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Symmetry of Ground States of p-Laplace Equations via the Moving Plane Method

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Abstract

In this paper we use the moving plane method to get the radial symmetry about a point $x_0 \in \mathbb{R}^N$ of the positive ground state solutions of the equation $-\text{div}\ (|Du|^{p-2}Du) = f(u)$ in $\mathbb{R}^N$, in the case $1 < p < 2$. We assume f to be locally Lipschitz continuous in $(0, +\infty)$ and nonincreasing near zero but we do not require any hypothesis on the critical set of the solution. To apply the moving plane method we first prove a weak comparison theorem for solutions of differential inequalities in unbounded domains.

1. Introduction

In this paper, we study the symmetry properties of positive solutions of the p-Laplace equation in $\mathbb{R}^N$, $N \geqq 2$, with the ground state condition at infinity, namely

$$\begin{gathered} -\Delta_p u(x) = f(u(x)) \quad \text{in } \mathbb{R}^N, \quad u > 0, \\ u(x) \to 0 \quad \text{as } |x| \to \infty \end{gathered} \tag{1.1}$$

for $1 < p < 2$, with $\Delta_p u = \text{div}(|Du|^{p-2}Du)$.

In the case $p = 2$, the radial symmetry of the solutions of (1.1) was proved first in [9] under certain assumptions on the behaviour of the solutions near infinity and for a class of C^1 nonlinearities. Later it was extended by LI [14] to fully nonlinear elliptic equations and by LI & NI [15] to general ground state solutions under the condition $f'(s) \leqq 0$, for s small. In all these papers the basic tool to get the symmetry results is the moving plane method of ALEXANDROV and SERRIN. This device, essentially based on maximum principles, goes back to ALEXANDROV and was first applied to the study of nonlinear differential equations by SERRIN [16]. Since then, the moving plane method has been widely used in many different problems, both in bounded and unbounded domains.

While trying to apply the moving plane method to problems involving the p-Laplace operator, $p \neq 2$, one encounters some serious difficulties. These are mainly due to the fact that the operator, for $p \neq 2$, is degenerate at the critical points of the solutions. In particular, comparison principles are not valid in the same form as for $p = 2$.

Nevertheless, some results through the moving plane method are available both for bounded domains and for solutions in $\mathbb{R}^N$. In [1], a symmetry theorem in the ball is obtained under the assumption that the solution has only one critical point. Later DAMASCELLI [4] proved some new comparison theorems from which he derived symmetry results in bounded domains again making some hypotheses on the critical set of a solution. Finally in [5], the authors prove a monotonicity and symmetry theorem, in the case $1 < p < 2$, without any assumption on the critical set of the solutions.

For problems in unbounded domains, nothing is known to our knowledge, except for the case of $\mathbb{R}^N$, i.e., problem (1.1). In [1] it is proved that if $f \in C^1[0, \infty)$, $f'(0) < 0$ and the solution u has only one critical point $0 \in \mathbb{R}^N$, then it is radially symmetric about 0, for any $p > 1$. Recently SERRIN & ZOU [17] obtained the same result but requiring f to be only locally Lipschitz continuous in $(0, +\infty)$ and nonincreasing near zero. Moreover, they consider C^1 nonnegative solutions of more general quasilinear equations and in particular study a class of quasilinear elliptic operators having a singularity at the origin, which includes as a special case, the p-Laplacian, $1 < p < 2$. In this case they get symmetry results under the weaker assumption that the set $\{x \in \mathbb{R}^N : |Du(x)| > 0\}$ is connected. All the above-mentioned papers use, as we already said, the moving plane method.

Another different approach to study symmetry properties of solutions of nonlinear problems involving the p-Laplace operators is through symmetrization methods. In [13] (for $p = N = 2$) and in [12] (for $p = N$) the symmetry theorem for solutions in the ball is obtained by using the Schwarz symmetrization and a Pohožaev-type identity. In [3], the author states two interesting symmetry results for problems in the ball and in $\mathbb{R}^N$ without any assumption on the critical set of the solutions. He uses a new rearrangement technique called continuous Steiner symmetrization.

In our paper we again use the moving plane method to prove that solutions u of (1.1) are radially symmetric about a point in $\mathbb{R}^N$, without making any hypothesis on the set where $Du = 0$. To be more precise we assume that

(H1) f is locally Lipschitz continuous in $(0, \infty)$,
(H2) there exists $s_0 > 0$ such that f is nonincreasing on $(0, s_0)$.

Our main result is

Theorem 1.1. *Under the assumptions* (H1) *and* (H2), *if* $u \in C^1(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)$ *is a solution of* (1.1), *for* $1 < p < 2$, *then* u *is radially symmetric about some point* $x_0 \in \mathbb{R}^N$, *i.e.,* $u = u(r)$, *with* $r = |x - x_0|$ *and* $u'(r) < 0$ *for all* $r > 0$.

The proof of this theorem is based on a weak comparison principle for solutions of differential inequalities in unbounded domains which extends a previous result of [4]. We follow quite closely the procedure of [5] which relies on simultaneously moving from infinity hyperplanes orthogonal to the directions in a neighbourhood of a fixed direction. We also should point out that an important tool (already used in [5]) to prove Theorem 1.1 is Proposition 5.1 which gives some geometrical properties of the critical set of a solution.

In the symmetry theorem for solutions of (1.1), proved in [3], the nonlinearity f need not be locally Lipschitz continuous but its growth must be controlled by a continuous function β satisfying the condition $\int_0^1 (s\beta(s))^{-1/p}\,ds = +\infty$. In particular, f can be only Hölder continuous with exponent $p-1$ in $[0,+\infty)$. Moreover, it is required to assume (H2) and

$$f(u)(.) \in L^1(\mathbb{R}^N) \tag{1.2}$$

or, alternatively, some a priori hypotheses on the decay of the solutions. Here we do not need (1.2). Further, by requiring f to be locally Lipschitz continuous only in $(0,+\infty)$ (as in [17]) we can allow any growth of f near zero. Yet we should point out that a stronger version of the growth condition used in [3], already appears in Proposition 1.3.2 of [8] where it is proved that it is necessary for the existence of radial positive solutions.

We also remark that in our paper, as well as in [3], the solutions are assumed to be in $C^1(\mathbb{R}^N)\cap W^{1,p}(\mathbb{R}^N)$ while in [17] the authors consider nonnegative solutions only in $C^1(\mathbb{R}^N)$. We conjecture that the condition $u \in W^{1,p}(\mathbb{R}^N)$ should not be necessary even with our approach.

Finally note that Theorem 1.1 also implies a regularity result since, from $Du \neq 0$ in $\mathbb{R}^N \setminus \{x_0\}$, by standard regularity results, we deduce that u belongs to $C^2(\mathbb{R}^N \setminus \{x_0\})$.

The paper is divided into five sections. In Section 2 we prove a Poincaré type inequality. This is used in Section 3 to get a weak comparison principle for solutions of differential inequalities in unbounded domains. In Section 4 we prove a partial symmetry result while Section 5 contains the important Proposition 5.1 together with the proof of Theorem 1.1.

