OPTIMAL TRANSPORT, ISOPERIMETRY, SMOOTHNESS AND CUT LOCUS

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MAIN THEME

Interplay between hard/soft, and smooth/nonsmooth, analysis/geometry

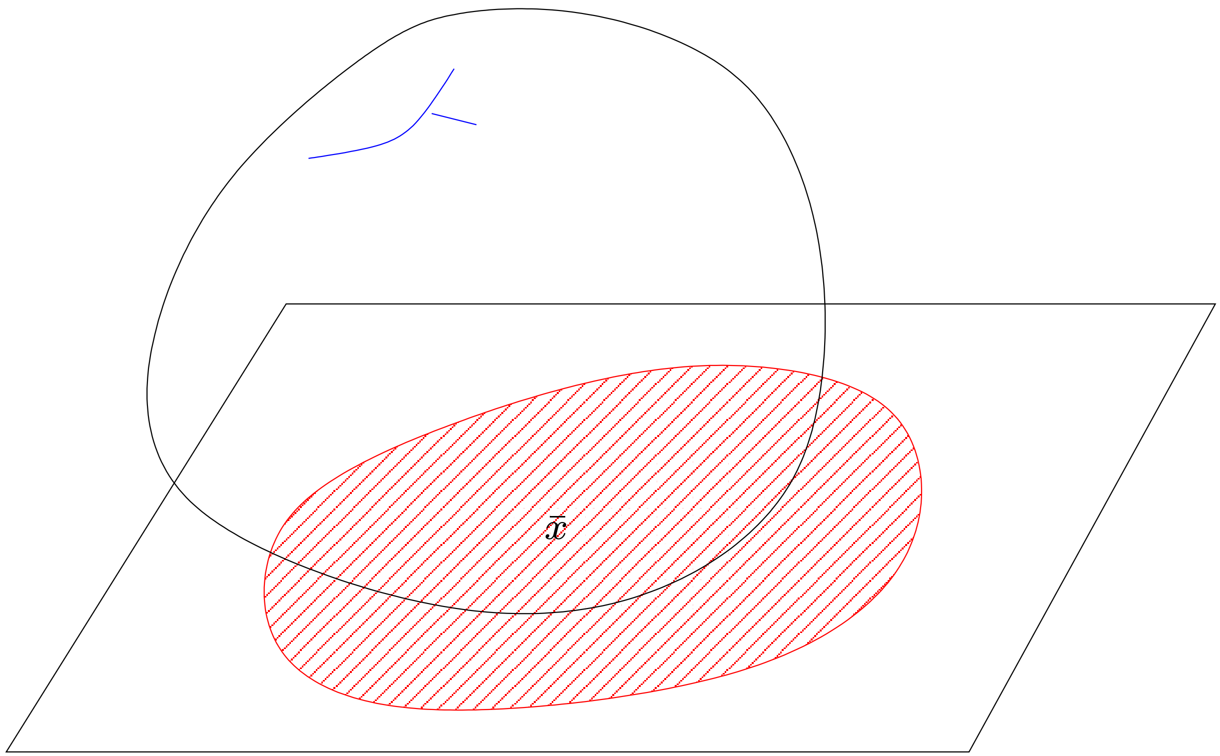
in relation with optimal transport

General References

- *Topics in Optimal Transportation* [**TOT**] (AMS, 2003): Introduction
- *Optimal transport, old and new* [**oldnew**] (Springer, 2008): Reference text, more probabilistic & geometric

Thm (convex Earth): A C^4 perturbation of $\mathbb{S}^n$ looks convex from any of its points

(Figalli–Rifford $n = 2$, Figalli–Rifford–V $n \geq 3$)



Has to do with PDEs and calculus of variations!??

PRELIMINARY: Push-forward, or change of variables

$\mu(dx)$, $\nu(dy)$ two (probability) measures

$y = T(x)$ measurable

Def: $T_{\#}\mu = \nu$ if $\forall B, \quad \mu[T^{-1}(B)] = \nu[B]$

Equivalently: $\forall \varphi, \quad \int \varphi \circ T \, d\mu = \int \varphi \, d(T_{\#}\mu)$

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Probabilistic formulation

$\text{law}\,(U) = \mu, \quad \text{law}\,(V) = \nu, \quad V = T(U)$

Analytic formulation

In $\mathbb{R}^n$, $T_\#(f(x)\,dx) = g(y)\,dy$, if T is 1-to-1, yields

$f(x) = g(T(x))\,|\det(dT)(x)|$

MONGE–KANTOROVICH PROBLEM

MÉMOIRE
SUR LA
THÉORIE DES DÉBLAIS ET DES REMBLAIS.

Par M. MONGE.

LORSQU'ON doit transporter des terres d'un lieu dans un autre, on a coutume de donner le nom de *Déblai* au volume des terres que l'on doit transporter, & le nom de *Remblai* à l'espace qu'elles doivent occuper après le transport.

Le prix du transport d'une molécule étant, toutes choses d'ailleurs égales, proportionnel à son poids & à l'espace qu'on lui fait parcourir, & par conséquent le prix du transport total devant être proportionnel à la somme des produits des molécules multipliées chacune par l'espace parcouru, il s'ensuit que le déblai & le remblai étant donnés de figure & de position, il n'est pas indifférent que telle molécule du déblai soit transportée dans tel ou tel autre endroit du remblai, mais qu'il y a une certaine distribution à faire des molécules du premier dans le second, d'après laquelle la somme de ces produits sera la moindre possible, & le prix du transport total sera un *minimum*.

C'est la solution de cette question que je me propose de donner ici. Je diviserai ce Mémoire en deux parties, dans la première je supposerai que les déblais & les remblais sont des aires contenues dans un même plan; dans le second, je supposerai que ce sont des volumes.

PREMIÈRE PARTIE.

Du transport des aires planes sur des aires comprises dans un même plan.

I.

QUELLE que soit la route que doive suivre une molécule

The Kantorovich problem (Kantorovich, 1942)

- $\mathcal{X}$, $\mathcal{Y}$ two Polish (= metric separable complete) spaces
- $\mu \in P(\mathcal{X})$, $\nu \in P(\mathcal{Y})$
- $c \in C(\mathcal{X} \times \mathcal{Y}; \mathbb{R})$, $\qquad c \geq \underline{c} \in L^1(\mu) + L^1(\nu)$

$$\Pi(\mu, \nu) = \Big\{ \pi \in P(\mathcal{X} \times \mathcal{Y});\ \text{marginals of } \pi \text{ are } \mu \text{ and } \nu \Big\}$$

$(\forall h,$

$$\int h(x)\,\pi(dx\,dy) = \int h\,d\mu; \qquad \int h(y)\,\pi(dx\,dy) = \int h\,d\nu)$$

$$\textbf{(K)} \qquad \boxed{\inf_{\pi \in \Pi(\mu,\nu)} \int c(x,y)\,\pi(dx\,dy)}$$

Prop: Infimum achieved by compactness of $\Pi(\mu, \nu)$

In the sequel, assume infimum is finite

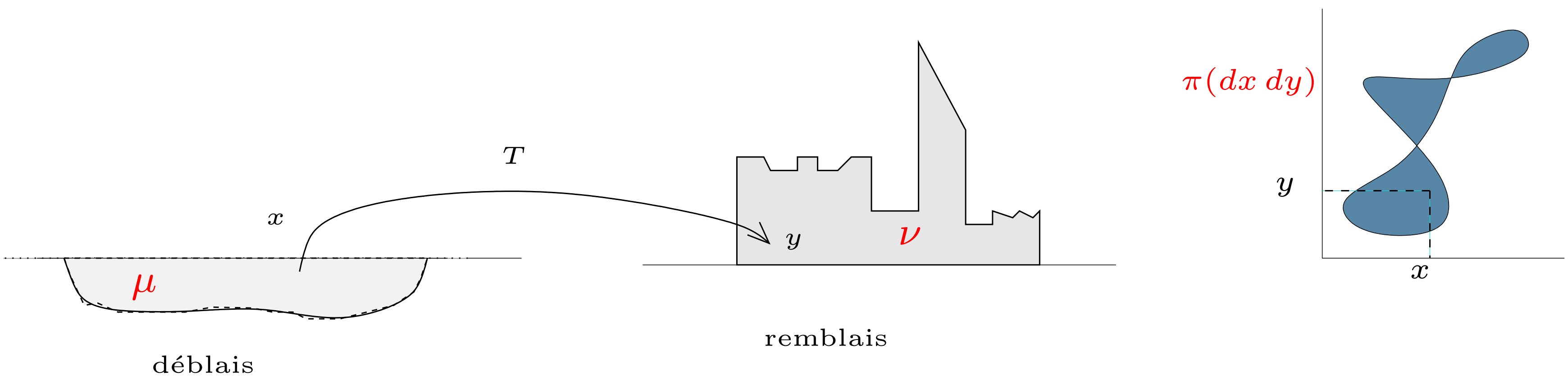
Probabilistic version

X and Y two given random variables (= with given laws)

$$\textbf{(K')} \qquad \boxed{\inf \; \mathbb{E}\, c(X,Y)}$$

(Infimum over all couplings of (X,Y))

