Logarithmic decay of hyperbolic equations with arbitrary small boundary damping

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Let Ω bounded domain in $\mathbb{R}^n$, $\partial\Omega \in C^2$.

Let $a^{jk}(\cdot) \in C^1(\overline{\Omega};\ \mathbb{R})$ be fixed functions satisfying

$$a^{jk}(x) = a^{kj}(x), \qquad \forall\, x \in \overline{\Omega},\ j,k = 1,2,\cdots,n, \tag{2.1}$$

and for some constant $s_0 > 0$,

$$\sum_{j,k=1}^{n} a^{jk}(x)\xi^j\overline{\xi}^k \geq s_0|\xi|^2, \qquad \forall\,(x,\xi) \in \overline{\Omega} \times \mathbb{C}^n, \tag{2.2}$$

where $\xi = (\xi^1, \cdots, \xi^n)$.

Fix a function $a(\cdot) \in L^\infty(\partial\Omega;\ \mathbb{R}^+)$ satisfying

$$\Gamma_0 \triangleq \{x \in \partial\Omega;\ a(x) > 0\} \neq \emptyset. \tag{2.3}$$

$$\begin{cases} u_{tt} - \displaystyle\sum_{j,k=1}^{n} (a^{jk} u_{x_j})_{x_k} = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ \displaystyle\sum_{j,k=1}^{n} a^{jk} u_{x_j} \nu_{x_k} + a(x) u_t = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ (u(0), u_t(0)) = (u^0, u^1) & \text{in } \Omega. \end{cases} \tag{2.4}$$

Put

$$H \triangleq \left\{ (f,g) \in H^1(\Omega) \times L^2(\Omega) \;\middle|\; \int_\Omega f dx = 0 \right\},$$

which is a Hilbert space, whose norm is given by

$$||(f,g)||_H = \sqrt{\int_\Omega \Big[\sum_{j,k=1}^n a^{jk} f_j \overline{f_k} + |g|^2 \Big] dx}, \qquad \forall\, (f,g) \in H.$$

Define an unbounded operator $\mathcal{A}: \ H \to H$ by (recalling that $u_j^0 = \frac{\partial u^0}{\partial x_j}$)

$$\begin{cases} \mathcal{A} \triangleq \begin{pmatrix} 0 & I \\ \sum_{j,k=1}^{n} \partial_k(a^{jk}\partial_j) & 0 \end{pmatrix}, \\ D(\mathcal{A}) \triangleq \left\{ u = (u^0, u^1) \in H;\ \mathcal{A}u \in H;\ \Big(\sum_{j,k=1}^{n} a^{jk} u_j^0 \nu_k + a u^1 \Big)\Big|_{\partial\Omega} = 0 \right\}. \end{cases}$$

It is easy to show that $\mathcal{A}$ generates a C_0-semigroup $\{e^{t\mathcal{A}}\}_{t\in\mathbb{R}}$ on H.

Therefore, system is well-posed in H.

By means of the classical energy method, it is easy to check that

$$\frac{d}{dt}||(u,u_t)||_H^2 = -2\int_{\Gamma_0} a(x)|u_t|^2 d\Gamma_0.$$

It shows that the only dissipative mechanism acting on the system is through the sub-boundary Γ_0.

Hence, the energy of every solution tends to zero as $t \to \infty$, without any geometric conditions on the domain Ω.

Our goal is devoted to analyze further the decay rate of solutions of system (2.4) tends to zero as $t \to \infty$.

In this respect, very interesting logarithmic decay result was given by Lebeau-Robbiano (1997) for the above system under the regularity assumption that $a^{jk}(\cdot)$ and $a(\cdot)$, and the boundary $\partial\Omega$ are C^∞-smooth.

Since the sub-boundary Γ_0 in which the damping $a(x)u_t$ is effective may be very "small" with respect to the whole boundary $\partial\Omega$, the "geometric optics condition" introduced by Bardos-Lebeau-Rauch is not guaranteed for system (2.4), and therefore, one can not expect exponential stability of this system.

Theorem 1. (X. Fu, Comm. PDE, 2009)

Let $a^{jk}(\cdot) \in C^2(\overline{\Omega};\ \mathbb{R})$ satisfy (2.1)–(2.2), and $a(\cdot) \in L^\infty(\partial\Omega;\ \mathbb{R}^+)$ satisfy (2.3). Then solutions $e^{t\mathcal{A}}(u^0, u^1) \equiv (u, u_t) \in C(\mathbb{R};\ D(\mathcal{A})) \cap C^1(\mathbb{R};\ H)$ satisfy

$$||e^{t\mathcal{A}}(u^0, u^1)||_H \le \frac{C}{\ln(2+t)}||(u^0, u^1)||_{D(\mathcal{A})}, \quad \forall\ (u^0, u^1) \in D(\mathcal{A}),\ \forall\ t > 0. \tag{3.1}$$

As pointed in “Lebeau-Robbiano (1997)”, for some special case of system (2.4), logarithmic stability is the best decay rate.