2. Poincaré-Type Inequality

Let $C \subseteq \mathbb{R}^N$ and let f be a real-valued function on C. Define the potential

$$V_C f(x) = \int_C \frac{f(y)}{|x-y|^{N-1}}\,dy, \quad x \in \mathbb{R}^N,$$

whenever the integral exists. We have

Lemma 2.1. *Suppose that C and K are subsets of $\mathbb{R}^N$ that are of finite measure. If $f \in L^q(C)$, $1 \leqq q \leqq \infty$, then $V_C f \in L^q(K)$ with*

$$||V_C f||_{L^q(K)} \leqq N\omega_N \left(\frac{|C|}{\omega_N}\right)^{1/Nq'} \left(\frac{|K|}{\omega_N}\right)^{1/Nq} ||f||_{L^q(C)}. \tag{2.1}$$

Proof. The inequality (2.1) is derived in the proof of Lemma 2.2 [4] with some change of notations. □

For the case when the set C is unbounded, we have

Lemma 2.2. *Suppose that C is a subset of $\mathbb{R}^N$, possibly unbounded, and suppose that $K \subset \mathbb{R}^N$ is bounded with* $\operatorname{dist}(K, C) \geqq r > 0$. *If $f \in L^p(C)$ with $1 \leqq p < N$, then $V_C f \in L^\infty(K)$ and*

$$||V_C f||_{L^\infty(K)} \leqq \left(N\omega_N \, \frac{p-1}{N-p}\right)^{1/p'} \left(\frac{1}{r}\right)^{(N-p)/p} ||f||_{L^p(C)} \;\; if \;\; p > 1,$$
$$||V_C f||_{L^\infty(K)} \leqq \left(\frac{1}{r}\right)^{N-1} ||f||_{L^1(C)} \qquad \text{if } p = 1. \tag{2.2}$$

Hence $V_C f \in L^q(K)$, for any $1 \leqq q < \infty$ and

$$||V_C(f)||_{L^q(K)} \leqq \begin{cases} |K|^{1/q}\left(N\omega_N \dfrac{p-1}{N-p}\right)^{1/p'}\left(\dfrac{1}{r}\right)^{(N-p)/p} ||f||_{L^p(C)} & \textit{for } p > 1, \\ |K|^{1/q}\left(\dfrac{1}{r}\right)^{N-1} ||f||_{L^1(C)} & \textit{for } p = 1. \end{cases}$$

Proof. For each $x \in K$, the function $h_x : y \to \frac{1}{|x-y|^{N-1}}$ belongs to $L^q(C)$ for any $q > \frac{N}{N-1}$, since $\operatorname{dist}(x, C) \geqq r > 0$. If $1 < p < N$, then $p' = \frac{p}{p-1} > N' = \frac{N}{N-1}$, so that $h_x \in L^{p'}(C)$.

By Hölder's inequality, we get

$$\begin{aligned} |V_C f(x)| &\leqq \left(\int_C |f(y)|^p \, dy\right)^{1/p} \left(\int_C \frac{dy}{|x-y|^{(N-1)p'}}\right)^{1/p'} \\ &\leqq ||f||_{L^p(C)} \left(\int_{\mathbb{R}^N \setminus B(x,r)} \frac{dy}{|x-y|^{(N-1)p'}}\right)^{1/p'} \\ &= ||f||_{L^p(C)} \left(N\omega_N \int_r^\infty \rho^{(N-1)(1-p')} \, d\rho\right)^{1/p'} \\ &= ||f||_{L^p(C)} \left(N\omega_N \frac{(p-1)}{(N-p)}\right)^{1/p'} r^{(p-N)/p}. \end{aligned}$$

This immediately gives (2.2). $\square$

From the previous estimates and from the following representation formula (see [10, Lemma 7.14]) for all $u \in C_c^\infty(\mathbb{R}^N)$:

$$u(x) = \frac{1}{N\omega_N} \int_{\mathbb{R}^N} \frac{Du(y) \, . \, (x-y)}{|x-y|^N} \, dy, \tag{2.3}$$

we get the following Poincaré-type inequality:

Proposition 2.1. *Let $1 \leqq p < N$. Assume that $K \subset \mathbb{R}^N$ is a bounded set. For a positive number r, define $K_r = K + B(0,r)$, where $B(0,r)$ is the ball with center at the origin and radius r. If $K_r = A \cup B$ with A and B measurable, then there exists a constant $\alpha = \alpha(N, p, |K|)$ such that*

$$||u||_{L^q(K)} \leqq \alpha \left(|A|^{1/Nq'} ||Du||_{L^q(A)} + |B|^{1/Nq'} ||Du||_{L^q(B)} + \frac{1}{r^{(N-p)/p}} ||Du||_{L^p(\mathbb{R}^N \setminus K_r)} \right) \tag{2.4}$$

for every $u \in W^{1,q}_{\text{loc}}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)$ with $q \in (1, \infty)$.

Proof. Let us first take $u \in C_c^\infty(\mathbb{R}^N)$. By (2.3), we get

$$|u(x)| \leqq \frac{1}{N\omega_N} (V_A\ (|Du|)(x) + V_B\ (|Du|)(x) + V_C\ (|Du|)(x)),$$

where $C = \mathbb{R}^N \setminus K_r$. Hence

$$||u||_{L^q(K)} \leqq \frac{1}{N\omega_N} \big(||V_A(|Du|)||_{L^q(K)} + ||V_B(|Du|)||_{L^q(K)} + ||V_C(|Du|)||_{L^q(K)} \big).$$

From Lemma 2.1 and 2.2, we get

$$||u||_{L^q(K)} \leqq \left(\frac{|K|^{\frac{1}{q}}}{\omega_N} \right)^{1/N} \left(|A|^{1/Nq'}\ ||Du||_{L^q(A)}\ + |B|^{1/Nq'}\ ||Du||_{L^q(B)} \right) + (N\omega_N)^{-1/p} \left(\frac{p-1}{N-p} \right)^{1-(1/p)} |K|^{1/q} \left(\frac{1}{r} \right)^{(N-p)/p} ||Du||_{L^p(C)}. \tag{2.5}$$

so that (2.4) follows for $u \in C_c^\infty(\mathbb{R}^N)$. In the general case of $u \in W^{1,q}_{\text{loc}}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)$, we get (2.4) by density, taking a sequence $\{u_n\}$ in $C_c^\infty(\mathbb{R}^N)$ which converges to u in $W^{1,p}(\mathbb{R}^N)$ and in $W^{1,q}(K_r)$. (Such a sequence can be easily constructed using convolutions; see, for example, [7].) □

3. Comparison Principles

We start by proving a weak comparison principle for solutions of p-Laplace differential inequalities in unbounded domains.

Let u, v be two positive functions in $C^1(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)$ weakly satisfying

$$\begin{aligned} -\Delta_p v &\leqq\ f(v)\ \text{ in } \mathbb{R}^N, \\ -\Delta_p u &\geqq\ f(u)\ \text{ in } \mathbb{R}^N, \\ u(x), v(x) &\to\ 0\ \text{ as } |x| \to \infty \end{aligned} \tag{3.1}$$

where $p > 1$ and $f : (0, \infty) \to \mathbb{R}$ satisfies (H1) and (H2) of Section 1.