Engineer's interpretation



Given the initial and final distributions, transport matter at lowest possible cost

The Monge problem (Monge, 1781)

Assume $\pi = (\mathrm{Id}, T)_{\#}\mu = \mu(dx)\,\delta_{y=T(x)}$

$\longrightarrow$ belongs to $\Pi(\mu,\nu)$ iff $T_{\#}\mu = \nu$

$\Longrightarrow$ the Kantorovich problem becomes

$$\textbf{(M)} \quad \boxed{\inf_{T_{\#}\mu=\nu} \int c(x, T(x))\,\mu(dx)} = \inf \mathbb{E}\, c(X, T(X))$$

- Interpretation: Don't split mass! $\quad Y = T(X)$
- No compactness $\Longrightarrow$ not clear if infimum achieved

History of the Monge problem

- Original Monge cost function: $c(x,y) = |x-y|$ in $\mathbb{R}^3$

- For this cost, existence of a minimizer proven around 1998–2003 !! (Ambrosio, Caffarelli, Evans, Feldman, Gangbo, McCann, Sudakov, Trudinger, Wang)

- Easier solution when the cost is “strictly convex”

Kantorovich duality

(Kantorovich 1942; still active research area)

(Kdual)

$$\inf_{\pi\in\Pi(\mu,\nu)} \int c(x,y)\,\pi(dx\,dy) \quad = \quad \sup_{(\psi,\phi)\in\Psi_c} \left\{ \int \phi\,d\nu - \int \psi\,d\mu \right\}$$

- $\pi \in \Pi(\mu,\nu)$ if π has marginals μ and ν
- $(\psi,\phi) \in \Psi_c$ if $\phi(y) - \psi(x) \le c(x,y) \quad (\forall x, y)$

- Economical interpretation: shipper's problem (buys at price $\psi(x)$ at x, sells at price $\phi(y)$ at y)
- Supremum achieved e.g. if $c \le \bar{c} \in L^1(\mu) + L^1(\nu)$

c-convexity (I)

(Rüschendorf, nineties)

- Fix $c : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$
- $\psi : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$, $\psi^c(y) := \inf_{x\in\mathcal{X}} \big[\psi(x) + c(x,y)\big]$
- $\phi : \mathcal{Y} \to \mathbb{R} \cup \{-\infty\}$, $\phi^c(x) := \sup_{y\in\mathcal{Y}} \big[\phi(y) - c(x,y)\big]$
- ψ is said c-convex if $(\psi^c)^c = \psi$
  ϕ is said c-concave if $(\phi^c)^c = \phi$
- **Ex:** $c(x,y) = -x \cdot y$ in $\mathbb{R}^n \times \mathbb{R}^n$:
  $\psi^c = -\psi^*$ (Legendre transform);
  c-convex $\Longleftrightarrow$ l.s.c. convex

Rks: (a) many conventions!!

(b) differential criterion for c-convexity?? Yes if the Ma–Trudinger–Wang condition is satisfied, see later.

c-convexity (II)

- If ψ is c-convex, define its c-subdifferential $\partial_c\psi$ by

$$\partial_c\psi(x) = \Big\{ y \in \mathcal{Y};\ \forall z \in \mathcal{X}, \quad \psi(z)+c(z,y) \geq \psi(x)+c(x,y) \Big\}$$

- $\Gamma \subset \mathcal{X} \times \mathcal{Y}$ is c-cyclically monotone (c-CM) if

$$\forall N \in \mathbb{N} \quad \forall (x_1,y_1),\ldots,(x_N,y_N) \in \Gamma^N,$$

$$\sum_{i=1}^{N} c(x_i,y_i) \leq \sum_{i=1}^{N} c(x_i,y_{i+1}) \qquad (y_{N+1} = y_1)$$

- $\partial_c\psi$ is c-CM (immediate)
- **Ex:** $c(x,y) = -x \cdot y \Longrightarrow \quad \partial_c\psi = \partial\psi$

c-convex analysis (Rockafellar; Rüschendorf)

- $\psi^c(y) - \psi(x) \leq c(x,y) \qquad \forall \psi$
- ψ is c-convex ($(\psi^c)^c = \psi$) iff $\quad \exists \zeta; \quad \psi = \zeta^c$
- $\partial_c \psi = \big\{(x,y);\ \psi^c(y) - \psi(x) = c(x,y)\big\}$ is c-CM
- If Γ is c-CM then $\exists \psi$ c-convex s.t. $\Gamma \subset \partial_c \psi$

Pf: Fix $(x_0, y_0) \in \Gamma$, define $\psi(x) :=$

$$\sup_{m \in \mathbb{N}} \sup \Big\{ \big[c(x_0,y_0) - c(x_1,y_0)\big] + \big[c(x_1,y_1) - c(x_2,y_1)\big] + \cdots + \big[c(x_m,y_m) - c(x,y_m)\big]; \quad (x_i,y_i) \in \Gamma \Big\}$$

- c-convex functions inherit some regularity from c, e.g.

$$\|\psi\|_{\mathrm{Lip}} \leq \sup_y \|c(\,\cdot\,,y)\|_{\mathrm{Lip}}, \qquad D^2\psi \geq \inf_{x,y} (-D_x^2 c)$$

Saddle point structure $\qquad \pi \in \Pi(\mu,\nu),\ \psi \quad c\text{-convex}$

$$\boxed{\operatorname{Spt}\pi \subset \partial_c\psi} \Longrightarrow \left\{\begin{matrix} \pi \text{ optimal} \\ \psi \text{ optimal} \end{matrix}\right\} \Longrightarrow \textbf{(Kdual)}$$

Pf: $\forall \widetilde{\pi}\ \forall \widetilde{\psi}$

$$\begin{aligned}
\int c(x,y)\,\widetilde{\pi}(dx\,dy) &\geq \int \left[\psi^c(y) - \psi(x)\right]\widetilde{\pi}(dx\,dy) \\
&= \int \psi^c\,d\nu - \int \psi\,d\mu \\
&= \int \left[\psi^c(y) - \psi(x)\right]\pi(dx\,dy) \\
&= \int c(x,y)\,\pi(dx\,dy) \\
&\geq \int \left[\widetilde{\psi}^c(y) - \widetilde{\psi}(x)\right]\pi(dx\,dy) \\
&= \int \widetilde{\psi}^c\,d\nu - \int \widetilde{\psi}\,d\mu \qquad \square
\end{aligned}$$

Complements (I)

- Criteria for optimality:

- If π is optimal then (ψ, ψ^c) is optimal iff $\mathrm{Spt}\,\pi \subset \partial_c \psi$

- If (ψ, ψ^c) is optimal then π is optimal iff $\mathrm{Spt}\,\pi \subset \partial_c \psi$

- π is optimal iff $\mathrm{Spt}\,\pi$ is c-CM (Pratelli, Schachermayer–Teichmann 2007-2008)

- This implies stability: If $\pi_k \in \Pi(\mu_k, \nu_k)$ optimal, $\pi_k \longrightarrow \pi \in \Pi(\mu, \nu)$ (weakly), then π is optimal

Complements (II)

- Link with Aubry–Mather theory:

$$c := \inf \Big\{ C(\mu,\mu);\ \mu \in P(\mathcal{X}) \Big\} \quad \Longrightarrow \exists \mu \text{ minimizer}$$

$$\mathcal{A} := \bigcap_{\psi \text{ opt.}} \partial_c \psi \qquad\qquad \mathcal{M} := \overline{\bigcup_{\pi \text{ opt.}} \operatorname{Spt} \pi}$$

These sets play an important role in dynamical systems theory [Fathi] [oldnew Chap. 8]

- Link with combinatorics: When $c : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$, **(Kdual)** reduces to a "continuous" (measure-theoretic) version of the marriage lemma [TOT Sect. 1.4]

Solution of the Monge problem under a twist condition

(Brenier, Rüschendorf, McCann, Gangbo, ...)

Let μ, ν, c s.t. the dual Kantorovich problem has a solution ψ

Assume

(0) $\mathcal{X}$ is a Riemannian manifold

(1) $c(x,y)$ is (uniformly) Lipschitz in x, uniformly in y

(2) $[\nabla_x c(x,y) = \nabla_x c(x,y')] \Longrightarrow y = y'$
(Twist: $\nabla_x c$ is a 1-to-1 function of y)

(3) c is superdifferentiable everywhere

(4) $\mu(dx) = f(x)\,\mathrm{vol}(dx)$

Then $\exists!$ solution to the Monge–Kantorovich problem

Structure of the solution

$\pi(dx\,dy)$-a.s.

$$\boxed{\nabla\psi(x) + \nabla_x c(x,y) = 0}$$

This determines the transport map:

$y = T(x) = (\nabla_x c)^{-1}\big(x, -\nabla\psi(x)\big)$

Ex: (Brenier, Rüschendorf, Cuesta-Albertos, Matràn... around 1990) $c(x,y) = |x-y|^2/2$ in $\mathbb{R}^n$, then

$T(x) = \nabla\Phi(x)$, where Φ is convex

Ex: (McCann 1999) $c(x,y) = d(x,y)^2/2$ on a compact Riemannian manifold, then

$T(x) = \exp_x(\nabla\psi(x))$, where ψ is $d^2/2$-convex

Proof of solution of Monge problem

- ψ c-convex $\overset{(1)}{\Longrightarrow}$ ψ Lipschitz
  $\overset{(0)}{\Longrightarrow}$ ψ differentiable a.e.
  $\overset{(4)}{\Longrightarrow}$ ψ differentiable μ-a.s.