Theorem 1 is a consequence of following resolvent estimate for operator $\mathcal{A}$:

Theorem 2. (X. Fu, Comm. PDE, 2009)

There exists a constant $C > 0$ such that for any

$$\mathrm{Re}\,\lambda \in \left[-\frac{e^{-C|\mathrm{Im}\,\lambda|}}{C}, 0\right],$$

we have

$$||(\mathcal{A} - \lambda I)^{-1}||_{\mathcal{L}(H)} \le Ce^{C|\mathrm{Im}\,\lambda|}, \qquad \text{for } |\lambda| > 1.$$

We shall develop an approach based on global Carleman estimate to

prove Theorem 2.

Based on which type of equations to obtain Carleman estimate?

Fix $f = (f^0, f^1) \in H$ and $u = (u^0, u^1) \in D(\mathcal{A})$. Then

$$(\mathcal{A} - \lambda I)u = f \tag{4.1}$$

is equivalent to

$$\begin{cases} -\lambda u^0 + u^1 = f^0, \\ \displaystyle\sum_{j,k=1}^{n} (a^{jk} u^0_j)_k - \lambda u^1 = f^1. \end{cases} \tag{4.2}$$

Therefore

$$u^1 = f^0 + \lambda u^0.$$

Noting the boundary condition

$$\Big(\sum_{j,k=1}^{n} a^{jk} u_j^0 \nu_k + a u^1\Big)\Big|_{\partial\Omega} = 0,$$

we have

$$\begin{cases} \displaystyle\sum_{j,k=1}^{n} (a^{jk} u_j^0)_k - \lambda^2 u^0 = \lambda f^0 + f^1 & \text{in } \Omega, \\ \displaystyle\sum_{j,k=1}^{n} a^{jk} u_j^0 \nu_k + a\lambda u^0 = -a f^0 & \text{on } \partial\Omega, \\ u^1 = f^0 + \lambda u^0 & \text{in } \Omega. \end{cases} \tag{4.3}$$

Put

$$v = e^{i\lambda s} u^0. \tag{4.4}$$

It is easy check that v satisfies the following equation:

$$\begin{cases} v_{ss} + \sum_{j,k=1}^{n} (a^{jk} v_j)_k = (\lambda f^0 + f^1) e^{i\lambda s} & \text{in } \mathbb{R} \times \Omega, \\ \sum_{j,k=1}^{n} a^{jk} v_j \nu_k - iav_s = -af^0 e^{i\lambda s} & \text{on } \mathbb{R} \times \partial\Omega. \end{cases} \tag{4.5}$$

Global Carleman estimate for elliptic equations with

nonhomogeneous Neumann-like boundary condition

$$\begin{cases} z_{ss} + \sum_{j,k=1}^{n} \left(a^{jk} z_j\right)_k = z^0 & \text{in } (-2,2) \times \Omega, \\ \sum_{j,k=1}^{n} a^{jk} z_j \nu_k - ia(x) z_s = a(x) z^1 & \text{on } (-2,2) \times \partial\Omega. \end{cases} \tag{4.6}$$

Theorem 3.

There exists a constant $C > 0$ such that, for any $\varepsilon > 0$, any solution z of system (4.6) satisfies

$$\begin{aligned} ||z||_{H^1(Y)} \leq Ce^{C/\varepsilon}\Big[||z^0||_{L^2(X)} + ||z^1||_{L^2(\Sigma)} + ||z||_{L^2(Z)} + ||z_s||_{L^2(Z)}\Big] \\ + Ce^{-2/\varepsilon}||z||_{H^1(X)}. \end{aligned} \tag{4.7}$$

where

$$X = (-2,2) \times \Omega, \quad \Sigma = (-2,2) \times \partial\Omega, \quad Y = (-1,1) \times \Omega, \quad Z = (-2,2) \times \Gamma_0.$$

The proof is based on the following point-wise estimate for elliptic operators.

Step 1. Point-wise estimate for elliptic operators.

Lemma 1.