In the proof of the weak comparison principle we will use the following standard estimate whose proof can be found, for example, in Lemma 2.1 of [4].

$$(|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta') \quad (\eta - \eta') \geqq c(|\eta| + |\eta'|)^{p-2} \, |\eta - \eta'|^2 \tag{3.2}$$

for any $\eta, \eta' \in \mathbb{R}^N$.

Theorem 3.1. *Let $N \geqq 2$, $1 < p < 2$, and let u and v satisfy* (3.1). *Let K be a bounded set and assume that there exists an open set Ω such that $v(x) \in (0, s_0)$, for any $x \in \Omega \backslash K$. For any open set $\Omega' \subseteq \Omega$ with $v \leqq u$ on $\partial\Omega'$,*

(a) *if $\Omega' \cap K = \emptyset$, then $v \leqq u$* in *Ω',*

(b) *if $\Omega' \cap K \neq \emptyset$, then there exists a number $\rho > 0$ depending on $p, N, |K|, f$ and the $W^{1,p}(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N)$ norms of v and u such that, for any r with $(\frac{1}{r})^{(N-p)/2p} < \rho$ there exist $\delta_1, M > 0$ depending on r and also on $p, N, |K|, f$ such that*

$$v \leqq u \quad \text{in } \Omega'$$

provided that $\Omega' \cap K_r = A \cup B$ with $|A \cap B| = 0$, $M_B < M$ and $|A| < \delta$ for a number $\delta < \delta_1 M_{K_r}^{N(p-2)}$, where for any set T, $M_T = M_T(u, v) = \sup_T(|Du| + |Dv|)$ and $K_r = K + B(0, r)$ are as defined in Proposition 2.1.

Proof. Since $v \leqq u$ on $\partial\Omega'$ (i.e., $(v-u)^+ \in W_0^{1,p}(\Omega')$), for any $\varepsilon > 0$, the function $(v - u - \varepsilon)^+ \in W^{1,p}(\Omega')$ and has compact support, because of the behaviour of u and v at ∞. Thus this function can be used as a test function, yielding

$$\begin{aligned} \int_{\Omega' \cap [v \geqq u + \varepsilon]} & (|Dv|^{p-2} Dv - |Du|^{p-2} Du).(Dv - Du) \, dx \\ & \leqq \int_{\Omega' \cap [v \geqq u+\varepsilon]} (f(v) - f(u))(v - u - \varepsilon) \, dx. \end{aligned} \tag{3.3}$$

Now if $\Omega' \cap K = \emptyset$, then from (3.3), (3.2) and the monotonicity of f we get

$$\int_{\Omega'} |D(v - u - \varepsilon)^+|^2 \, (|Dv| + |Du|)^{p-2} dx \leqq 0,$$

from which, letting $\varepsilon \to 0$, we get $v \leqq u$ in Ω'.

If, instead, $\Omega' \cap K \neq \emptyset$ and $x \in D = (\Omega' \setminus K) \cap [v \geqq u + \varepsilon]$, then both $v(x)$ and $u(x)$ are smaller than s_0. Hence, by (H2), $(f(v(x)) - f(u(x)) \leqq 0$ in D. Using this, (3.2) and the local Lipschitz condition on f, from (3.3) we get

$$\int_{\Omega' \cap [v \geqq u+\varepsilon]} |Dv - Du|^2 \, (|Dv| + |Du|)^{p-2} dx \leqq \frac{L}{c} \int_{K \cap [v \geqq u+\varepsilon]} (v-u)(v-u-\varepsilon) \, dx$$

where L is the Lipschitz constant for f in $[s_0, ||v||_{L^\infty(K)} + ||u||_{L^\infty(K)}]$ and c comes from (3.2). Passing to the limit for $\varepsilon \to 0$ we find

$$\int_{[v \geqq u] \cap \Omega'} |Dv - Du|^2 \, (|Dv| + |Du|)^{p-2} \, dx \leqq \frac{L}{c} \int_K [(v - u)^+]^2 \, dx. \tag{3.4}$$

Now, for any $r > 0$, writing Ω' as $\Omega' = A \cup B \cup C$ with $A \cup B = \Omega' \cap K_r$ and $C = \Omega' \backslash K_r$, from (3.4) we obtain

$$\begin{aligned} M_{K_r}^{p-2} \int_A |D(v-u)^+|^2 dx + M_B^{p-2} \int_B |D(v-u)^+|^2\, dx \\ + \int_C |D(v-u)^+|^2 (|Dv| + |Du|)^{p-2}\, dx \\ \leqq \frac{L}{c} \int_K [(v-u)^+]^2\, dx. \end{aligned} \tag{3.5}$$

Next, using Hölder's inequality, we have

$$\begin{aligned} &\int_C |D(v-u)^+|^p dx \\ &= \int_C [|D(v-u)^+]^p\ (|Dv| + |Du|)^{(p-2)p/2} (|Dv| + |Du|)^{(2-p)p/2}\, dx \\ &\leqq \left(\int_C |D(v-u)^+|^2\ (|Dv| + |Du|)^{p-2} dx \right)^{p/2} \left(\int_C (|Du| + |Dv|)^p \right)^{(2-p)/2} \end{aligned}$$

so that

$$\begin{aligned} &\int_C |D(v-u)^+|^2 (|Dv| + |Du|)^{p-2} dx \\ &\geqq \left(\int_C |D(v-u)^+|^p dx \right)^{2/p} \left(\int_C (|Dv| + |Du|)^p \right)^{(p-2)/p}. \end{aligned} \tag{3.6}$$

Applying the Poincaré inequality (2.4) with $q = 2$, we estimate the right side of (3.5) by

$$\begin{aligned} &\frac{L}{c} \int_K [(v-u)^+]^2 dx \\ &\leqq \frac{L\alpha}{c} \left[|A|^{\frac{1}{N}} \int_A |D(v-u)^+|^2 dx + |K_r|^{1/N} \int_B |D(v-u)^+|^2\, dx \right. \\ &\quad \left. + \left(\frac{1}{r} \right)^{2(N-p)/p} \left(\int_C |D(v-u)^+|^p \right)^{2/p} \right]. \end{aligned} \tag{3.7}$$

By (3.5), (3.6) and (3.7) we have

$$\begin{aligned} &M_{K_r}^{p-2} \int_A |D(v-u)^+|^2\, dx + M_B^{p-2} \int_B |D(v-u)^+|^2\, dx \\ &\quad + \left(\int_C |D(v-u)^+|^p\, dx \right)^{2/p} \left[\int_C (|Dv| + |Du|)^p \right]^{(p-2)/p} \\ &\leqq \frac{L\alpha}{c} \left[|A|^{1/N} \int_A |D(v-u)^+|^2\, dx + |K_r|^{1/N} \int_B |D(v-u)^+|^2\, dx \right. \\ &\quad \left. + \frac{1}{r}^{(2(N-p))/p} \left(\int_C |D(v-u)^+|^p\, dx \right)^{2/p} \right]. \end{aligned} \tag{3.8}$$