- $\operatorname{Spt}\pi \subset \partial_c \psi \implies \pi(dx\,dy) -$ a.s.,
  $\psi(z) + c(z,y)$ is minimum at $z = x$

$\implies -\nabla\psi(x) \in \nabla_x^- c(\,\cdot\,,y)$
$\overset{(3)}{\Longrightarrow} \nabla_x c(\,\cdot\,,y) = -\nabla\psi(x)$
$\overset{(2)}{\Longrightarrow} y = (\nabla_x c)^{-1}\big(x, -\nabla\psi(x)\big)$

Now let us see some applications — in $\mathbb{R}^d$

Recall: Solution of the Euclidean optimal transport

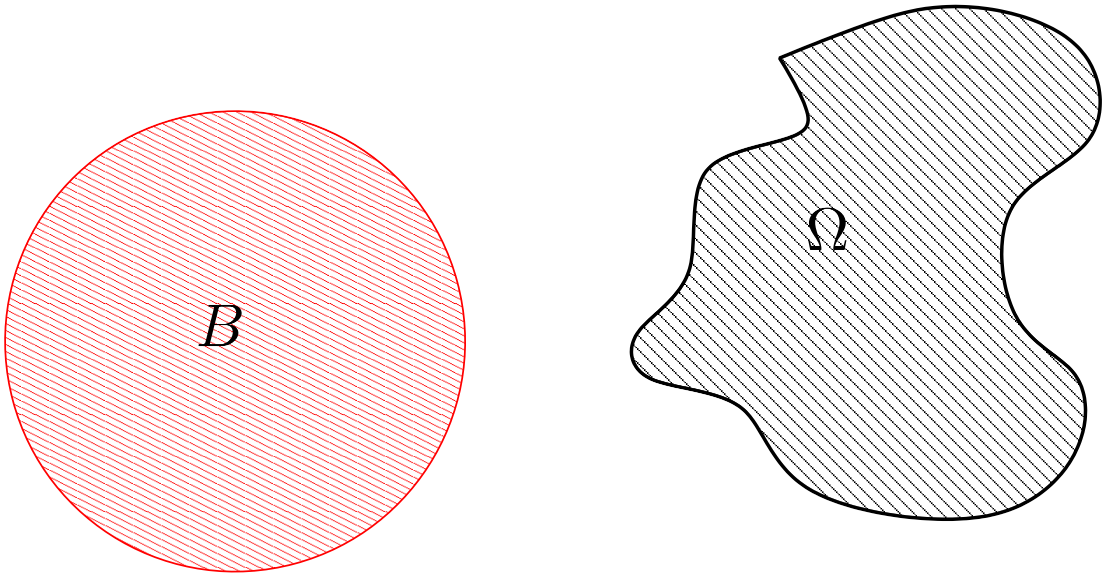
$$T = \nabla\Phi, \qquad \Phi \text{ convex.}$$

In particular T is monotone.

Monotone changes of variables are powerful!

Euclidean isoperimetric problem

Given the volume, the sphere has minimal surface



$$\begin{cases} |\Omega| = \mathcal{L}^n[\Omega] \\ \\ |\partial\Omega| = \mathcal{H}^{n-1}[\partial\Omega] \end{cases}$$

$$x \in \Omega \quad \longrightarrow \quad y = T(x) \in B = B(0,1)$$

Assume

$$\begin{cases} T \text{ pushes uniform measure forward to uniform measure} \\ dT \textbf{ has nonnegative eigenvalues} \end{cases}$$

Then $f(x) = 1/|\Omega|$, $g(y) = 1/|B|$, so $\det(dT) = |B|/|\Omega|$

$$\left(\frac{|B|}{|\Omega|}\right)^{\frac{1}{n}} = (\det dT)^{\frac{1}{n}} = \left(\prod_{i=1}^{n} \lambda_i\right)^{\frac{1}{n}} \leq \frac{\sum_{i=1}^{n} \lambda_i}{n} = \frac{\text{div } T}{n}$$

$$|\Omega| \times \left(\frac{|B|}{|\Omega|}\right)^{\frac{1}{n}} \leq \int_\Omega \frac{\text{div } T}{n} = \frac{1}{n}\int_{\partial\Omega} T\cdot\nu \leq \frac{1}{n}\int_{\partial\Omega} \cancel{\|T\|} = \frac{|\partial\Omega|}{n}$$

(Gromov)

Now let us prove Sobolev (Cordero–Nazaret–V 2004)

$\mathbb{R}^n, \quad 1 < p < n,$

$$\|u\|_{L^{p^\star}(\mathbb{R}^n)} \leq S_n(p)\,\|\nabla u\|_{L^p(\mathbb{R}^n)} \qquad p^\star = \frac{np}{n-p}$$

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W.l.o.g. $u \geq 0$ and $\int u^{p^\star} = 1$: becomes

$$0 < K \leq \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

$$\int g = 1, \qquad T : u^{p^\star}\,dx \longrightarrow g(y)\,dy, \qquad g(T(x)) = \frac{u(x)^{p^\star}}{\det(dT(x))}$$

$$\int g^{1-\frac{1}{n}} = \int g(y)^{-\frac{1}{n}}\, g(y)\,dy = \int g(T(x))^{-\frac{1}{n}}\, u^{p^\star}(x)\,dx$$

$$= \int (\det dT(x))^{\frac{1}{n}}\, (u^{p^\star})^{1-\frac{1}{n}}\,(x)\,dx$$

$$\leq \frac{1}{n}\int (\operatorname{div} T(x))(u^{p^\star\left(1-\frac{1}{n}\right)})(x)\,dx$$

$$= -\frac{p^\star}{n}\left(1-\frac{1}{n}\right)\int u^{p^\star\left(1-\frac{1}{n}\right)-1}\,\nabla u\cdot T\,dx$$

$$\int g = 1, \qquad T : u^{p^\star} dx \longrightarrow g(y)\, dy, \qquad g(T(x)) = \frac{u(x)^{p^\star}}{\det(dT(x))}$$

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$$\leq \frac{1}{n} \int (\operatorname{div} T(x))(u^{p^\star\left(1-\frac{1}{n}\right)})(x)\, dx$$

$$= -\frac{p^\star}{n}\left(1 - \frac{1}{n}\right) \int u^{p^\star/p'} \quad \nabla u \cdot T\, dx \qquad\qquad \frac{1}{p} + \frac{1}{p'} = 1$$

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$$= -\frac{p^\star}{n}\left(1 - \frac{1}{n}\right) \int u^{p^\star/p'} \quad \nabla u \cdot T\, dx \qquad \frac{1}{p} + \frac{1}{p'} = 1$$

$$\leq \frac{p^\star}{n}\left(1 - \frac{1}{n}\right) \left(\int u(x)^{p^\star}\, |T(x)|^{p'}\right)^{\frac{1}{p'}} \left(\int |\nabla u|^p\right)^{\frac{1}{p}}$$

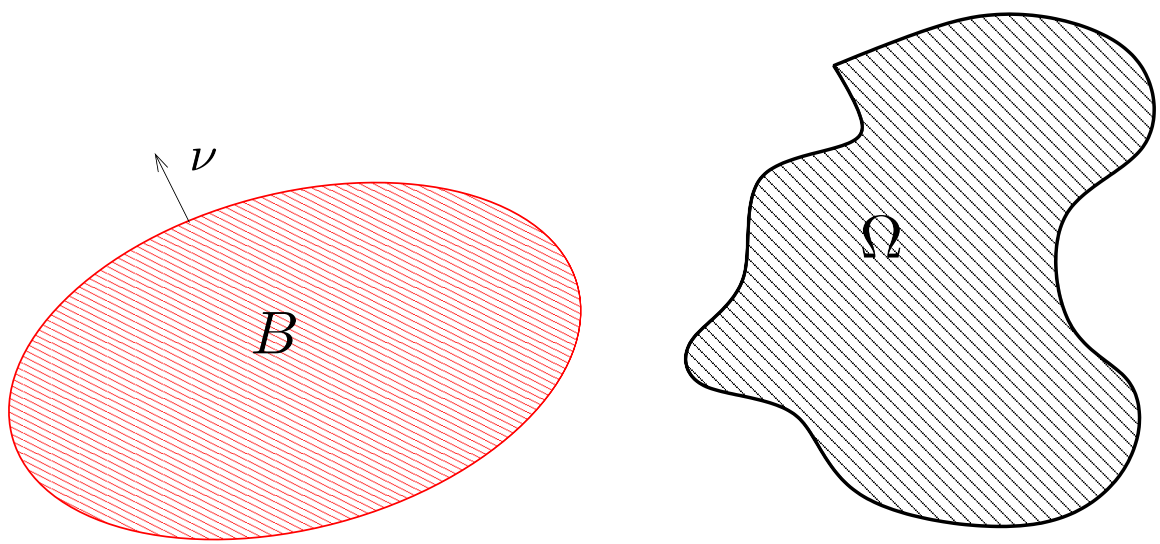
$$= \frac{p^\star}{n}\left(1 - \frac{1}{n}\right) \left(\int g(y)\, |y|^{p'}\right)^{\frac{1}{p'}} \left(\int |\nabla u|^p\right)^{\frac{1}{p}}$$