Let $b^{jk} \in C^2(\mathbb{R}^n;\ \mathbb{R})$ satisfy $b^{jk} = b^{kj}$. Assume that $w \in C^2(\mathbb{R}^{1+n}; \mathbb{C})$ and

$\ell \in C^2(\mathbb{R}^{1+n}; \mathbb{R})$. Set $\theta = e^{\ell}, \quad v = \theta w$. Then

$$\begin{aligned}
&\theta^2 \Big| w_{ss} + \sum_{j,k=1}^{n} (b^{jk} w_{x_j})_{x_k} \Big|^2 + M_s + \mathrm{div}\, V \\
&\geq 2\Big(3\ell_{ss} + \sum_{j,k=1}^{n} (b^{jk}\ell_{x_j})_{x_k}\Big)|v_s|^2 + 4\sum_{j,k=1}^{n} b^{jk}\ell_{x_j s}(v_{x_k}\overline{v}_s + \overline{v}_{x_k} v_s) \qquad (4.8) \\
&\quad + \sum_{j,k=1}^{n} c^{jk}(v_{x_k}\overline{v}_{x_j} + \overline{v}_{x_k} v_{x_j}) + B|v|^2,
\end{aligned}$$

- The above Lemma is a consequence of Theorem 2.1 in reference

  "X. Fu, *Null controllability for the parabolic equation with a complex principal part, J. Func. Anal.,* **257** (2009), 1333–1354."

- The regularity of the coefficients b^{jk} can be improved to C^1.

  "X. Fu, X. Liu and X. Zhang, *Well-posedness and local controllability for quasilinear complex Ginzburg-Landau equations, Preprint.*"

By $\theta = e^{\ell},\ v = \theta z$, we have

$$\begin{aligned}\theta\mathcal{P}z &= \sum_{j,k=1}^{m}(b^{jk}v_j)_k + \sum_{j,k=1}^{m} b^{jk}\ell_j\ell_k v - 2\sum_{j,k=1}^{m} b^{jk}\ell_j v_k - \sum_{j,k=1}^{m}(b^{jk}\ell_j)_k v \\ &= I_1 + I_2,\end{aligned} \tag{4.9}$$

Hence, by (4.9), it is easy to see that

$$2|\theta\mathcal{P}z|^2 + \frac{1}{2}|I_1|^2 \geq \theta(\mathcal{P}z\overline{I_1} + \overline{\mathcal{P}z}I_1) = 2|I_1|^2 + (I_1\bar{I}_2 + I_2\bar{I}_1). \tag{4.10}$$

Note however that there is no boundary condition for z at $s = \pm 2$. Therefore, we need to introduce a cut-off function $\varphi(s) \in C_0^\infty(-b, b) \subset C_0^\infty(\mathbb{R})$ such that

$$\begin{cases} 0 \le \varphi(s) \le 1 & |s| < b, \\ \varphi(s) = 1, & |s| \le b_0, \end{cases} \qquad (1 < b_0 < b \le 2). \tag{4.11}$$

Put

$$\hat{z} = \varphi z. \tag{4.12}$$

Then, noting that φ does not depend on x, by (4.6), it follows

$$\begin{cases} \hat{z}_{ss} + \sum_{j,k=1}^{n} \left(a^{jk} \hat{z}_{x_j}\right)_{x_k} = \varphi_{ss} z + 2\varphi_s z_s + \varphi z^0 & \text{in } (-2, 2) \times \Omega, \\ \sum_{j,k=1}^{n} a^{jk} \hat{z}_{x_j} \nu_{x_k} - ia(x)\hat{z}_s = -ia(x)\varphi_s z + a(x)\varphi z^1 & \text{on } (-2, 2) \times \partial\Omega. \end{cases} \tag{4.13}$$

Step 2. Choose of the weight function.

$$\begin{cases} \psi(s,x) \triangleq \dfrac{\hat{\psi}(x)}{||\hat{\psi}||_{L^\infty(\Omega)}} + b^2 - s^2, & \phi = e^{\mu\psi}, \quad \theta = e^{\ell} = e^{\lambda\phi}, \\ \tilde{\psi}(s,x) \triangleq -\dfrac{\hat{\psi}(x)}{||\hat{\psi}||_{L^\infty(\Omega)}} + b^2 - s^2, & \tilde{\phi} = e^{\mu\tilde{\psi}}, \quad \tilde{\theta} = e^{\tilde{\ell}} = e^{\lambda\tilde{\phi}}, \end{cases}$$

where $\hat{\psi} \in C^2(\overline{\Omega})$ satisfying (see Fursikov– Imanuvilov (1994))

$$\begin{array}{ll} \hat{\psi} > 0 \text{ in } \Omega, & |\nabla\hat{\psi}| > 0 \text{ in } \overline{\Omega}, \\ \hat{\psi} = 0 \text{ on } \partial\Omega \setminus \Gamma_0, & \displaystyle\sum_{j,k=1}^{n} a^{jk}\hat{\psi}_j\nu_k \leq 0 \text{ on } \partial\Omega \setminus \Gamma_0. \end{array} \tag{4.14}$$