Now fix $\rho = \frac{c}{L\alpha}\left[\int_{\mathbb{R}^N}(|Dv| + |Du|)^p\right]^{(p-2)/p}$ and take r with $(\frac{1}{r})^{(N-p)2/p} < \rho$. Thus if we set $\delta_1 = (\frac{c}{L\alpha})^N$ and choose any $\delta > 0$ and $M > 0$ such that $\delta < \delta_1 M_{K_r}^{N(p-2)}$ and $M^{p-2} > \frac{L\alpha}{c}|K_r|^{\frac{1}{N}}$, we deduce from (3.8) that if $|A| < \delta$ and $M_B < M$, then $(v-u)^+ = 0$ in Ω and hence the assertion (b) is proved. □

We end this section recalling some known results about solutions of differential inequalities involving the p-Laplace operator. We begin with a strong comparison principle whose proof can be found in [4].

Suppose that Ω is a domain in $\mathbb{R}^N$ and $u, v \in C^1(\Omega)$ weakly satisfy

$$\begin{aligned} -\Delta_p u &\geqq f(u) \quad \text{in } \Omega, \\ -\Delta_p v &\leqq f(v) \quad \text{in } \Omega, \end{aligned} \tag{3.9}$$

with $f : \mathbb{R} \to \mathbb{R}$ locally Lipschitz continuous.

Theorem 3.2. *Suppose that $1 < p < \infty$ and define $Z_v^u = \{x \in \Omega : Du(x) = Dv(x) = 0\}$. If $u \geqq v$ in Ω and there exists $x_0 \in \Omega \setminus Z_v^u$ with $u(x_0) = v(x_0)$, then $u \equiv v$ in the connected component of $\Omega \setminus Z_v^u$ containing x_0.*

Finally we recall a version of the strong maximum principle and Hopf's lemma for the p-Laplacian which is a particular case of a result proved in [18]. (See also Theorem 2.2, the strong maximum principle, in [4].)

Theorem 3.3. *Let Ω be a domain in $\mathbb{R}^N$ and suppose that $u \in C^1(\Omega)$, $u \geqq 0$ in Ω weakly satisfies*

$$-\Delta_p u + cu^q = g \geqq 0 \quad \text{in } \Omega \tag{3.10}$$

with $1 < p < \infty$, $q \geqq p-1$, $c \geqq 0$ and $g \in L^\infty_{\text{loc}}(\Omega)$. If $u \not\equiv 0$, then $u > 0$ in Ω. Moreover for any point $x_0 \in \partial\Omega$ where the interior sphere condition is satisfied and $u \in C^1(\Omega \cup \{x_0\})$ and $u(x_0) = 0$ the inequality $\frac{\partial u}{\partial s} > 0$ holds for any inward directional derivative. (*This means that if y approaches x_0 in a ball $B \subset \Omega$ that has x_0 on its boundary then* $\lim_{y \to x_0} \frac{u(y)-u(x_0)}{|y-x_0|} > 0$.)

4. A Partial Symmetry Result

Here, as an intermediate step towards proving Theorem 1.1, we use the moving plane method to get a first partial symmetry result. We start with some notations.

Let ν be a direction in $\mathbb{R}^N$, i.e, $|\nu| = 1$. Define for $\lambda \in \mathbb{R}$ the half space

$$\Sigma_\lambda^\nu = \{x \in \mathbb{R}^N : x.\nu < \lambda\},$$

and the hyperplane

$$T_\lambda^\nu = \{x \in \mathbb{R}^N : x.\nu = \lambda\}.$$

Let R_λ^ν be the reflection through T_λ^ν, i.e.,

$$R_\lambda^\nu(x) = x_\lambda^\nu = x + 2(\lambda - x.\nu)\nu$$

for any $x \in \mathbb{R}^N$. We also set

$$(\Sigma_\lambda^\nu)' = R_\lambda^\nu(\Sigma_\lambda^\nu).$$

For a function $u \in C^1(\mathbb{R}^N)$, we define the reflected function

$$u_\lambda^\nu(x) = u(x_\lambda^\nu) \quad \forall\, x \in \mathbb{R}^N,$$

and the sets

$$\begin{aligned} Z &= \{x \in \mathbb{R}^N : Du(x) = 0\}, \\ Z_\lambda^\nu &= \{x \in \Sigma_\lambda^\nu : Du(x) = Du_\lambda^\nu(x) = 0\}, \\ \Lambda(\nu) &= \{\lambda \in \mathbb{R} : u \geqq u_\mu^\nu \quad \text{in} \quad \Sigma_\mu^\nu \quad \forall\, \mu > \lambda\}. \end{aligned}$$

Then if $\Lambda(\nu) \neq \emptyset$, we consider

$$\lambda_0(\nu) = \inf \Lambda(\nu).$$

Finally let us fix a ball $K = B(P, R)$ centered in a point P belonging to $\Sigma_{\lambda_0(\nu)}^\nu$ with distance $d > 0$ from $T_{\lambda_0(\nu)}^\nu$ and radius $R > 0$ such that $u(x) < s_0$ (s_0 is defined in (H2) of Section 1) for any $x \in \mathbb{R}^N \setminus K$. Note that if $x \notin K$, then $x_\lambda^\nu \notin K$ for each $\lambda \geqq \lambda_0(\nu) - d$.

Then define the set $K_r = K + B(0, r)$, with $r < \rho$, ρ as given in Theorem 3.1, depending on the set K and consider the numbers δ_1 and M which appear in the same theorem.

Theorem 4.1. *Let $u \in C^1(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)$ be a weak solution of* (1.1) *with $1 < p < 2$ and f satisfying* (H1) *and* (H2). *Then $\Lambda(\nu) \neq \emptyset$ for any direction ν and there exists at least one connected component C^ν of $\Sigma_{\lambda_0(\nu)}^\nu \setminus Z_{\lambda_0(\nu)}^\nu$ such that $C^\nu \cap K_r \neq \emptyset$ and*

$$u \equiv u_{\lambda_0(\nu)}^\nu \quad in \quad C^\nu. \tag{4.1}$$

Moreover,

$$Du(x) \neq 0 \quad \forall\, x \in C^\nu, \quad Du(x) = 0 \quad \forall\, x \in \partial C^\nu \setminus T_{\lambda_0(\nu)}^\nu, \tag{4.2}$$

$$u > u_\lambda^\nu \quad in \quad \Sigma_\lambda^\nu \setminus Z_\lambda^\nu, \quad \forall\, \lambda, \quad \lambda_0(\nu) < \lambda, \tag{4.3}$$

$$\frac{\partial u}{\partial \nu}(x) < 0 \quad \forall\, x \in (\Sigma_{\lambda_0(\nu)}^\nu)' \setminus Z. \tag{4.4}$$

Proof. We divide the proof into three steps.

