \sum_{j \neq i} \beta_{ij} u_j}_{\text{Lotka-Volterra}} .$$

Densities $u_i \in H^1(\Omega, \mathbb{R})$, $i = 1, \dots, k$, with $\Omega \subset \mathbb{R}^N$

Competition parameter $\beta > 0$.

Bounds in Hölder spaces and convergence

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With mild assumptions on the functions f_i segregation occurs: let $(u_{1,\beta}, \dots, u_{k,\beta})$ be solutions of the system with β fixed, then

$$\begin{aligned} u_{i,\beta} \to u_i \quad & \text{as } \beta \to +\infty \text{ in } H^1 \cap \mathcal{C}^{0,\alpha} \\ u_i \cdot u_j = 0 \quad & \text{for } i \neq j, \text{ almost everywhere on } \Omega \\ -\Delta u_i = f_i(x, u_i) \quad & \text{where } u_i > 0. \end{aligned}$$

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$$-\Delta u_i = f_i(x, u_i) \quad \text{where } u_i > 0.$$

The limiting configuration may not be solution of any differential equation anymore; moreover the supports of the limiting u_i are also unknown $\Rightarrow$ free boundary problem. We call nodal the associated partition and nodal set its boundary.

Segregation and nodal sets

The main issues are

- Hölder (or Lipschiz) bounds uniform in β;
- regularity of the limiting configuration; Lipschitz-continuous
- extremality conditions, satisfied by the functions along the nodal set;
- regularity of the nodal set;
- uniqueness of the limiting configuration;
- variational characterization of the supports, in terms of an optimal partition of the domain.

Interaction through the nodal set

For the Lotka-Volterra case with symmetric interactions (i.e. $\beta_{ij} = \beta_{ji}=1$), we have a set of differential inequalities. Denote $\widehat{u}_i = u_i - \sum_{j\neq i} u_j$ (and similarly for $\widehat{f}_i$), then

$$-\Delta u_i \leq f_i(x, \widehat{u}_i) \quad \text{and} \quad -\Delta \widehat{u}_i \geq \widehat{f}_i(x, \widehat{u}_i) \qquad (\mathcal{S})$$

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While, for all type of interaction with a gradient structure, we have the weak reflection law:
$U \in \mathrm{Lip}(\Omega)$ and, for any ball $B_r(x_0, r) \subset \Omega$,

$$\frac{d}{dr} \frac{1}{r^{N-2}} \int_{B_r(x_0)} |\nabla U|^2 = \frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\nu U)^2 \, d\sigma + \frac{2}{r^{N-1}} \int_{B_r(x_0)} \sum_i f_i(x, u_i) \langle \nabla u_i, x - x_0 \rangle. \quad (WRL)$$

More references

- Regularity for energy minimizing configurations:

L. Caffarelli, F.-H- Lin, Singularly perturbed elliptic systems and multi-valued harmonic functions with free boundaries, *J. Amer. Math. Soc.* (2008)

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L. A. Caffarelli, A. L. Karakhanyan, and F. Lin, The geometry of solutions to a segregation problem for non-divergence systems, *Journal of Fixed Point Theory and Applications* (2009)

- Lotka-Volterra interactions:

K. Wang and Z. Zhang,Some New Results in Competing Systems with Many Species, *Ann. Inst. H. Poincaré Anal. Non Linéaire* (2010)

Theorem

$$\textit{Energy minimizing} \quad \Longrightarrow \quad (\mathcal{S}) \quad \Longrightarrow \quad (\textit{WRL})$$

Next, we have extended the result by Caffarelli and Lin for minimizers and $(\mathcal{S})$ to $(\textit{WRL})$.

Regularity of the nodal set under the weak reflection law

Let $U=(u_1,\ldots,u_k)$ the limiting profile as before and $\mathcal{N}$ its nodal set:

$$\mathcal{N}=\{x\in\Omega\ :\ u_i(x))=0\ ,\forall i=1,\ldots,k\}\ .$$

Assume (WRL) holds. Then,

Theorem (Tavares-T, 2010)

$$\mathcal{N}=\mathcal{N}_s\cup\mathcal{N}_r$$

where

- $\mathcal{N}_s$ *has Hausdorff dimension at most* $N-2$*;*
- $\mathcal{N}_r$ *is a union of* $\mathcal{C}^{1,\alpha}$ *codimension 1 surfaces.*
- *for every* $x\in\mathcal{N}_r$ *the nodal set locally locally separates the domain in two parts. Rename* u *and* v *the components supported in the two subdomains. Then,*