Step 3. Estimation of the energy terms and divergence terms.
It is easy to see that

$$\begin{cases} \ell_s = \lambda\mu\phi\psi_s, \quad \ell_j = \lambda\mu\phi\psi_j, \quad \ell_{js} = \lambda\mu^2\phi\psi_s\psi_j \\ \ell_{ss} = \lambda\mu^2\phi\psi_s^2 + \lambda\mu\phi\psi_{ss}, \quad \ell_{jk} = \lambda\mu^2\phi\psi_j\psi_k + \lambda\mu\phi\psi_{jk} \end{cases} \tag{4.15}$$

Integrating inequality (4.13) (with w replaced by $\hat{z}$) in $(-b, b) \times \Omega$, recalling that φ vanishes near $s = \pm b$, we end up with (recalling $\hat{z} = \varphi z,\ v = \theta\hat{z}$)

$$\begin{aligned} &\lambda\mu^2 \int_{-b}^{b}\int_\Omega \theta^2\phi(|\nabla z|^2 + |z_s|^2) dxds + \lambda^3\mu^4 \int_{-b}^{b}\int_\Omega \theta^2\phi^3|z|^2 dxds \\ &\le C \int_{-b}^{b}\int_\Omega e^{2\lambda\phi}|\varphi_{ss}z + 2\varphi_s z_s + \varphi z^0|^2 dxds \\ &\quad + Ce^{C\lambda} \int_{-b}^{b}\int_{\Gamma_0} (|z|^2 + |z_s|^2 + |z^1|^2) dxds. \end{aligned} \tag{4.16}$$

Step 4. End of the proof.

It is easy to check that

$$\begin{cases} \phi(s,\cdot) \geq 2 + e^{\mu}, & \text{for any } s \text{ satisfying } |s| \leq 1, \\ \phi(s,\cdot) \leq 1 + e^{\mu}, & \text{for any } s \text{ satisfying } b_0 \leq |s| \leq b. \end{cases} \tag{4.17}$$

enote $c_0 = 2 + e^{\mu} > 1$, and recalling that $b_0 \in (1, b)$. Fixing the parameter μ, one finds

$$\begin{aligned} &\lambda e^{2\lambda c_0} \int_{-1}^{1} \int_{\Omega} (|\nabla z|^2 + |z_s|^2 + |z|^2) dxds \\ &\leq Ce^{C\lambda} \left\{ \int_{-2}^{2} \int_{\Omega} |z^0|^2 dxds + \int_{-2}^{2} \int_{\partial\Omega} |z^1|^2 dxds + \int_{-2}^{2} \int_{\Gamma_0} (|z|^2 + |z_s|^2) dxds \right\} \\ &\quad + Ce^{2\lambda(c_0-1)} \int_{(-b,-b_0)\bigcup(b_0,b)} \int_{\Omega} (|z|^2 + |z_s|^2) dxds. \end{aligned} \tag{4.18}$$

We consider a linear evolution equation on Hilbert space H:

$$\begin{cases} \dfrac{du}{dt} = Au, \\ u(0) = u_0. \end{cases} \tag{5.1}$$

Let B be an unbounded operator, closed on H, with domain $D(B)$ dense in H.

Assume that

(H1). $A = iB$ generates a bounded C_0 semigroup $S(t) = e^{tA}$ on H.

(H2). $i\mathbb{R} \cap \sigma(A) = \emptyset$.

Recall that for any $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\,\lambda > 0$, $\lambda I - A$ is continuous bijective from $D(A)$ to H and

$$||(\lambda I - A)^{-1}||_{\mathcal{L}(D(A),H)} \leq |\mathrm{Re}\,\lambda|^{-1}. \tag{5.2}$$

Also, from Hille-Yosida theorem, for $k \in N$ and $s \geq 0$, the operator $\frac{e^{sA}}{(I-A)^k} = \frac{e^{isB}}{(I-iB)^k}$ is well-defined. We denote $R(\xi) = (\xi I - B)^{-1}$ the resolvent of B which is well-defined and holomorphic with respect to ξ for $\mathrm{Im}\,\xi < 0$. We need the following condition.