Step 1. We start by proving that $\Lambda(\nu) \neq \emptyset$ and that it actually contains an interval $(a, +\infty)$. Since $u(x) \to 0$ as $|x| \to \infty$, we can find a ball $B_a = B(0, a)$ with centre at the origin and radius $a > 0$ sufficiently large such that $\max_{\mathbb{R}^N \setminus B_a} u < s_0$.

Therefore if $\lambda > a$, the function $v = u_\lambda^\nu$ takes its values $v(x)$ in the interval $(0, s_0)$, for any $x \in \Sigma_\lambda^\nu$. Since $u \equiv v$ on $\partial \Sigma_\lambda^\nu$, applying the easy form of the weak comparison principle (part (a) of Theorem 3.1 with $\Omega = \Omega' = \Sigma_\lambda^\nu$) we get $u \geqq v$ in Σ_λ^ν so that $\lambda \in \Lambda(\nu)$ for any $\lambda > a$.

Step 2. Let us note that by the behaviour of u at infinity the set $\Lambda(\nu)$ is bounded from below; hence $\lambda_0 = \inf \Lambda(\nu) > -\infty$. Now, at the minimal position $\lambda_0(\nu)$, either $u \equiv u^\nu_{\lambda_0(\nu)}$ in the whole half space $\Sigma^\nu_{\lambda_0(\nu)}$, which of course proves the theorem, or there exists a point $x_0 \in \Sigma^\nu_{\lambda_0(\nu)}$ such that $u(x_0) > u^\nu_{\lambda_0(\nu)}(x_0)$. Let us assume that this is the case and proceed. Arguing by contradiction we suppose that there do not exist connected components of $\Sigma^\nu_{\lambda_0(\nu)} \setminus Z^\nu_{\lambda_0(\nu)}$ where $u \equiv u^\nu_{\lambda_0(\nu)}$ which intersect K_r. Thus, by the strong comparison principle (Theorem 3.2) we get $u > u^\nu_{\lambda_0(\nu)}$ on $U \setminus Z^\nu_{\lambda_0(\nu)}$ with $U = K_r \cap \Sigma^\nu_{\lambda_0(\nu)}$.

Then, let us choose an open set O such that $(U \cap Z^\nu_{\lambda_0(\nu)}) \subset O \subset \Sigma^\nu_{\lambda_0(\nu)}$ and $M_{O,\lambda_0} = \sup_O(|Du| + |Du^\nu_{\lambda_0(\nu)}|) < \frac{1}{2}M$ (this is possible since $Du = Du^\nu_{\lambda_0(\nu)} = 0$ in $Z^\nu_{\lambda_0(\nu)}$).

We also fix a compact set $S \subset U$ such that $|U \setminus S| < \frac{1}{2}\delta$ where $\delta \leqq \delta_1(M')^{N(P-2)}$, $M' = \sup_{|\lambda-\lambda_0(\nu)|<d}(\sup_{x\in K_r}(|Du| + Du^\nu_\lambda|)$. If $S \setminus O \neq \emptyset$, then the function $u - u^\nu_{\lambda_0(\nu)}$ is positive there, and since $S \setminus O$ is compact we have that

$$\min_{S\setminus O}(u - u^\nu_{\lambda_0(\nu)}) = m > 0.$$

By continuity there exists $\varepsilon > 0 (\varepsilon < d)$ such that for $\lambda_0(\nu) - \varepsilon < \lambda < \lambda_0(\nu)$ we have $|(K_r \cap \Sigma^\nu_\lambda) \setminus S| < \delta$, $\quad M_{O,\lambda} = \sup_O(|Du| + |Du^\nu_\lambda|) < M$ and $u^\nu_\lambda(x) < s_0$ for any $x \in \Sigma^\nu_\lambda \setminus K$. Moreover, if $S \setminus O \neq \emptyset$,

$$u - u^\nu_\lambda > \tfrac{1}{2}m > 0 \quad \text{in } S \setminus O \tag{4.5}$$

(in particular $u - u^\nu_\lambda > 0$ on $\partial(S \setminus O)$). Moreover, for such values of λ we have that $u \geqq u^\nu_\lambda$ on the boundary of the set $\Omega' = \Sigma^\nu_\lambda \setminus (S \setminus O)$, because $\partial\Omega' = T^\nu_\lambda \cup \partial(S \setminus O)$. Since $\Omega' \cap K_r = A \cup B$ with $A = (\Sigma^\nu_\lambda \cap K_r) \setminus S$ and $B = S \cap O$ and $|A| < \delta$, $M_B < M$, we can apply the weak comparison principle (Theorem 3.1 with $\Omega = \Sigma^\nu_\lambda$) to the functions u and u^ν_λ in Ω' to get

$$u \geqq u^\nu_\lambda \quad \text{in} \quad \Omega' \quad \text{for any} \quad \lambda_0(\nu) - \varepsilon < \lambda < \lambda_0(\nu). \tag{4.6}$$

Since $\Sigma^\nu_\lambda = \Omega' \cup (S \setminus O)$, from (4.5) and (4.6) we get a contradiction to the definition of $\lambda_0(\nu)$. Hence we have proved the first part of the assertion.

Step 3. Finally we prove (4.2)–(4.4). If C^ν is a connected component of $\Sigma^\nu_{\lambda_0(\nu)} \setminus Z^\nu_{\lambda_0(\nu)}$ where (4.1) holds, then the inequality $|Du| + |Du^\nu_{\lambda_0(\nu)}| > 0$ in C^ν is equivalent to $|Du| > 0$ in C^ν. Thus $\partial C^\nu \subset T^\nu_{\lambda_0(\nu)} \cup Z$ and (4.2) follows.

To prove (4.3) we argue as in [5] (Step 3 of Theorem 3.1 there) and observe that it is enough to show that

$$u > u^\nu_\lambda \quad \text{in } \Sigma^\nu_\lambda \setminus Z \quad \text{if} \quad \lambda > \lambda_0(\nu). \tag{4.7}$$

In fact, if (4.3) is false and $u(x_0) = u^\nu_\lambda(x_0)$ for a point x_0 in $\Sigma^\nu_\lambda \setminus Z^\nu_\lambda$, then $u \equiv u^\nu_\lambda$ in the component of $\Sigma^\nu_\lambda \setminus Z^\nu_\lambda$ to which x_0 belongs and this implies that both $|Du(x_0)|$ and $|Du^\nu_{\lambda_0(\nu)}(x)|$ are not zero, i.e., $x_0 \in \Sigma^\nu_\lambda \setminus Z$ so that (4.7) does not hold.

To get (4.7) we assume, for simplicity, that $\nu = e_1 = (1, 0, \dots, 0)$ and write coordinates in $\mathbb{R}^N$ as $x = (y, z)$ with $y \in \mathbb{R}$, $z \in \mathbb{R}^{N-1}$ and we omit the superscript $\nu = e_1$ in Σ^ν_λ, u^ν_λ, etc.