$$|\nabla u(x)|=|\nabla v(x)|$$

Minimal partitions with respect to Dirichlet energy

Given suitably smooth boundary traces γ_i, with $\gamma_i(x)\gamma_j(x) = 0$ a.e. if $i \neq j$ and functions F_i, with we minimize

$$\min \left\{ \int_\Omega \sum_{i=1}^{k} \frac{1}{2}|\nabla u_i|^2 + F_i(x, u_i) \right.$$

$$\left. u_i = \gamma_i \text{ on } \partial\Omega\ ,\ u_i(x)u_j(x) = 0 \text{ a.e. if } i \neq j \right\}$$

Theorem (Conti, T., Verzini)

Let $\lambda_1(\Omega)$ be the first eigenvalue of the Laplace operator with Dirichlet boundary conditions in Ω. If $\frac{\partial F_i}{\partial s}(x, 0) = 0$ and

$$\frac{\partial^2 F_i}{\partial s^2}(x, s) < \lambda_1(\Omega), \forall x \in \Omega$$

then, for each fixed boundary data, the minimizer is unique.

Extremality Conditions

Denote

$$\widehat{u}_i = u_i - \sum_{j \neq i} u_j$$

and similarly, being $f_i(x,s) = \frac{\partial F_i}{\partial s}(x,s)$,

$$\widehat{f}_i(x,\widehat{u}_i) = \sum_j f_j(x,\widehat{u}_i)\chi_{\mathrm{supp}(u_j)} = \begin{cases} f_i(x,u_i) & \text{if } x \in \mathrm{supp}(u_i) \\ -f_h(x,u_h) & \text{if } x \in \mathrm{supp}(u_h). \end{cases}$$

Theorem (Conti-T-Verzini, 2006)

Let U be a minimizer. Then, for every i, we have, in distributional sense,

$$-\Delta u_i \leq f_i(x,\widehat{u}_i)$$

$$-\Delta \widehat{u}_i \geq \widehat{f}_i(x,\widehat{u}_i)$$

Uniqueness?

Problem:

For given bounday trace $\Gamma = (\gamma_1, \ldots, \gamma_k)$, with $\gamma_i \in H^{1/2}(\partial\Omega)$ having mutually disjoint supports, find how many extensions $U = (u_1, \ldots, u_k)$ of Γ do satisfy either:

- $-\Delta u_i \leq f_i(x, \widehat{u}_i) \qquad -\Delta \widehat{u}_i \geq \widehat{f}_i(x, \widehat{u}_i)$

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- $-\Delta u_i \leq f_i(x, \widehat{u}_i) \qquad -\Delta \widehat{u}_i \geq \widehat{f}_i(x, \widehat{u}_i)$
- (WRL)?

Consequences

We focus on the following planar system of *three* equations with strongly competing interactions

$$\begin{cases} -\Delta u_i + V u_i = -\beta g_i(u_1, \dots, u_k) & \text{in } D \\ u_i \geq 0 & \text{in } D \\ u_i = \gamma_i & \text{on } \partial D. \end{cases}$$

Remark that this system does not need to have a gradient structure.
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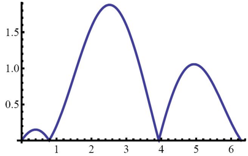

$\gamma_i \in C^1(\partial D),\ \gamma_i \geq 0$

$\gamma_i \cdot \gamma_j = 0$

$\mathrm{supp}\gamma_i$ connected

We know that the solution converges, as $\beta \to +\infty$ to a limiting profile, that satisfies the weak reflection law *WRL*.

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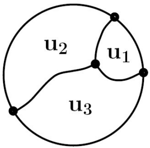
This condition contains a lot of significant information about the regularity of the limiting configuration and its nodal set. If Ω is a planar domain, then the limiting nodal set consists of three smooth arcs meeting ar a single point. In addition they share the angle in equal parts.

- How can we predict the position of the triple intersection?
- Does its location depend on the particular form of the competitive interaction?
- Does it fulfill a variational principle?


u2
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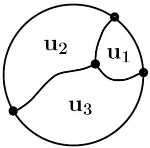
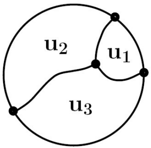


$$-\Delta u_i + V u_i = 0 \quad \text{in } \{u_i > 0\}$$



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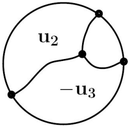
extremality condition

$$-\Delta(u_2 - u_3) + V(u_2 - u_3) = 0 \quad \text{in } \{u_2 > 0\} \cup \{u_3 > 0\}$$



We stablish uniqueness of the limiting configuration and its variational characterization.

