Functional-anatytic theory of concentration compactness and the Trudinger-Moser inequality

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Outline:

- The next best thing when compactness is missing: cocompactness (a property of imbedding of Banach spaces weaker than compactness). Sobolev imbeddings on $\mathbb{R}^N$ are not compact, but they are cocompact relative to suitable transformations.

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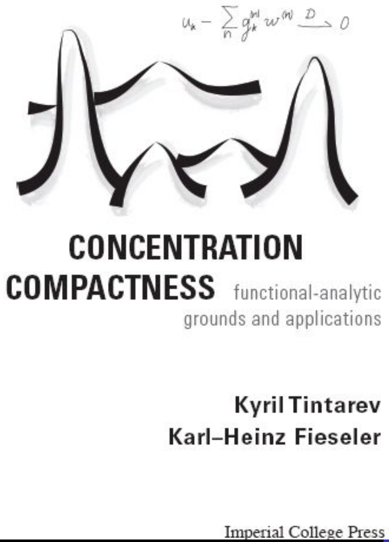
- Cocompactness of Strichartz imbedding for NLS (Terence Tao).

Collaborations: Adimurthi, Michael Cwikel, Djairo de Figueiredo, Giovanni Mancini, João Marcos do O, Kunnath Sandeep, Ian Schindler. Monograph with participation of Karl-Heinz Fieseler.



$u_k - \sum_n g_k^{(n)} w^{(n)} \xrightarrow{D} 0$

CONCENTRATION
COMPACTNESS functional-analytic
grounds and applications

Kyril Tintarev
Karl–Heinz Fieseler

Imperial College Press

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  $$\|u\|_{L^{2^*}(\mathbb{R}^N)} \leq C\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}, \quad 2^* = \frac{2N}{N-2},$$

  is not compact: for any $u \in \mathcal{D}^{1,2}$, one has $t_k^{\frac{N-2}{2}} u(t_k x) \rightharpoonup 0$ whenever $t_k \to 0$ or $t_k \to \infty$.

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- Lieb 1982, rephrased: If a sequence $u_k \in W^{1,p}(\mathbb{R}^N)$, $N > p$, satisfies, with any $y_k \in \mathbb{Z}^N$, $u_k(\cdot - y_k) \rightharpoonup 0$, then it converges in $L^q(\mathbb{R}^N)$, $p < q < p^*$.

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- Lions 1985, modified: If a sequence $u_k \in \mathcal{D}^{1,p}(\mathbb{R}^N)$, $N > p$, satisfies, with any $j_k \in \mathbb{Z}$, $y_k \in \mathbb{R}^N$, $2^{\frac{N-p}{p}j_k} u_k(2^{j_k} \cdot - y_k) \rightharpoonup 0$, then it converges in $L^{p^*}(\mathbb{R}^N)$.

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$$u_k \overset{X,D}{\rightharpoonup} 0 \Rightarrow u_k \overset{Y}{\to} 0.$$
- Definition 3: The group D of operators is called a dislocation group if $g_k \in D, g_k \not\rightharpoonup 0 \Rightarrow \exists k_j, g \in D, g_{j_k} \overset{s}{\to} g$.

Application: selfsimilar critical nonlinearity

Example

Let $N \geq 3$ and let $f \in C([0,\infty))$, $f \geq 0$, $\gamma > 1$, satisfy

$$f(\gamma^j s) = \gamma^{\frac{N+2}{N-2}j} f(s), s > 0, j \in \mathbb{Z}.$$

Consider a functional $J : \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathbb{R}$:

$$J(u) \overset{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} F(u), \text{ where } F(t) = \int_0^t f(s)ds.$$

If $J'(u_k) \to 0$, $J(u_k) \to c \neq 0$ and u_k is bounded, then $u_k$converges weakly to a solution of

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Proof: If $\gamma^{\frac{N-2}{2}j_k} u_k(\gamma^{j_k}x - y_k) \rightharpoonup 0$ for all j_k, y_k, then $u_k \to 0$ in L^{2^*} and, by $J'(u) \to 0$, in $\mathcal{D}^{1,2}$.

Profile decomposition

Theorem

Let u_k be a bounded sequence in the Hilbert space H with a dislocation group D. There exist $w^{(n)} \in H$, $g_k^{(n)} \in D$, $k, n \in \mathbb{N}$, such that for a renumbered subsequence

$$g_k^{(1)} = id,\ {g_k^{(n)}}^* g_k^{(m)} \rightharpoonup 0 \text{ for } n \neq m,$$

$$w^{(n)} = \text{ w-lim } {g_k^{(n)}}^{-1} u_k,$$

$$\sum_{n \in \mathbb{N}} \|w^{(n)}\|^2 \leq \limsup \|u_k\|^2,$$

$$\text{and } u_k - \sum_{n \in \mathbb{N}} g_k^{(n)} w^{(n)} \overset{D}{\rightharpoonup} 0. \left[\overset{Y}{\rightarrow} 0 \text{ if } H \overset{D}{\subset} Y \right]$$

Cocompactness as concentration compactness

Concentration compactness (Uhlenbeck, Lions, Brezis, Coron, Lieb 1981-1984) evaluates, typically, the weak limits of measure sequences $d\mu_k = |u_k|^p dx$ and $d\nu_k = |\nabla u_k|^p dx$. If μ_∞has no singular part, L^p-convergence follows. Adaptation of Lions' CC to critical problems on unbounded domains by Chabrowski, 1994. Profile decomposition (started by Struwe's "global compactness" in 1984) gives a more detailed description of concentration, which makes it easier to prove that it does not occur.