Suppose, for contradiction, that there exist $\mu > \lambda_0(\nu)$ and $P = (y_0, z_0) \in \Sigma_\mu \setminus Z$ such that $u(P) = u_\mu(P) = u(P_\mu)$. Then, by the strong comparison principle (see Theorem 3.2), $u \equiv u_\mu$ in the component C of $\Sigma_\mu \setminus Z$ to which P belongs. Thus we can find a neighbouring point $Q = (y, z_0)$ with $y < y_0$ and $u(Q) = u(Q_\mu)$.

Now choose $\lambda_1 \in (\lambda_0, \mu)$ such that $Q_{\lambda_1} = P_\mu$. Then

$$u(Q) \geqq u(Q_{\lambda_1}) = u(P_\mu) \geqq u(Q_\mu) = u(Q)$$

because $u \geqq u_{\lambda_1}$ in Σ_{λ_1} and u is decreasing in $(\Sigma_\mu)'$. Thus $u(y, z_0) = u(P_\mu)$ for points (y, z_0) near $P_\mu = (2\mu - y_0, z_0)$ with $y > 2\mu - y_0$. Then we define

$$y_1 = \sup\{y > 2\mu - y_0 \ : \ u(y, z_0) = u(P_\mu)\}.$$

If $y_1 < \infty$, we take $\tilde{\lambda} = \frac{1}{2}[(2\mu - y_0) + y_1]$ so that $(y_1, z_0) = (P_\mu)_{\tilde{\lambda}}$ and we have $u_{\tilde{\lambda}}(P_\mu) = u(y_1, z_0) = u(P_\mu)$ by continuity. Therefore, since $Du(P_\mu) \neq 0$ (because $P \notin Z$), we can apply the strong comparison principle, claiming that $u \equiv u_{\tilde{\lambda}}$ in the connected component of $\Sigma_{\tilde{\lambda}} \setminus Z_{\tilde{\lambda}}$ to which P_μ belongs. Repeating the previous arguments, with μ and P replaced by $\tilde{\lambda}$ and P_μ we get the existence of points (y, z_0) with $y > y_1$ and $u(y, z_0) = u(y_1, z_0) = u(P_\mu)$, contradicting the definition of y_1. Thus $y_1 = \infty$, which is not possible since $u(x) \to 0 \quad$ as $|x| \to \infty$. This proves that

$$u > u_\lambda^\nu \quad \text{in} \quad \Sigma_\lambda^\nu \setminus Z_\lambda^\nu \quad \text{for} \quad \lambda > \lambda_0.$$

The final assertion (4.4) follows as in [5] from (4.3) and the usual Hopf lemma for strictly elliptic operators. In fact, if we again assume that $\nu = e_1$ if $x = (\lambda, z) \in (\Sigma_{\lambda_0})' \setminus Z$, then $\lambda > \lambda_0$ and $Du(x) \neq 0$. In a ball $B = B_r(x)$ we have that $|Du| \geqq \varepsilon > 0$ so that $|Du|, |Du_\lambda^\nu| \geqq \varepsilon > 0$ in $B \cap (\Sigma_\lambda)$. This implies that $u \in C^{2,\alpha}(B)$ and the difference $u - u_\lambda$ satisfies an equation of the type $L(u - u_\lambda) = 0$ where L is a linear strictly elliptic operator (see [16]). On the other hand, we have by (4.3) that $u - u_\lambda > 0$ in $B \cap \Sigma_\lambda$ while $u(x) = u_\lambda(x)$ because x belongs to T_λ. Hence by the usual Hopf lemma we get $0 > \frac{\partial(u - u_\lambda)}{\partial x_1}(\lambda, z) = 2\frac{\partial u}{\partial x_1}(x)$, which implies (4.4). □

5. Proof of the Symmetry Result

We start with a proposition which tells us how the set Z of critical points of a solution of (1.1) can intersect the limiting half space $(\Sigma_{\lambda_0(\nu)}^\nu)'$. It is the analogue of a result of [5], extended for the case of unbounded domains. We include the proof here for the reader's convenience. This proposition along with Theorem 4.1 will help us to prove the symmetry result arguing by contradiction.

Proposition 5.1. *Suppose that $u \in C^1(\mathbb{R}^N)$ is a weak solution of* (1.1) *for* $1 < p < 2$. *For any direction ν, the half space $(\Sigma_{\lambda_0(\nu)}^\nu)'$ cannot contain a subset Γ of Z on which u = constant and whose projection on the hyperplane $T_{\lambda_0(\nu)}^\nu$ contains a set open in $T_{\lambda_0(\nu)}^\nu$.*

Proof. For simplicity, take $\nu = e_1$ and write $x = (y, z)$, $y \in \mathbb{R}$, $z \in \mathbb{R}^{N-1}$. As usual, we omit the superscript $\nu = e_1$ in the notation. To argue by contradiction, assume that $(\Sigma_{\lambda_0})'$ contains a subset Γ of Z on which $u = \alpha > 0$ and that there exists $\gamma > 0$ and $z_0 \in \mathbb{R}^{N-1}$ such that for each point $(\lambda_0, z) \in T_{\lambda_0}$ with $z \in \omega = \{z \in \mathbb{R}^{N-1} : |z - z_0| < \gamma\}$, there exists $(y, z) \in \Gamma$ with $y > \lambda_0$.

Let S be the intersection of the cylinder $\mathbb{R} \times \omega$ with $(\Sigma_{\lambda_0})'$. Define the "right" and "left" part of this cylinder (with respect to Γ) as

$$S_r = \{(y, z) : z \in \omega,\, y \in (\sigma_1(z), \bar{\lambda})\},$$
$$S_l = \{(y, z) : z \in \omega,\, y \in (\lambda_0, \sigma_1(z))\},$$

where

$$\sigma_1(z) = \inf\{y : (y', z) \notin \overline{\Gamma} \text{ for any } y' > y\}$$

and $\bar{\lambda} > \lambda_0$ is such that

$$0 < u(\bar{\lambda}, z) < \alpha - \varepsilon \quad \forall\, z \in \omega,$$

for some $\varepsilon > 0$. Note that since $u(x) \to 0$ when $|x| \to \infty$ and $u = \alpha > 0$ on $\overline{\Gamma}$, the set $\{y : (y', z) \notin \overline{\Gamma} \quad \text{for any} \quad y' > y\}$ is nonempty and therefore $\lambda_0 < \sigma_1(z) < \infty$. For the same reason $\bar{\lambda}$ is well defined.

We consider two cases.