Development: Wulff isoperimetric inequality with error term

(Figalli–Maggi–Pratelli)

$\|\cdot\|$ an arbitrary norm, $\quad B = \{\|x - x_0\| \leq r\}$,

$$|\partial\Omega| = \int_{\partial\Omega} \|\nu_\Omega\|_* \, \mathcal{H}^{n-1}(dx)$$



$$|\partial\Omega| \geq \inf_{|B|=|\Omega|} \left\{ |\partial B| \left(1 + \text{const.} \left(\frac{|\Omega \Delta B|}{|\Omega|} \right)^2 \right) \right\}$$

OPTIMAL TRANSPORT, ISOPERIMETRY, SMOOTHNESS AND CUT LOCUS PART II

Basic notions of Riemannian geometry

- geodesics
- exponential map
- sectional curvature

More Riemannian geometry

(M,g) compact Riemannian manifold, $c(x,y) = \dfrac{d(x,y)^2}{2}$

$\mathrm{cut}(M) = \Big\{(x,y) \in M \times M$ where d fails to be smooth$\Big\}$

$(x,y) \notin \mathrm{cut}(M)$; $x^1, \ldots, x^n, y^1, \ldots, y^n$ local coordinates

$$c_{i_1 \ldots i_k, j_1 \ldots j_\ell} := \frac{\partial^{k+\ell} c(x,y)}{\partial x^{i_1} \ldots \partial x^{i_k} \, \partial y^{j_1} \ldots \partial y^{j_\ell}} \qquad [c^{i,j}] = [c_{i,j}]^{-1}$$

For $(\xi, \eta) \in T_x M \times T_y M$, define

$$\mathfrak{S}_{x,y}(\xi,\eta) := \frac{3}{2} \sum_{ijk\ell rs} \Big(c_{ij,r}\, c^{r,s}\, c_{s,k\ell} - c_{ij,k\ell} \Big) \, \xi^i \, \xi^j \, \eta^k \, \eta^\ell$$

(Ma–Trudinger–Wang) $\forall x, y, \xi, \eta$

$$\boxed{\Big[-\textstyle\sum_{i,j} c_{i,j}\, \xi^i\, \eta^j = 0\Big] \Longrightarrow \quad \mathfrak{S}_{x,y} \cdot (\xi,\eta) \geq 0}$$

The Ma–Trudinger–Wang tensor $\mathfrak{S}$

- is a fourth-order, nonlocal, nonlinear expression of the Riemannian metric

- is covariant (independent of coordinate change) [Loeper, Kim–McCann]

- has an alternative expression:

$$-\frac{3}{2}\,\frac{d^2}{ds^2}\bigg|_{s=0}\,\frac{d^2}{dt^2}\bigg|_{t=0}\; c\big(\exp_x(t\xi), \exp_x(v+s\widetilde{\eta})\big)$$

$$\widetilde{\eta} = (d_v \exp_x)^{-1}\eta$$

- generalizes sectional curvature (Loeper):

ξ, η two orthogonal unit vectors in T_xM

$$\Longrightarrow \quad \mathfrak{S}_{x,x}(\xi,\eta) = \mathrm{Sect}(\{\xi,\eta\})$$

So the MTW condition is stronger than ($\mathrm{Sect} \geq 0$).

Influence on regularity theory

$c = d^2/2$, optimal transport $T = \exp \nabla\psi$, is ψ smooth??

In general no!

What are the good conditions for regularity?? PDE problem

$$\boxed{\det\left[\nabla^2\psi(x) + \nabla_x^2 c\big(x, \exp_x(\nabla\psi(x))\big)\right] = \frac{f(x)}{g(T(x))\big|\det(d_{\nabla\psi(x)}\exp_x)\big|}}$$

Note: c-convexity of ψ $\quad\Longrightarrow$

$$\nabla^2\psi(x) + \nabla_x^2 c\big(x, \exp_x(\nabla\psi(x))\big) \geq 0$$

Negative result

Loeper, Kim: Even on a compact surface with positive sectional curvature, and probability densities f, g $C^\infty > 0$, O.T. may be discontinuous

Ma, Trudinger, Wang, Loeper ....

(MTW) comes close to be equivalent to the smoothness of optimal transport.

- If it is violated, then $\exists\, f, g \in C^\infty(M)$, positive probability densities, such that the optimal transport map T between $\mu = f\,\mathrm{vol}$ and $\nu = g\,\mathrm{vol}$, for the cost $c = d^2/2$, is discontinuous.

- If it is satisfied, one can hope that T is C^∞. This has been proven under "slightly" stronger assumptions.

References: [oldnew, Chap. 12]

+ recent papers by Delanoë, Figalli, Ge, Kim, Loeper, Ma, McCann, Rifford, Trudinger, V., Wang...

Influence on geometry

$$\mathcal{V}(x) = \Big\{ v = \dot\gamma(0);\ \gamma : [0,1] \to M \text{ geodesic} \Big\}$$

- $\mathcal{V}(x)$ is the manifold M, "written in $T_x M$"
- boundary of $\mathcal{V}(x)$ = tangent cut locus of x, $\mathrm{TCL}(x)$
- interior of $\mathcal{V}(x)$ = injectivity domain, $I(x)$

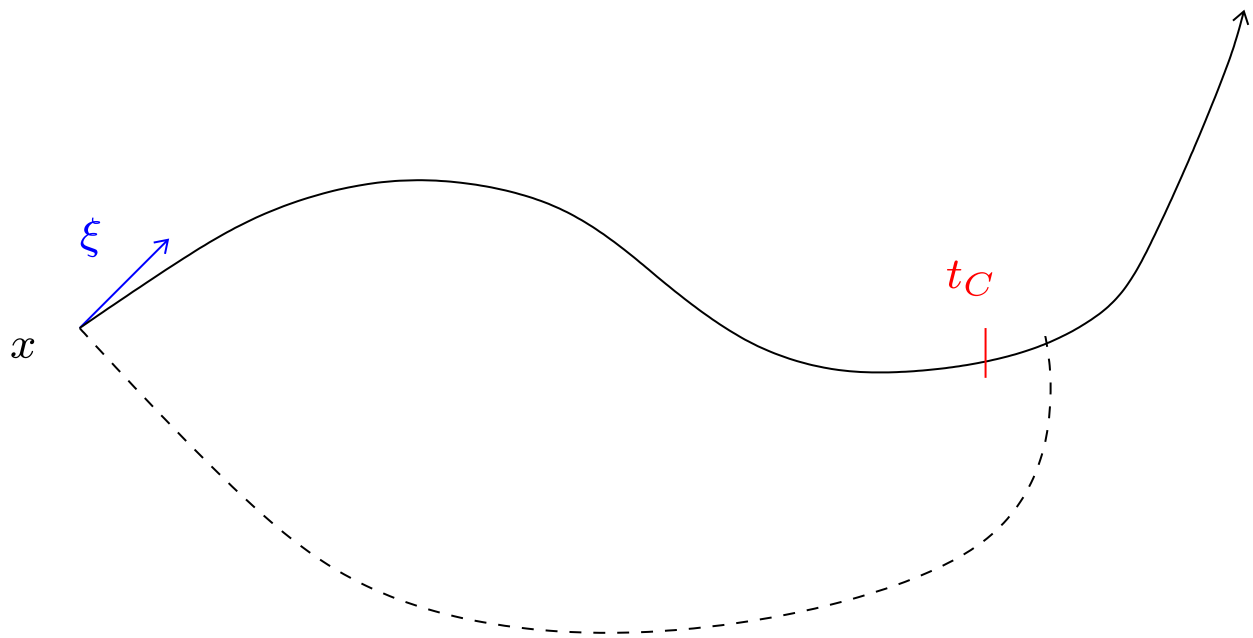
Open problem (Itoh–Tanaka): Is $\mathrm{TCL}(x)$ an Alexandrov space?

Possible conjecture:

(MTW) $\Longrightarrow$ **(CI)** $\forall x$, $I(x)$ is convex

Proven (Loeper–V.) under some additional assumptions ("nonfocalization" + strict MTW) Also Figalli–Rifford–V.