- Though the Lotka-Volterra system does not, the limiting problem possesses a variational structure.

Outline

Uniqueness for the Dirichlet Problem

We study the problem of uniqueness of the limiting configuration and its variational characterization.

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Theorem (Conti-T-Verzini, Noris-T)

For admissible boundary data, there exists a unique solution to either

$$-\Delta u_i + V u_i \leq 0 \qquad -\Delta \widehat{u}_i + V \widehat{u}_i \geq 0 \qquad (*)$$

or (WRL) having that prescribed trace .

Ideas of the proof:

- for topological reasons, if the traces' supports are connected, then there is a unique triple point;
- if the triple point happens to be on the boundary then there is uniqueness;
- if two solutions of (*) share the same triple point then they coincide;
  - there is local unqueness and continuous dependence between the trace and the triple point;
  - use a local expansion at the triple point and the local uniqueness to prove that, as the solution is asymptotically a minimizer it is the unique minimizer.

limiting configuration as $\beta \to +\infty \implies (*)$

$\updownarrow$

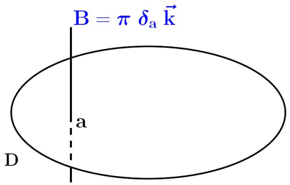
variational characterization of the configuration, in terms of minimal energy

$\updownarrow$

stationary Schrödinger equation, with a magnetic field of Aharonov–Bohm type

Noris, Terracini, *Nodal sets of magnetic Schrödinger operators of Aharonov–Bohm type and energy minimizing partitions*, Indiana Un. Math. J., to appear.

Auxiliary singular magnetic operator and the Aharonov-Bohm effect



magnetic potential A with half integer flux

$A : D \to \mathbb{R}^2$ such that

$B = \nabla \times A$

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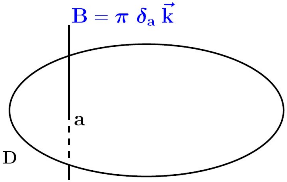

magnetic potential A with half integer flux

$A : D \to \mathbb{R}^2$ such that

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We consider the magnetic Schrödinger equation

$$\begin{cases} (i\nabla + A)^2 U + VU = 0 & \text{in } D \\ U = \Gamma & \text{on } \partial D. \end{cases}$$

where $U(x) \in L^2(D, \mathbb{C})$ stationary wave function

$\Gamma \in W^{1,\infty}(\partial D, \mathbb{C})$ suitable boundary trace

Some remarks

- The boundary trace Γ is a "complexification" of γ (see below):

$$\Gamma = \Psi\gamma, \qquad \Psi : \partial\Omega \to \mathbb{S}^1, \qquad \deg(\Psi^2) = 1$$

hence Γ has exactly three zeroes on ∂D.

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- Associated quadratic form:

$$Q_{A,V}(U) = \int_\Omega \left(|(i\nabla + A)U|^2 + V|U|^2\right) dx_1 dx_2$$

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We are interested in the nodal set of U, that is $\overline{\{x \ : U(x) = 0\}}$.

K_ψ-real solutions

We are concerned with the analysis of K_ψ–real solutions:

Definition

We say that $U : \Omega \to \mathbb{C}$ is K_ψ–real if there exists a function $\psi \in C^1(\Omega, \mathbb{C})$ with

$$|\psi| = 1, \qquad \deg(\psi, a) = 2n + 1, \; n \in \mathbb{Z}$$

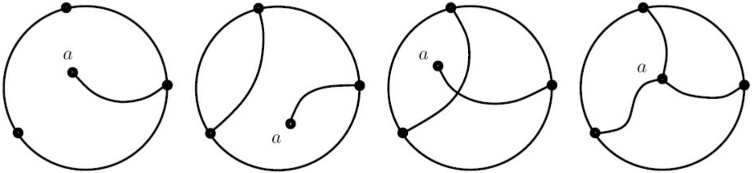
such that $U = \psi \bar{U}$.