Case 1. $f(\alpha) \leqq 0$. In this case, we have $u(x) \leqq \alpha$ in S_r and $u \not\equiv \alpha$ in S_r because u is nonincreasing in the x_1 direction in $(\Sigma_{\lambda_0})'$ and $u(\bar{\lambda}, z) < \alpha - \varepsilon$, for all $z \in \omega$. The function $u - \alpha$ satisfies

$$-\Delta_p(u - \alpha) + L(u - \alpha) = f(u) - f(\alpha) + L(u - \alpha) + f(\alpha) \leqq 0$$

in S_r, where L is the Lipschitz constant corresponding to f on $[\delta, ||u||_{L^\infty(\mathbb{R}^N)}]$ for some suitable $\delta > 0$. Then by the strict maximum principle (Theorem 3.3), we get $u < \alpha$ in S_r. Observe that since f is locally Lipschitz continuous on $(0, \infty)$, there exists a constant $L = L(\delta)$ such that $f(s) + Ls$ is nondecreasing in $[\delta, ||u||_{L^\infty(\mathbb{R}^N)}]$. Since for some point $x' \in \partial S_r \cap \overline{\Gamma}$ the interior sphere condition is satisfied (see the argument in [5], Proposition 3.1), applying Hopf's lemma at that point, we obtain $Du(x') \not\equiv 0$, which is a contradiction because $Du(x) \equiv 0$ on $\overline{\Gamma}$.

Case 2. $f(\alpha) > 0$. In this case, we consider the "left" part S_l and we have $u \geqq \alpha$ on S_l, by the monotonicity of u in $(\Sigma_{\lambda_0})'$. If $u \equiv \alpha$ on S_l, then the equation cannot hold unless $f(\alpha) = 0$. Thus $u \not\equiv \alpha$ on S_l and by exploiting the local Lipschitz continuity of f, we get

$$-\Delta_p(u - \alpha) + L(u - \alpha) = f(u) - f(\alpha) + L(u - \alpha) + f(\alpha) > 0,$$

for some suitable constant $L > 0$. Then by the strict maximum principle, $u > \alpha$, and as before at some point $x' \in \partial S_l \cap \overline{\Gamma}$ Hopf's lemma gives a contradiction. Thus the proposition follows. □

Remark 5.1. If the critical set Z of u were regular, then, in particular, the boundary of the connected component C^ν, given by Theorem 4.1, would be regular. Thus u would be constant on ∂C^ν, and exploiting Proposition 5.1, we could prove that $u = u^\nu_{\lambda_0(\nu)}$ in $\Sigma^\nu_{\lambda_0(\nu)}$ (see Remark 3.1 in [5] for a more detailed discussion on this). Since, in general, we do not have a priori information on the regularity of Z, we proceed as in [5] to prove the symmetry of u.

In order to show that for any direction ν, $u \equiv u^\nu_{\lambda_0(\nu)}$ on $\Sigma^\nu_{\lambda_0(\nu)}$ at the limiting position, we argue by contradiction as in [5]. If $u \not\equiv u^\nu_{\lambda_0(\nu)}$, then the component C^ν given by Theorem 4.1 can be used to get Γ, a subset of Z, on which $u =$ constant and whose projection on $T^\nu_{\lambda_0(\nu)}$ contains an open set.

We follow closely the procedure of [5], using the method of simultaneously moving hyperplanes orthogonal to directions close to ν. For the reader's convenience we repeat the main steps of the proof leaving out some technical details that can be found in [5]. We start with a topological lemma whose simple proof is contained in [5] (Corollary 4.1).

Proposition 5.2. *Let A and B be open connected sets in a topological space and assume that $A \cap B \neq \emptyset$, and $A \not\equiv B$. Then $(\partial A \cap B) \cup (\partial B \cap A) \neq \emptyset$.*

Given a direction ν, define $\mathscr{F}_\nu$ as the collection of the connected components C_ν of $(\Sigma^\nu_{\lambda_0(\nu)} \backslash Z^\nu_{\lambda_0(\nu)})$ such that $C^\nu \cap K_r \neq \emptyset$ (where K_r is the set defined in the previous section) and $u \equiv u^\nu_{\lambda_0(\nu)}$ in C^ν. By (4.2), we know that $Du \neq 0$ in C^ν and $Du = 0$ on $(\partial C^\nu \backslash T^\nu_{\lambda_0(\nu)})$. We need to introduce a symmetrized version $\tilde{\mathscr{F}}_\nu$ of $\mathscr{F}_\nu$. If $Du(x) = 0$ for all x on ∂C^ν, then define $\tilde{C}^\nu = C^\nu$. If not, there are points $x \in (\partial C^\nu \cap T^\nu_{\lambda_0(\nu)})$ such that $Du(x) \neq 0$. In such cases, define

$$\tilde{C}^\nu = C^\nu \cup R^\nu_{\lambda_0(\nu)}(C^\nu) \cup \{x \in (\partial C^\nu \cap T^\nu_{\lambda_0(\nu)}) : Du(x) \neq 0\}.$$

Let $\tilde{\mathscr{F}}_\nu$ be the collection of all such $\tilde{C}^\nu$s. Define

$$\mathscr{B}_\nu = \{B_\nu = C^\nu \cap K_r : C^\nu \in \mathscr{F}_\nu\}$$

and define $\tilde{\mathscr{B}}_\nu$ similarly.

It is important to note that if ν_1 and ν_2 are any two directions, then for $C^{\nu_1} \in \mathscr{F}_{\nu_1}$ and $C^{\nu_2} \in \mathscr{F}_{\nu_2}$, we have either $\tilde{C}^{\nu_1} \equiv \tilde{C}^{\nu_2}$ or $\tilde{C}^{\nu_1} \cap \tilde{C}^{\nu_2} = \emptyset$. In fact, if $\tilde{C}^{\nu_1} \cap \tilde{C}^{\nu_2} \neq \emptyset$ and $\tilde{C}^{\nu_1} \not\equiv \tilde{C}^{\nu_2}$, then by Proposition 5.2, either $\partial\tilde{C}^{\nu_1} \cap \tilde{C}^{\nu_2}$ or $\tilde{C}^{\nu_1} \cap \partial\tilde{C}^{\nu_2}$ is nonempty and this is not possible since $Du \neq 0$ in $\tilde{C}^{\nu_i}$, and $Du = 0$ on $\partial\tilde{C}^{\nu_i}$, $i = 1, 2$. Of course, the same conclusion holds for $\tilde{B}^{\nu_1}$ and $\tilde{B}^{\nu_2}$ also.

Proof of Theorem 1.1. Given a direction ν, define

$$I_\eta(\nu) = \{\mu \in \mathbb{R}^N : |\mu| = 1, |\mu - \nu| < \eta\}.$$

We want to prove now that, for any direction ν, u is symmetric with respect to the hyperplane $T^\nu_{\lambda_0(\nu)}$, i.e., $u \equiv u^\nu_{\lambda_0(\nu)}$ in $\Sigma^\nu_{\lambda_0(\nu)}$.

Arguing by contradiction, suppose now that ν_0 is a direction such that $u \not\equiv u^{\nu_0}_{\lambda_0(\nu_0)}$ on $\Sigma^{\nu_0}_{\lambda_0(\nu_0)}$. Then by Theorem 4.1, we know that the equality $u \equiv u^{\nu_0}_{\lambda_0(\nu_0)}$

has to hold in some component C^{ν_0} such that $C^{\nu_0} \cap K_r \neq \emptyset$. Hence $\mathscr{F}_{\nu_0} \neq \emptyset$ and it contains at most countably many components of $(\Sigma^{\nu_0}_{\lambda_0(\nu_0)} \setminus Z^{\nu_0}_{\lambda_0(\nu_0)})$ and so does $\mathscr{B}_{\nu_0}$. Further, if $\mathscr{B}_{\nu_0} = \{B_i^{\nu_0}, i \in I \subseteq \mathbb{N}\}$, then

$$\sum_{i \in I} |B_i^{\nu_0}| \leqq |K_r| < \infty,$$

so that we can choose $n_0 \geqq 1$ for which

$$\sum_{i=n_0+1}^{\infty} |B_i^{\nu_0}| < \frac{\delta}{6}$$

where $\delta > 0$ will be fixed later in a suitable way.