Cut locus and tangent cut locus



$$\mathrm{cut}(x) = \bigcup_{|\xi|=1} \{\exp_x(t_C\,\xi)\}$$ — important but mysterious!

$$\mathrm{TCL}(x) = (\exp_x)^{-1}(\mathrm{cut}(x)) = \{t_C(\xi)\,\xi\} \subset T_xM$$

"Interior" (TCL) [before cut] = injectivity domain $I(x)$

manuscripta math. 114, 247–264 (2004)

Jin-ichi Itoh · Kazuyoshi Kiyohara

The cut loci and the conjugate loci on ellipsoids

Received: 7 October 2003 / Published online: 10 May 2004

Abstract. We prove that the cut locus of any point on any ellipsoid is an arc on the curvature line through the antipodal point. Also, we prove that the conjugate locus has exactly four cusps, which is known as the last geometric statement of Jacobi.

1. Introduction

On a riemannian manifold any geodesic γ emanating from a point p has the property that the segment $\gamma(s)$, $0 \leq s \leq t$, is minimal, i.e., its length is equal to the distance between p and $\gamma(t)$, if t is small. If $t = t_0$ is the largest positive time possessing such a property, then the point $\gamma(t_0)$ is called the cut point of the geodesic $\gamma(t)$ $(t \geq 0)$. Similarly, if $t = t_1$ is the largest positive time such that the segment $\gamma(s)$, $0 \leq s \leq t$, is minimal among the geodesics from p infinitesimally close to γ, then the point $\gamma(t_1)$ is called the first conjugate point of p along the geodesic $\gamma(t)$. The cut locus (resp. the conjugate locus) of the point p is defined as the set of cut points (resp. first conjugate points) of all the geodesics emanating from p.

The cut locus of a point on a convex surface was first investigated by H. Poincaré [11]. S. Myers [10] proved that the cut locus of a point on a real analytic surface homeomorphic to the sphere is a finite tree and its edges are analytic curves. Moreover, he proved that the end points of the cut locus of a point p are also conjugate points of p and they are cusps in the conjugate locus.

Although the behavior of geodesics on ellipsoids is well investigated since the nineteenth century, the cut locus of a general point on two-dimensional ellipsoids seems to have not been known explicitly (see [2]). For an umbilical point, the cut locus and the conjugate locus are reduced to its antipodal point ([9], [8]). For the other points, it seems to be believed that the cut locus is a topological segment. This depends on the unproved Jacobi's geometric statement: The conjugate locus of a non-umbilical point on ellipsoids has exactly four cusps ([1], [2], [3], [4], [6], [7], see also [12], which contains historical remarks).

Recently in an experimental work [5] a software named Thaw conjectured that on any ellipsoid the cut locus of a point $p_0 = (\lambda_1^0, \lambda_2^0)$ is a subarc of a curvature

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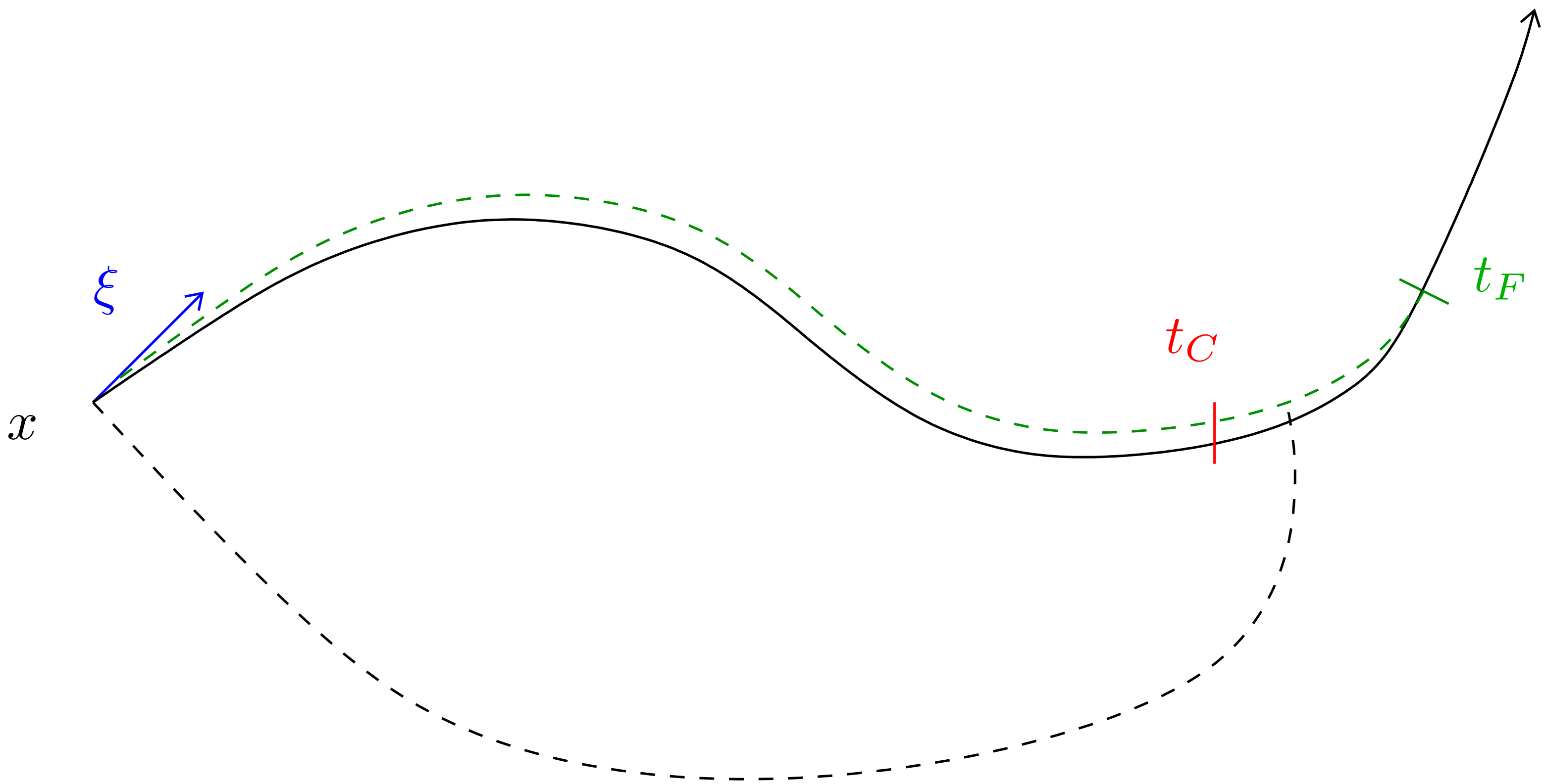
Focal/nonfocal cut locus



$$\det(d_{t_F\,\xi} \exp_x) = 0$$

$$\begin{cases} t_F = t_C \longrightarrow & \text{focalization at cut locus} \\ t_F > t_C \longrightarrow & \text{nonfocalization} \end{cases}$$

Ex: $\mathbb{S}^n$, ellipsoid are focal; $\mathbb{RP}^n = \mathbb{S}^n/\{\pm \mathrm{Id}\}$ is nonfocal

Poincaré, Klingenberg, Weinstein & others:

- If M is simply connected, positive sectional curvature

- and either even-dimensional or 1/4-pinched,

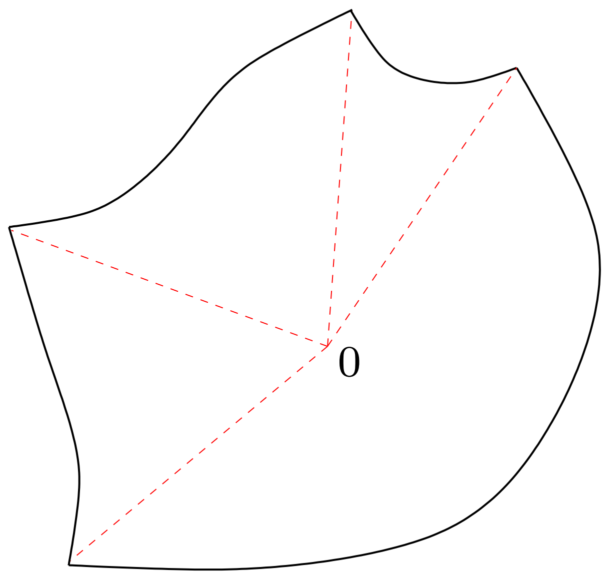
... then cut(M) is focal at some point.

Cut locus and regularity of distance

- d^2 is always semiconcave
- $\mathrm{cut}(x)$ is where $d^2(\,\cdot\,, x)$ fails to be smooth/semiconvex
- d^2 is differentiable if and only if there is only one minimizing velocity
- Two kinds of cut velocities: either nonunique minimizing velocity (not differentiable), or focalization (not C^2)
- The nondifferentiability is in general impossible to detect locally

Differential structure of TCL

Nonfocal case: complete description



Focal case: much wilder, and wild dependence

Known: TCL is Lipschitz (Itoh–Tanaka, Li–Nirenberg)

Open problem: Is $I(x)$ semiconvex??