This is a generalization of the notion of K–real function given by Helffer, Hoffman–Ostenhof and Owen, where they consider the special case $\psi = e^{i\theta}$ (θ is the angular coordinate centered at the pole). This condition arises quite naturally in this context. In fact they prove that the the spectrum of the associated magnetic Schrödinger operator (with Dirichler boundary conditions) consists of eigenvalues corresponding to K_ψ–real eigenfunctions.

Main results

Proposition (Nodal set)

For K_ψ–real boundary trace with three zeros on the boundary, the nodal set of the solution U is nontrivial. Moreover it consists of at most three regular arcs and at least one nodal arc ends at a.



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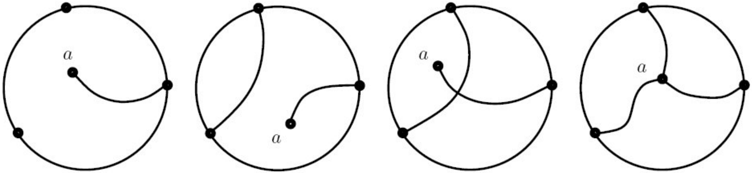

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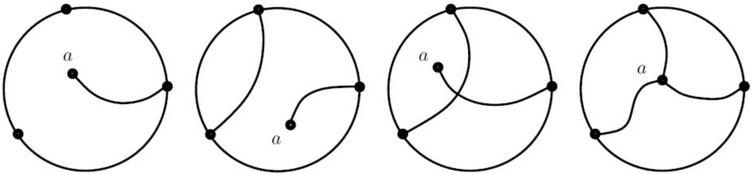

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Proposition

Given Γ, there exists at most one point $a \in D$ such that the nodal lines of U have a triple point configuration.

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Theorem (Uniqueness)

The triple point configuration coincides with the limiting configuration as $\beta \to +\infty$; in particular, it is unique.

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Moreover we consider the energy associated to the problem

$$J(w) = \int_D \left(|\nabla w|^2 + V(x) w^2\right) dx_1 dx_2, \qquad w \in H^1(D, \mathbb{R}),$$

and the following related optimal partition problem

$$\min \left\{ \sum_{i=1}^{3} J(u_i) : u_i \geq 0, u_i \cdot u_j = 0, u_i = \gamma_i \right\}$$

Proposition (Variational characterization)

The triple point configuration is the only solution of this optimal partition problem.

Triple points are critical points of the energy

Proposition (Variational characterization of the triple point)

The only critical points of the function

$$a \mapsto \varphi(a) = \min\{Q_{A,V}(U) \ : \ U \in H^1(\Omega),\ U = \Gamma \text{ on } \partial\Omega\}$$

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Proposition (Regularity)

(i) *The position of the triple point depends continuously on the L^∞–norm of the boundary data.*

(ii) *Also the nodal lines depend continuously on the L^∞–norm of the boundary data.*

An interesting consequence of the criticality property of multiple junctions is the following: let $\lambda_k(a)$ be the k-th eigenvalue of the magnetic A-B operator with pole at a; if you assume smoothness of the map

$$a \mapsto \lambda_k(a)$$

Then, critical points of $\lambda_k(a)$ correspond to points of multiple intersections of the nodal arcs.

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B. Helffer, V. Bonnaillie-Noël and T. Hoffmann-Ostenhof, *Aharonov-Bohm Hamiltonians, isospectrality and minimal partitions.* Journal of Physics A, to appear.

Bonnaillie-Noël, V., Helffer, B., and Vial, G. Numerical simulations for nodal domains and spectral minimal partitions

Recent numerical results by Bonnaillie-Noël and Helffer.

Theorem (Noris-T, 2009)

Let λ_a^k be the k–eigenvalue of the Aharonov Bohm operator with half integer circulation and Dirichlet boundary condition in a planar domain. Here a is the singularity of the magnetic potential. Assume that the function $a \mapsto \lambda_a^k$ is differentiable and that a is a critical point. Then there is a multiple intersection of the nodal lines at a.

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As a consequence, we have the following:

Corollary

- *The function $a \mapsto \lambda_1(a)$ has a global interior maximum corresponding to an eigenfunction of multiplicity two.*

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As a consequence, we have the following:

Corollary

- *The function $a \mapsto \lambda_1(a)$ has a global interior maximum corresponding to an eigenfunction of multiplicity two.*
- *If a_0 is a local extremum of a simple eigenvalue $\lambda_k(a)$, then, there is a multiple junction of the nodal lines of the corresponding eigenfunction at a_0.*

Sketch of the proof

Key observation: The three problems are all related to a real elliptic equation on a twofold covering manifold.