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Main potential of the profile decomposition is that stimuates working in further spaces with further groups: e.g. Heisenberg group (K.T. 2001), Besov spaces (M.Cwikel and K.T. in progress), Sobolev spaces on fractals (Schindler & K.T. 2009) and a very hard cocompactness proof for Strichartz imbeddings by Terence Tao (2009).

(Pohozhaev)-Trudinger-Moser inequality

- $$\sup_{\|\nabla u\|_2=1} \int_B e^{4\pi u^2} \,\mathrm{d}x < \infty, u \in H_0^1(B)$$

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- The contant 4π is optimal, so the nonlinearity appears to be critical. However, the Trudinger-Moser functional is weakly continuous on $\{\|\nabla u\|_2 \leq 1, u \neq 0\}$ and also weakly continuous on any sequence $u_k \rightharpoonup 0$ unless $\|u_k - \mu_{t_k}\|_{H^1} \to 0$, where $\mu_t(r) \overset{\text{def}}{=} (2\pi)^{-\frac{1}{2}}(\log\frac{1}{t})^{\frac{1}{2}} \min\left\{\frac{\log\frac{1}{r}}{\log\frac{1}{t}}, 1\right\}, \quad r, t \in (0,1).$

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- Conjecture: Trudinger-Moser nonlinearity is only partially critical (cf. Hardy potential term) and there is a stronger version of the inequality (cf. Adimurthi & Druet: $4\pi \mapsto 4\pi(1 + \lambda\|u\|^2)$, $\lambda < \lambda_1(B)$.)

Conformally invariant Trudinger-Moser inequality

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- and the functional is invariant with respect to Möbius transformations (Mancini and Sandeep; Adimuthi and K. T., preprints).

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- Version for N-Laplacian on $\mathbb{H}^N$: Mancini - Sandeep - Tintarev (in preparation). A radial compactness result: de Figueiredo - do O - Tintarev (in preparation).

Dilations on the unit disk

Let $H^1_{0,r}(B)$ denote the subspace of radially symmetric functions.

$$h_s u(r) \overset{\mathrm{def}}{=} s^{-\frac{1}{2}} u(r^s),\ s > 0.$$

Operators h_s preserve the gradient norm and the Hardy-type potential term in the inequality

$$\|\nabla u\|_2^2 \geq \frac{1}{4} \int_B \frac{u^2}{(r \log \frac{1}{r})^2} \, \mathrm{d}x \quad \text{[Leray,1933]}$$

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Creating an invariant functional by limit: $\lim_{s\to 0} J(h_s u) - J(0) = 0$ and $\lim_{s\to\infty} J(h_s u) - J(0) = 2\pi \mathbf{1}_M(u)$ where $M = \{\mu_t\}_{t\in(0,1)}$.

Moser functions and convergence

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$$\int_B (e^{\lambda u_k^2} - 1)\,\mathrm{d}x \to 0.$$

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- The functional

$$\sup_{r \in (0,1)} \frac{|u(r)|}{(\log \frac{1}{r})^{\frac{1}{2}}}$$

converges to zero whenever $h_{s_k} u_k \rightharpoonup 0$ and is dilation-invariant.

Inequality of Hardy-Sobolev type

Hölder interpolation

$$\left(\int_B \frac{|u|^p}{(r\log\frac{1}{r})^2(\log\frac{1}{r})^{\frac{p-2}{2}}}\right)^{\frac{1}{p}} \le C\|\nabla u\|_2, \quad 2\le p\le 2^*=\infty,$$

between $\sup_r \frac{|u(r)|}{(\log\frac{1}{r})^{\frac{1}{2}}} \le C\|\nabla u\|_2$ and $\left(\int_B \frac{|u|^2}{(r\log\frac{1}{r})^2}\right)^{\frac{1}{2}} \le C\|\nabla u\|_2$.
For all $p\ge 2$ it remains dilation-invariant and therefore is the natural analog of the functional in the Hardy-Sobolev inequality in the case of higher dimension. The true critical nonlinearity in $H^1_{0,r}(B)$ is $\|u\|_\infty = \sup_r \frac{|u(r)|}{(\log\frac{1}{r})^{\frac{1}{2}}}$

Cocompactness and interpolation (K. T. and Michael Cwikel)

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Cocompactness of a Strichartz estimate (Terence Tao, 2009)

Imbedding

$$\|e^{it\Delta}u\|_{L^q(\mathbb{R}^{N+1})} \leq C\|u\|_{L^2(\mathbb{R}^N)},\ q = \frac{2N+2}{N},$$

is cocompact with respect to the product of operator groups:

$$\text{Dilations: } u(x) \mapsto 2^{\frac{Nj}{2}}u(2^j x),\ j \in \mathbb{Z};$$

$$\text{Translations: } u(x) \mapsto u(x-y),\ y \in \mathbb{R}^N;$$

$$\text{Time shifts: } \hat{u}(\xi) \mapsto i\tau|\xi|^2\hat{u}(\xi),\ \tau \in \mathbb{R};$$

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Restriction of functions to a cell in Fourier domain turns weak convergence into uniform convergence. The rest of the proof is a “reassembly”.