NB: Injectivity domain $\subset$ Nonfocal domain

Theorem

(M, g) s.t. $M = \mathbb{S}^n$ and $\|g - g^0\|_{C^4} \leq \varepsilon(n)$

$\Longrightarrow \forall x, \;\; I(x)$ is uniformly convex

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- Unexpected geometric stability property
- Good effect of positive curvature
- C^4 is "natural" for preservation of convexity, since C^2

is natural for continuity of cut locus. Sketch of argument:

$g_k \to g$ in $C^2 \Longrightarrow \exp_k \to \exp$ in C^1

$v_k \in \mathrm{TCL}(x_k)$, $(x_k, v_k) \to (x, v)$, then

- either $\det(d_{v_k} \exp_{x_k}) = 0$ and pass to the limit

- or $\exp_{x_k} v_k = \exp_{x_k} w_k$ and $x_k \to x$, $v_k \to v$, $w_k \to w$,

if $v \ne w$ then done; if $v = w$ then $\det(d_v \exp_x) = 0$.

Plan of the theorem

Three steps:

1 [easy] $\mathbb{S}^n$ has convex nonfocal domains and this is stable under perturbation (IFT)

2 [hard] $\mathbb{S}^n$ satisfies a strong form of the MTW condition and this is stable under perturbation

3 [local-to-global] strong MTW + convexity of nonfocal domains implies convexity of injectivity domains

Rk: Property 3 has a spirit of “nonlocal” Myers theorem

Why expect MTW related to the convexity of I?

- (Believe in miracles!) Both conditions play a key role in the regularity theory of optimal transport

- Cheat a bit.... MTW looks like

$F(x,v) := \nabla_x^2 c(x, \exp_x v)$ concave in v

If $F(x,v_0) > -\infty$ and $F(x,v_1) > -\infty$ then
$F(x,(1-t)v_0 + tv_1) > -\infty$

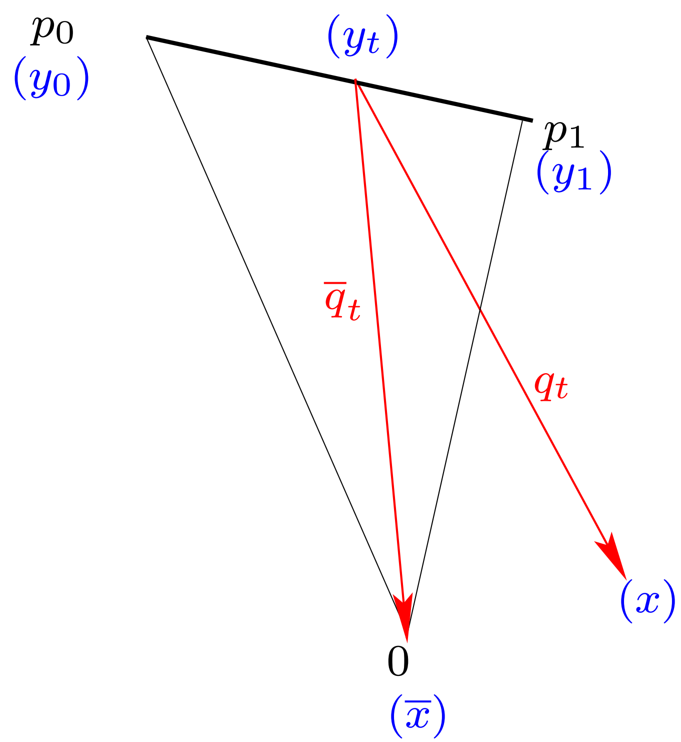
This argument does not work but gives us a hint that there could be a link.

- In spirit nonlocal version of Myers theorem

An integral version of (MTW)

$\overline{x} \in M \qquad [p_0, p_1] \subset I(\overline{x}) \qquad y_t = \exp_{\overline{x}} p_t$

$p_t = (1-t)p_0 + t p_1 \qquad x \in \bigcap_{0 \le t \le 1} M \setminus \mathrm{cut}(y_t)$



$$d(x,y_t)^2 - d(\overline{x},y_t)^2 \ge \min\Big(d(x,y_0)^2 - d(\overline{x},y_0)^2,\ d(x,y_1)^2 - d(\overline{x},y_1)^2 \Big)$$

Sketch of proof (Kim & McCann....)

$c = d^2/2$

$h(t) = c(\overline{x}, y_t) - c(x, y_t)$

$\dot{h}(t) = \nabla^2_{x,y} c(x, y_t) \cdot (q_t - \overline{q}_t, \dot{y}_t)$

$q_t = \log_{y_t} x, \;\; \overline{q}_t = \log_{y_t} \overline{x}$

If $\dot{h}(t) = 0$ then

$$\ddot{h}(t) = \frac{2}{3} \int_0^1 \mathfrak{S}_{\exp_{y_t}((1-s)q + s\overline{q}), y_t} (q_t - \overline{q}_t, \dot{y}_t)\,(1-s)\,ds \geq 0$$

... so expect the maximum value of h at $t = 0$ or $t = 1$.

A reinforced MTW condition

Reinforce

$$\big[\langle \xi, \eta \rangle = 0\big] \Longrightarrow \quad \mathfrak{S}_{x,y}(\xi, \eta) \geq 0$$

into $\quad \mathrm{MTW}(K_0, C_0)$

$$\mathfrak{S}_{(x,y)}(\xi, \eta) \geq K_0 |\xi|^2 \, |\widetilde{\eta}|^2 - C_0 \langle \xi, \widetilde{\eta} \rangle^2$$

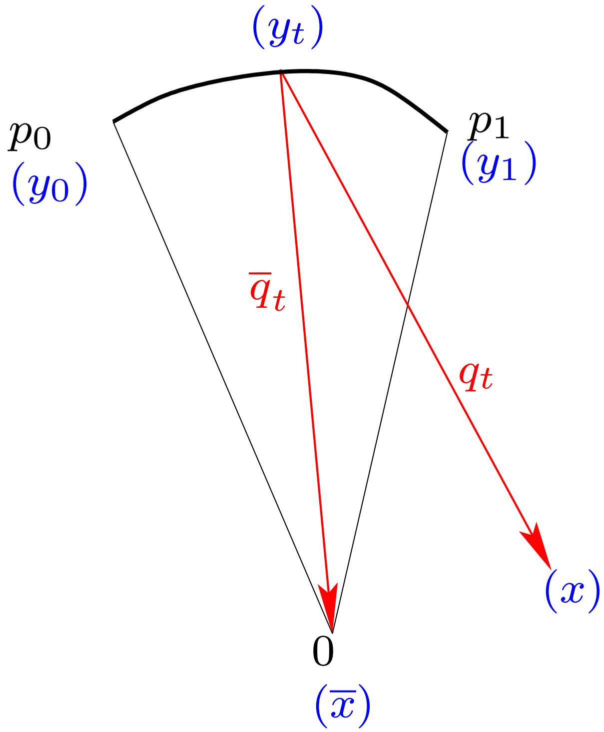
If nonfocalization, basically equivalent to strict MTW!

An integral version of $\mathrm{MTW}(K_0, C_0)$

$\overline{x} \in M \qquad (p_t)_{0\le t\le 1} \subset I(\overline{x}) \qquad y_t = \exp_{\overline{x}} p_t$

$x \in \bigcap_{0\le t\le 1} M \setminus \mathrm{cut}(y_t);$

$\overline{q}_t = (\exp_{y_t})^{-1}(\overline{x}) \qquad q_t = (\exp_{y_t})^{-1}(x)$



$$\forall t \in (0,1), \qquad \begin{cases} [\overline{q}_t, q_t] \subset I(y_t) \\ |\ddot{p}_t| \leq \varepsilon_0 \, |q_t - \overline{q}_t| \, |\dot{p}_t|^2 \end{cases} \qquad \Longrightarrow$$

$$d(x,y_t)^2 - d(\overline{x},y_t)^2 \geq \min\Big(d(x,y_0)^2 - d(\overline{x},y_0)^2, \; d(x,y_1)^2 - d(\overline{x},y_1)^2 \Big)$$
$$+ 2 \, \lambda \, t(1-t) \left(\inf_{0 \leq s \leq 1} |q_s - \overline{q}_s|^2 \right) |p_1 - p_0|^2$$

for some $\varepsilon_0, \lambda > 0$

Rk: $d(\overline{x}, x) \leq |q - \overline{q}|$ **but** keep the latter expression to detect cut locus!!