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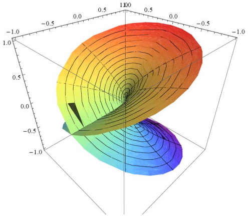

The twofold covering manifold Σ is the Riemann surface of the complex square root, centered at a

$$\Sigma = \{(x, y) \in \mathbb{C}^2,\ x \in D \setminus \{a\} : y^2 = x - a\}$$

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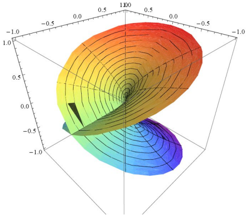

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In particular, the magnetic Schrödinger equation is related to an elliptic equation on Σ thanks to gauge invariance.
Helffer, Hoffmann-Ostenhof M. and T., Owen, (1999).

Gauge invariance

$$\begin{cases} (i\nabla + A)^2 U + VU = 0 & \text{in } D \\ U = \Gamma & \text{on } \partial D. \end{cases}$$

$$\updownarrow \qquad u := e^{-i\frac{\vartheta}{2}} U$$

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Gauge invariance

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- there is a one–to–one correspondence between the solutions U of the magnetic Schrödinger equation and the antisymmetric, real solutions u of the related elliptic equation on Σ;
- the nodal lines of U are obtained from those of u by applying a complex square root;

Proof of uniqueness

Notice that a is a triple point of U if and only if

$$u(a) = \nabla u(a) = 0$$

We apply the implicit function theorem to the following Cauchy–type formula

$$\nabla u(a) = -\frac{i}{2\pi}\int_{\partial D} \frac{e^{-i\vartheta/2}}{(x-a)^{1/2}}\left(\frac{\partial U}{\partial x} - iAU\right)\, dx + \\ + \frac{1}{2\pi}\int_{D} \frac{e^{-i\vartheta/2}}{(x-a)^{1/2}}(i\nabla + A)^2 U \; dx_1 dx_2.$$

Hartman, Wintner, *Amer. J. Math.* (1953).

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- At a triple point it does not vanish: $\Longrightarrow$ local uniqueness.
- We exploit the uniqueness of the minimal partition with respect to the lagrangian energy and a continuation argument to obtain global uniqueness.

Outline

Optimal partition problems related to eigenvalues

Let $\Omega \subset \mathbb{R}^N$ be a connected, open bounded domain with regular boundary $\partial\Omega$. For any measurable $\omega \subset \Omega$, define its first eigenvalue of the Schrödinger operator in $H^1_0(\omega)$, namely

$$\lambda_1(\omega) = \min_{\substack{u \in H^1_0(\Omega)\setminus\{0\} \\ u\equiv 0 \text{ a.e. on } \Omega\setminus\omega}} \frac{\int_\omega \left(|\nabla u(x)|^2 - V(x)|u(x)|^2\right) dx}{\int_\omega |u(x)|^2 dx}.$$

We assume the operator to be positive. For a fixed $p > 0$, let us consider the following class of optimal partition problems

$$\mathfrak{L}^p_{k,p}(\Omega) = \inf_{\mathfrak{D}_k} \frac{1}{k} \sum_{i=1}^{k} \left(\lambda_1(\omega_i)\right)^p, \qquad \mathfrak{L}_k(\Omega) = \inf_{\mathfrak{D}_k} \max_{i=1,\dots,k} \lambda_1(\omega_i)$$

Whe take the minimization over the class of partitions in k "disjoint" measurable subsets of Ω:

$$\mathcal{D}_k := \left\{ \mathcal{D} = (\omega_1, \ldots, \omega_k) : \bigcup_{i=1}^{k} \omega_i \subset \emptyset, \ |\omega_i \cap \omega_j| = 0 \text{ if } i \neq j \right\} .$$

Minimal partitions for eigenvalues:
Bucur-Buttazzo-Henrot
Buttazzo-Timofte
Conti-T-Verzini
Caffarelli-Lin
Helffer-Hoffmann-Ostenhof-T
Bourdin-Bucur-Oudet

Spectrum and spectral optimal partitions

In general, an easy consequence of the Courant nodal theorem for connected domains is that

$$\lambda_k \leq \mathfrak{L}_{k,p} \leq \mathfrak{L}_k \ .$$

For one dimensional problems we have equality (Sturm oscillation principle) for every k. On the other hand, in more space dimensions, the k-th eigenfunction may possess less than k nodal domains. Given k, we denote by L_k the smallest eigenvalue whose eigenspace contains an eigenfunction with k nodal domains ($L_k = +\infty$ if no such an eigenfunction exists).
Then we obviously have

$$\lambda_k \leq \mathfrak{L}_k \leq L_k \ , \qquad \forall k.$$

For planar domains, we can give the full picture of the equality case.