Assumption

There exists a positive function $g(t) : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

(i)
$$g'(t) > 0, \quad \lim_{t\to\infty} g(t) = \infty; \tag{5.3}$$

(ii) For some constant $c > 0$, there exists $c' > 0$ such that
$$e^{-ct^2} g(t) \leq e^{-c't}; \tag{5.4}$$

(iii) There exist $C > 0$ such that
$$||R(\xi)|| = ||(\xi I - B)^{-1}|| \leq Cg(|\mathrm{Re}\,\xi|). \tag{5.5}$$

Recalling that

$$\frac{du}{dt} = Au, \qquad u(0) = u_0.$$

The solution to (5.1) is

$$u(t) = S(t)u_0.$$

We say the solution of (5.1) decays at a rate of $h(t)$ if there exists a positive function $h(t)$ with $\lim_{t \to \infty} h(t) = 0$ such that

$$||u(t)||_H \leq h(t)||u_0||_{D(A)}, \qquad t > 0,$$

for all $u_0 \in D(A)$.

Denote $f(t)$ is inverse function of $g(t)$, we have the following result.

Theorem

For any $k > 0$, $k \in \mathbb{R}$, there exist two constants $C_k > 0$ and $\beta \geq 0$ such that

$$||e^{tA}u||_H \leq \frac{C_k}{[(\ln t)^{-\beta} f(t)]^k} ||u||_{D(A^k)} \tag{5.6}$$

for all $u \in D(A^k)$.

Here β depending on the form of $g(\cdot)$.

For the case $g(t) = e^t$, we choose $\beta = 0$, and get the solution of system (5.1) decay at a rate of $\frac{1}{\ln t}$.

Ideas of the proof.

Let $u \in H$. Let $\rho = \rho(t) \in C^\infty(\mathbb{R}_t)$ be such that

$$\begin{cases} \rho = 0 & \text{for } t < \frac{1}{3}, \\ \rho = 1 & \text{for } t > \frac{2}{3}. \end{cases}$$

Put

$$V(t) = e^{itB}u, \qquad U(t) = \frac{1}{(I - iB)^k}(\rho V). \tag{5.7}$$

It ie easy to check that $U(t)$ solves

$$\begin{cases} (\partial_t - iB)U = \rho_t \frac{1}{(I - iB)^k} V, \\ U(0) = 0. \end{cases} \tag{5.8}$$

Therefore,

$$U(t) = \int_0^t e^{i(t-s)B} \rho_t(s) \frac{1}{(I - iB)^k} V(s) ds \tag{5.9}$$

There exists standard method to prove the decay rate if we have the estimation of the resolvent operator. We refer to:

- N. Burq, Acta Math., (1998);
- H. Christianson;
- Z. Liu & B. Rao, Z. angew. Math. Phys., (2005).

Damping terms act on arbitrary sub-domain.

Let $a(\cdot) \in L^\infty(\Omega)$ be a non-negative bounded function such that

$$a(x) \geq a_0 > 0 \qquad \text{a.e. in } \omega \tag{6.1}$$

where ω is an open non-empty subset of Ω.

$$\begin{cases} u_{tt} - \sum_{j,k=1}^{n} (a^{jk} u_{x_j})_{x_k} + a(x) u_t = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ \sum_{j,k=1}^{n} a^{jk} u_{x_j} \nu_{x_k} = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ (u(0), u_t(0)) = (u^0, u^1) & \text{in } \Omega. \end{cases} \tag{6.2}$$

By means of the classical energy method, it is easy to check that

$$\frac{d}{dt}||(u, u_t)||_H^2 = -2\int_\Omega a(x)|u_t|^2 dx. \tag{6.3}$$

Hence, we conclude that the energy of every solution tends to zero as t tends to infinity, without any geometric conditions on the domain Ω.

Interpolation inequality for elliptic equations

$$\begin{cases} z_{ss} + \sum_{j,k=1}^{n} \left(a^{jk} z_j\right)_k + ia(x) z_s = z^0 & \text{in } X, \\ \sum_{j,k=1}^{n} a^{jk} z_j \nu_k = 0 & \text{on } \Sigma. \end{cases}$$

Theorem

There exists a constant $C > 0$ such that, for any $\varepsilon > 0$, any solution z of the above system satisfies

$$||z||_{H^1(Y)} \leq CRe^{CR/\varepsilon}\left[||z^0||_{L^2(X)} + ||z||_{L^2(Z)} + ||z_s||_{L^2(Z)}\right] + Ce^{-2/\varepsilon}||z||_{H^1(X)},$$

where

$$R = 1 + \sum_{j,k=1}^{n} ||a^{jk}||^2_{C^1(\overline{\Omega})}. \tag{6.4}$$

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Thank you!