Use the inequality to rule out flat parts

$$d(x,y_t)^2 - d(\overline{x},y_t)^2 \geq \min\Big(d(x,y_0)^2 - d(\overline{x},y_0)^2,\ d(x,y_1)^2 - d(\overline{x},y_1)^2 \Big)$$
$$+\, 2\,\lambda\, t(1-t) \left(\inf_{0\leq s\leq 1} |q_s - \overline{q}_s|^2 \right) |p_1 - p_0|^2$$

Assume nonfocalization for simplicity

- Imagine $(p_t)_{0\leq t\leq 1}$ is a segment in $\mathrm{TCL}(\overline{x})$
- Assume each p_t is a true cut velocity: y_t is joined to $\overline{x}$ by two distinct velocities $\overline{q}_t$, $\widetilde{q}_t$.
- Apply the inequality with $\overline{x} = x$, $q_t = \widetilde{q}_t \neq \overline{q}_t$.

$\Longrightarrow$ Contradiction!

- Use the integral inequality as an **a priori estimate** of uniform convexity for $I(x)$

- A vicious circle!? To prove the uniform convexity of $I(x)$, need to know a priori the convexity of $I(y)$!?

- Put this in a continuity method with ordering parameter $r = |(\exp_x)^{-1}(y)|$:

$$J := \left\{ r \geq 0;\ \forall x \in M, \quad I(x) \cap B(0,r) \text{ is } \kappa\text{-convex} \right\}$$

The uniform convexity allows to continue!

How to rule out concave parts?

The a priori estimate rules out flat parts, but not "strongly concave" parts.

For this use a more precise inequality. And, if the ball B_r touches $\mathrm{TCL}(x)$ at a concave part, then the contact is a unique minimizer of the norm. Then apply
Klingenberg: A norm-minimizing, nonfocal cut velocity is the velocity of a closed loop, so $q_t = -\overline{q}_t$

..... then work a bit.

And approximate to stay away from cut locus.....

All this established

Theorem (Loeper–V)

(M, g) compact with nonfocal cut locus, sat. **strict MTW** $\quad \big[\langle \xi, \eta\rangle = 0,\ \xi, \eta \neq 0\big] \Longrightarrow \big[\mathfrak{S}_{x,y} \cdot (\xi, \eta) > 0\big]$

Then $\forall x$, $I(x)$ is uniformly convex

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Now what about focalization?

Computing the Hessian of squared distance

$$\gamma(t) = \exp_x(tv), \qquad (e_1(t), \dots, e_n(t)) \qquad e_1(0) = v/|v|$$

$$R_{ij}(t) = \Big\langle \mathrm{Riem}_{\gamma(t)}\big(\dot\gamma(t), e_i(t)\big)\dot\gamma(t), e_j(t)\Big\rangle_{\gamma(t)}$$

$$\ddot J(t) + R(t)J(t) = 0$$

$$\begin{cases} J_0(0) = 0_n & \dot J_0(0) = I_n \\ J_1(0) = I_n & \dot J_1(0) = 0_n \end{cases}$$

Hessian operator:

$$S_{(x,v)} = J_0(1)^{-1}\, J_1(1)$$

$$v \in I(x) \implies \big\langle S_{(x,v)}\xi, \xi\big\rangle_x = \left\langle \left(\nabla_x^2 \frac{d(\,\cdot\,,y)^2}{2}\right)\cdot \xi, \xi \right\rangle_x$$

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Reference: *Jacobi fields forever* [oldnew, Chap. 14]

Generalized Ma–Trudinger–Wang tensor

$$\overline{\mathfrak{S}}_{(x,v)}(\xi,\eta) = -\frac{3}{2}\;\frac{d^2}{ds^2}\bigg|_{s=0}\left\langle S_{(x,v+s\eta)}\xi,\xi\right\rangle_x$$

So this can be computed in terms of Jacobi fields, and extended beyond the injectivity domain, all over the nonfocal domain (Figalli–Rifford)

The example of the Sphere

$$S(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{\tau \cos\tau}{\sin\tau}\right) I_{n-1} \end{bmatrix}$$

$v = t\, e_1,\ \eta = \eta_1\, e_1 + \eta_2\, e_2,\ \xi = \xi_1\, e_1 + \xi_2\, e_2 + \xi_3\, e_3,$ then

$$\begin{aligned} \overline{\mathfrak{S}}_{(x,v)}(\xi,\eta) = {} & 2\left(\frac{1}{t^2} - \frac{\cos t}{t\ \sin t}\right)\xi_1^2\,\eta_2^2 + 4\left(\frac{1}{t^2} - \frac{1}{\sin^2 t}\right)\xi_1\,\xi_2\,\eta_1\,\eta_2 \\ & + 2\left(\frac{1}{\sin^2 t} - \frac{t\ \cos t}{\sin^3 t}\right)\xi_2^2\,\eta_1^2 + \left(\frac{1}{\sin^2 t} + \frac{\cos t}{t\ \sin t} - \frac{2}{t^2}\right)\xi_2^2\,\eta_2^2 \\ & + 2\left(\frac{1}{\sin^2 t} - \frac{t\ \cos t}{\sin^3 t}\right)\xi_3^2\,\eta_1^2 + \left(\frac{1}{\sin^2 t} - \frac{\cos t}{t\ \sin t}\right)\xi_3^2\,\eta_2^2. \end{aligned}$$

..... $\overline{\mathfrak{S}}_{(x,v)}(\xi,\eta) \geq \kappa\Big(|\xi|^2\,|\eta|^2 - \langle\xi,\eta\rangle^2\Big) + \kappa\,|v|^2\,|\widetilde{\xi}|^2\,|\eta|^2$

Condition MTW(κ,κ) and even better!

Now for a varying metric...

Variations of Jacobi fields

dot $\longrightarrow \quad d/dt$

prime $\longrightarrow \quad d/d\alpha$, independent parameter

$$J_1^{-1} J_0' = A_0 - KC \qquad J_1^{-1} J_1' = C^* - KA_1$$

$$J_1^{-1} \dot{J}_0 = I + T_0 - KD \qquad J_1^{-1} \dot{J}_1 = D^* - KT_1 - KR^0$$

$$K(t) = J_1(t)^{-1} J_0(t) \qquad I = I_n \qquad R^0 = R(0)$$

$$A_0(t) = \int_0^t J_0(s)^* \, R'(s) \, J_0(s) \, ds$$

$$A_1(t) = \int_0^t J_1(s)^* \, R'(s) \, J_1(s) \, ds$$

$$T_0(t) = \int_0^t J_0(s)^* \, \dot{R}(s) \, J_0(s) \, ds \qquad T_1(t) = \int_0^t J_1(s)^* \, \dot{R}(s) \, J_1(s) \, ds$$

$$C(t) = \int_0^t J_1(s)^* \, R'(s) \, J_0(s) \, ds \qquad D(t) = \int_0^t J_1(s)^* \, \dot{R}(s) \, J_0(s) \, ds$$

$$
\begin{aligned}
\frac{2}{3}\,\overline{\mathfrak{S}}_{(x,v)}(\xi,\eta) = \quad & \Big(-2|v|\,\Big\langle K^{-1}(I+T_0)K^{-1}\xi,\ (I+T_0)K^{-1}\xi\Big\rangle - 2|v|\big\langle K^{-1}D^*\xi,\ D^*\xi\big\rangle \\
& +4|v|\big\langle K^{-1}(I+T_0)K^{-1}\xi, D^*\xi\big\rangle + 2\langle K^{-1}\xi, K^{-1}\xi\rangle + \langle(2T_0+2|v|(I+T_0-KD)(D-T_1K-R^0K) \\
& -4D^*K+2K(T_1+R^0)K)K^{-1}\xi, K^{-1}\xi\rangle\Big)\,\eta_1^2 + \Big(-\frac{2}{|v|}\langle K^{-1}A_0K^{-1}\xi, A_0K^{-1}\xi\rangle + \langle K^{-1}\xi, K^{-1}\xi\rangle \\
& +\frac{4}{|v|}\langle K^{-1}C^*\xi, A_0K^{-1}\xi\rangle - \frac{1}{|v|}\langle K^{-1}\xi,\xi\rangle - \frac{2}{|v|}\langle K^{-1}C^*\xi, C^*\xi\rangle - \frac{2}{|v|}\langle K^{-1}(P\xi)^\perp, (P\xi)^\perp\rangle \\
& +\frac{2}{|v|}\langle K^{-1}\xi, P\xi\rangle - \frac{4}{|v|}\langle (P\xi)^\perp, K^{-1}C^*\xi\rangle + \frac{4}{|v|}\langle K^{-1}A_0K^{-1}(P\xi)^\perp,\xi\rangle - \frac{4}{|v|}\langle C^*(P\xi)^\perp, K^{-1}\xi\rangle \\
& +\frac{4}{|v|}\langle A_1(P\xi)^\perp,\xi\rangle + \langle(-2D^*K+K(R^0+T_1)K+T_0-\frac{1}{|v|}\Big(J_1^{-1}J_1''K+2A_0A_1K+2KC^2-2A_0C \\
& -J_1^{-1}J_0''-2KCA_1K\Big))K^{-1}\xi, K^{-1}\xi\rangle\Big)\,\eta_2^2 + \Big(4\big\langle (I+T_0)K^{-1}\xi, K^{-1}(P\xi)^\perp\big\rangle - 4\big\langle K^{-1}C^*\xi, D^*\xi\big\rangle \\
& -4\big\langle DK^{-1}\xi, (P\xi)^\perp\big\rangle - 4\big\langle K^{-1}C^*\xi, D^*\xi\big\rangle + 4\big\langle (R^0+T_1)\xi, (P\xi)^\perp\big\rangle - 4\big\langle K^{-1}D^*\xi, (P\xi)^\perp\big\rangle \\
& +4\big\langle K^{-1}A_0K^{-1}\xi, D^*\xi\big\rangle + 4\big\langle K^{-1}(I+T_0)K^{-1}\xi, C^*\xi\big\rangle - 4\big\langle K^{-1}A_0K^{-1}\xi, (I+T_0)K^{-1}\xi\big\rangle \\
& +\big\langle(\frac{2}{|v|}\Big[|v|(I+T_0)C+|v|J_1^{-1}\dot{J}_0'+|v|A_0D-KC-|v|\Big(A_0(T_1+R^0)+(I+T_0)A_1+J_1^{-1}\dot{J}_1'\Big)K \\
& +C^*K-|v|K(CD+DC)+|v|K\Big(C(T_1+R^0)+DA_1\Big)K\Big])K^{-1}\xi, K^{-1}\xi\big\rangle\Big)\,\eta_1\,\eta_2
\end{aligned}
$$