Theorem (Helffer-Hoffman-Ostenhof-T,2009–2010)

Suppose $\Omega \subset \mathbb{R}^N$ regular. If $\mathfrak{L}_k = L_k$ then

$$\lambda_k = \mathfrak{L}_k = L_k \ .$$

In addition, one can find in the eigenspace associated to λ_k an eigenfunction u_k having extactly k nodal domains.

- The k–th eigenfunction has k nodal domains (i.e. is sharp with respect to the Courant nodal Theorem) if and only if the associated nodal k–partition is optimal.

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- The k–th eigenfunction has k nodal domains (i.e. is sharp with respect to the Courant nodal Theorem) if and only if the associated nodal k–partition is optimal.

As a consequence, every time we know (for instance for the symmetries of the problem) that the second eigenvalue is degenerate, then the minimal spectral 3-partition has necessarily a nontrivial clustering point.

Outline

Bishop Conjecture

We consider the Laplace-Beltrami operator on the two-sphere.

Conjecture (Bishop 1992)

The minimal 3-partition for $\frac{1}{3}(\sum_{i=1}^{3}\lambda_1(D_i))$ *corresponds to the* **Y**-*partition, whose boundary is given by the intersection of* $\mathbb{S}^2$ *with the three half-planes defined respectively by* $\phi = 0, \frac{2\pi}{3}, \frac{-2\pi}{3}$

The conjecture can be restated as

$$\mathfrak{L}_{3,1}(\mathbb{S}^2) = \frac{15}{4}$$

and implies

$$\mathfrak{L}_{3,p}(\mathbb{S}^2) = \mathfrak{L}_3(\mathbb{S}^2) = \frac{15}{4}\ , \forall p$$

Uniqueness for $\mathfrak{L}_3$ in two dimensions

Bishop's Conjecture was motivated by the analysis of the properties of harmonic functions in conic sets. A reference paper in this context is that by Friedland-Hayman. It is proved there that the optimal two-partition is achieved by the two half spheres. We are not able to prove the full Bishop conjecture, but we have the following result.

Theorem (Helffer-Hoffman Ostenhof-T)

Any minimal spectral 3-partition of $\mathbb{S}^2$ is (up to a rotation) obtained by the **Y***-partition. Hence*

$$\mathfrak{L}_3(\mathbb{S}^2) = \frac{15}{4}\,.$$

Some remarks

Consider a homogeneous function in $\mathbb{R}^3$ of the form

$$u(x) = r^\alpha g(\theta, \phi)$$

which is harmonic outside its nodal set:

$$-\Delta u = 0 \,, \qquad u > 0$$

and such that the complementary of the nodal set divides the sphere in three parts, then

$$\alpha(\alpha+1) \geq \mathfrak{L}_3(\mathbb{S}^2)\,.$$

Hence our theorem implies that $\alpha \geq 3/2$.

Ideas of the proof:

- first, minimal partitions on $\mathbb{S}^2$ in three parts exist and share the same properties as for planar domains: regularity and equal angle meeting property. Hence the nodal set is a finite union of arcs.

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- use Euler's formula and deduce that the nodal line s of a minimal 3-partitions consists exactly two points x_1 and x_2 and three arcs joining these two points.
- use Borsuk (or Ljusternik-Schnirelman) theorem to prove that the nodal set contains a pair of antipodal points.

- the next point is that any minimal 3-partition which contains two antipodal points in its boundary can be lifted to a symmetric 6-partition on the double covering $\mathbb{S}^2_{\mathcal{C}}$. Or, equivalently, to add a singular magnetic field with half integer flux.

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- finally, the last point is to show that on the double covering a minimal symmetric 6-partition is necessarily the litfing on the double covering of the **Y**-partition.
  - use the knowledge of the spectrum of the Laplace-Beltrami on the double covering and classify all odd and even spectrum.
  - use again the characterization of the eigenvalues whose nodal partition is minimal: this holds if and only the minimal eiegenvalue whose nodal partition has k nodal domain is the minimal one,