$$\frac{2}{3}\,\overline{\mathfrak{S}}_{(x,v)}(\xi,\eta)= \Big(-2|v|\,\Big\langle K^{-1}(I+T_0)K^{-1}\xi,\ (I+T_0)K^{-1}\xi\Big\rangle - 2|v|\Big\langle K^{-1}D^*\xi,\ D^*\xi\Big\rangle$$

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$$-4D^*K+2K(T_1+R^0)K)K^{-1}\xi, K^{-1}\xi\rangle\Big)\,\eta_1^2 + \Big(-\frac{2}{|v|}\langle K^{-1}A_0K^{-1}\xi, A_0K^{-1}\xi\rangle + \langle K^{-1}\xi, K^{-1}\xi\rangle$$

$$+\frac{4}{|v|}\langle K^{-1}C^*\xi, A_0K^{-1}\xi\rangle - \frac{1}{|v|}\langle K^{-1}\xi,\xi\rangle - \frac{2}{|v|}\langle K^{-1}C^*\xi, C^*\xi\rangle - \frac{2}{|v|}\langle K^{-1}(P\xi)^\perp, (P\xi)^\perp\rangle$$

$$+\frac{2}{|v|}\langle K^{-1}\xi, P\xi\rangle - \frac{4}{|v|}\langle(P\xi)^\perp, K^{-1}C^*\xi\rangle + \frac{4}{|v|}\langle K^{-1}A_0K^{-1}(P\xi)^\perp, \xi\rangle - \frac{4}{|v|}\langle C^*(P\xi)^\perp, K^{-1}\xi\rangle$$

$$+\frac{4}{|v|}\langle A_1(P\xi)^\perp, \xi\rangle + \langle(-2D^*K+K(R^0+T_1)K+T_0-\frac{1}{|v|}\Big(J_1^{-1}J_1''K+2A_0A_1K+2KC^2-2A_0C$$

$$-J_1^{-1}J_0''-2KCA_1K\Big))K^{-1}\xi, K^{-1}\xi\rangle\Big)\,\eta_2^2 + \Big(4\Big\langle(I+T_0)K^{-1}\xi, K^{-1}(P\xi)^\perp\Big\rangle - 4\Big\langle K^{-1}C^*\xi, D^*\xi\Big\rangle$$

$$-4\Big\langle DK^{-1}\xi, (P\xi)^\perp\Big\rangle - 4\Big\langle K^{-1}C^*\xi, D^*\xi\Big\rangle + 4\Big\langle(R^0+T_1)\xi, (P\xi)^\perp\Big\rangle - 4\Big\langle K^{-1}D^*\xi, (P\xi)^\perp\Big\rangle$$

$$+4\Big\langle K^{-1}A_0K^{-1}\xi, D^*\xi\Big\rangle + 4\Big\langle K^{-1}(I+T_0)K^{-1}\xi, C^*\xi\Big\rangle - 4\Big\langle K^{-1}A_0K^{-1}\xi, (I+T_0)K^{-1}\xi\Big\rangle$$

$$+\langle(\frac{2}{|v|}\Big[|v|(I+T_0)C+|v|J_1^{-1}\dot{J}_0'+|v|A_0D-KC-|v|\Big(A_0(T_1+R^0)+(I+T_0)A_1+J_1^{-1}\dot{J}_1'\Big)K$$

$$+C^*K-|v|K(CD+DC)+|v|K\Big(C(T_1+R^0)+DA_1\Big)K\Big])K^{-1}\xi, K^{-1}\xi\rangle\Big)\,\eta_1\,\eta_2$$

$$\frac{2}{3}\,\overline{\mathfrak{S}}_{(x,v)}(\xi,\eta) \geq \kappa\Big[\,\big(|\xi|^2\,|\eta|^2 - \langle \xi,\eta\rangle^2\big) + |v|^2\,|\Lambda^{-1}\xi|^2\,|\eta|^2\Big]$$

$$\kappa\left|\Lambda^{-\frac{1}{2}}\Big(\big(((I+T_0)\Lambda^{-1}\xi + |v|D^*\xi\big)\eta_1 + \Big(\big(|v|^{-1}A_0\Lambda^{-1}\xi + C^*\xi\big) + (P\xi)^{\perp}\Big)\eta_2\Big)\right|^2$$

$$\Lambda := -S^{-1}\big\lfloor_{e_1^{\perp}}$$

This implies robust positivity near the focal locus!

Evaluating the difference between true and extended cost

Let $\overline{x}, x \in M$, and let $(p_t)_{t_0<t<t_1}$ be a C^2 curve valued in $\mathrm{NF}(\overline{x})$, such that $\exp_{\overline{x}} p_t$ never meets $\mathrm{cut}(x)$.

$$h(t) = \frac{|p_t|_{\overline{x}}^2}{2} - \frac{d\big(\overline{x}, \exp_{\overline{x}} p_t\big)^2}{2}$$

$$y_t = \exp_{\overline{x}} p_t, \quad q_t = (\exp_{y_t})^{-1}(x), \quad \overline{q}_t = -(d_{p_t} \exp_{\overline{x}})(p_t),$$

If $[\overline{q}_t, q_t] \subset \mathrm{NF}(y_t)$ for all t then

$$\dot{h}(t) = \big\langle q_t - \overline{q}_t, \dot{y}_t \big\rangle_{y_t}$$

$$\ddot{h}(t) = \frac{2}{3} \int_0^1 (1-s)\, \overline{\mathfrak{S}}_{(y_t,(1-s)\overline{q}_t + s q_t)}(\dot{y}_t, q_t - \overline{q}_t)\, ds + \big\langle (d_{\overline{q}_t} \exp_{y_t})(q_t - \overline{q}_t), \ddot{p}_t \big\rangle_{\overline{x}}$$

Crossing the cut locus: Approximating a path

x given; $(p_t)_{0\le t\le 1} \subset \mathrm{NF}(\overline{x})$.

Then exists $(\widetilde{p}_t)_{0\le t\le 1}$ s.t. $\|p_t - \widetilde{p}_t\|_{C^2}$ is very small, and

$$\Big\{t;\ \exp_{\overline{x}} \widetilde{p}_t \in \mathrm{cut}(x)\Big\} \qquad \text{is finite.}$$

Pf: Fubini-type argument (co-area formula) using $\mathcal{H}^{n-1}(\mathrm{cut}(x)) < \infty$ (Li–Nirenberg)

A maximum principle

$f : [0,1] \to \mathbb{R}$ semiconvex function, C^2 except at a finite number of points t_j, s.t. $\ddot{f}(t) \ge -C\,|\dot{f}(t)|$, then

$$\forall t, \qquad f(t) \le \max\,(f(0), f(1)).$$

Putting everything together

Choose $p(t) \in T_{\overline{x}}M$ s.t.

$$\begin{cases} p(0), p(1) \in I(\overline{x}) \\ |\ddot{p}| \leq \delta \, |p(0) - p(1)|^2 \end{cases}$$

$$y_t := \exp_{\overline{x}} p(t)$$

$$\ell(t) := \frac{d(\overline{x}, y_t)^2}{2} - \frac{|p(t)|^2}{2} \leq 0$$

At $t = 0$, $t = 1$ this is 0.

Using the approximation, semiconcavity of squared distance, maximum principle, interpretation of MTW, $\ell(t) \geq \min(\ell(0), \ell(1)) = 0$.

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Using the approximation, semiconcavity of distance, maximum principle, interpretation of MTW, show $\ell(t) \geq \min(\ell(0), \ell(1)) = 0$.

So $p(t)$ is a minimizing velocity!