If I is finite, let n_0 be its cardinality. Denote $U^{\nu_0}_{\lambda_0(\nu_0)} = \Sigma^{\nu_0}_{\lambda_0(\nu_0)} \cap K_r$. Let us choose a compact set $S_0 \subset U^{\nu_0}_{\lambda_0(\nu_0)} \setminus (\cup_{i \in I} B_i^{\nu_0})$ so that

$$|U^{\nu_0}_{\lambda_0(\nu_0)} \setminus (\cup_{i \in I} B_i^{\nu_0}) \setminus S_0| < \frac{\delta}{6}$$

and n_0 compact sets $S_i \subset B_i^{\nu_0}$, $1 \leqq i \leqq n_0$, such that

$$|B_i^{\nu_0} \setminus S_i| < \frac{\delta}{6n_0}, \; i = 1, \ldots, n_0.$$

Setting $S = \cup_{i=0}^{n_0} S_i$, we have

$$|U^{\nu_0}_{\lambda_0} \setminus S| < \frac{3\delta}{6} = \frac{\delta}{2}.$$

Let $O = \{x \in U^{\nu_0}_{\lambda_0(\nu_0)} : |Du(x)| + |Du^{\nu_0}_{\lambda_0(\nu_0)}(x)| < \frac{1}{2}M\}$, where M is the constant given by Theorem 3.1. Then $(S_0 \setminus O)$ is compact and $u > u^{\nu_0}_{\lambda_0(\nu_0)}$ there and hence there exists $m > 0$ such that

$$u - u^{\nu_0}_{\lambda_0(\nu_0)} \geqq m > 0 \quad \text{on} \quad S_0 \setminus O.$$

By continuity with respect to λ and ν, there are ε_0 and $\eta_0 > 0$ such that if $|\lambda - \lambda_0(\nu_0)| < \varepsilon_0$ and $|\nu - \nu_0| < \eta_0$, then

$$M_O^{\nu,\lambda} := \sup_O \{|Du| + |Du^\nu_\lambda|\} < M, \tag{5.1}$$

$$S_i \subset U^\nu_\lambda := \Sigma^\nu_\lambda \cap K_r, \, 0 \leqq i \leqq n_0, \tag{5.2}$$

$$|U^\nu_\lambda \setminus \cup_{i=0}^{n_0} S_i| < \delta \tag{5.3}$$

$$u - u^\nu_\lambda \geqq \tfrac{1}{2}m > 0 \quad \text{in } S_0 \setminus O. \tag{5.4}$$

We divide the rest of the proof of Theorem 1.1 into four steps:

Step 1. $\lambda_0(\nu)$ is a continuous function of ν at ν_0.

Proof. Fix $\varepsilon, 0 < \varepsilon < \varepsilon_0$. By definition of $\lambda_0(\nu_0)$, there exist $\lambda \in (\lambda_0(\nu_0) - \varepsilon, \lambda_0(\nu_0))$ and $x \in \Sigma_\lambda^{\nu_0}$ such that $u(x) < u_\lambda^{\nu_0}(x)$. By continuity there is $\delta_2 > 0$ such that $x \in \Sigma_\lambda^\nu$ and $u(x) < u_\lambda^\nu(x)$ for every $\nu \in I_{\delta_2}(\nu_0)$. Hence for all $\nu \in I_{\delta_2}(\nu_0)$,

$$\lambda_0(\nu_0) - \varepsilon < \lambda < \lambda_0(\nu).$$

Now we claim that there is $\delta_3 > 0$ such that $\lambda_0(\nu) < \lambda_0(\nu_0) + \varepsilon$ for any $\nu \in I_{\delta_3}(\nu_0)$. If this were not true, then there would be a sequence $\{\nu_n\}$ of directions such that $\nu_n \to \nu_0$ and $\lambda_0(\nu_n) \geqq \lambda_0(\nu_0) + \varepsilon$ for all n. By Step 1 of the proof of Theorem 4.1 we have that $\lambda_0(\nu) < a$ for any direction ν. Thus the sequence $\lambda_0(\nu_n)$ is bounded and hence up to a subsequence, converges to a number $\bar{\lambda} \geqq \lambda_0(\nu_0) + \varepsilon$. Then by (4.3) of Theorem 4.1, we have

$$u > u_{\bar{\lambda}}^{\nu_0} \quad \text{in } \Sigma_{\bar{\lambda}}^{\nu_0} \setminus Z_{\bar{\lambda}}^{\nu_0}. \tag{5.5}$$

Arguing as in Step 2 of Theorem 4.1, we can construct an open set $O \subset \Sigma_{\bar{\lambda}}^{\nu_0}$ and a compact set $S \subset U_{\bar{\lambda}}^{\nu_0}$ such that

$$Z_{\bar{\lambda}}^{\nu_0} \cap K_r \subset O \quad , M_0^{\nu_0, \bar{\lambda}} < \tfrac{1}{2} M \quad , |U_{\bar{\lambda}}^{\nu_0} \setminus S| < \tfrac{1}{2}\delta,$$

and if $S \setminus O \neq \emptyset$, then

$$u - u_{\bar{\lambda}}^{\nu_0} > 0$$

in $S \setminus O$, where $\delta < \delta_1 (M')^{N(p-2)}$, $M' = \sup_{|\lambda - \lambda_0(\nu_0)| < d} (\sup_{x \in K_r} (|Du| + |Du_\lambda^\nu|))$ and M as given in Theorem 3.1 and d as chosen in Section 4. Then by continuity, there are $\eta, \sigma > 0$ such that

$$M_0^{\nu, \lambda} < M, \quad |U_\lambda^\nu \setminus S| < \delta \quad \text{and} \quad \mathrm{u} - \mathrm{u}_\lambda^\nu > 0$$

in $S \setminus O$, for all $\nu \in I_\eta(\nu_0)$ and $\lambda \in (\bar{\lambda} - \sigma, \bar{\lambda} + \sigma)$. For such values of ν and λ, applying Theorem 3.1 as in Step 2 of Theorem 4.1 with $\Omega = \Sigma_\lambda^\nu$ and $\Omega' = \Sigma_\lambda^\nu \setminus (S \setminus O)$, we get $u \geqq u_\lambda^\nu$ in Ω_λ^ν. (Note that since by definition the set $K = B(P, R)$ has its centre on the hyperplane $T_{\lambda_0(\nu_0) - d}^{\nu_0}$, the condition $v = u_\lambda^\nu < s_0$ in $\Omega \setminus K = \Sigma_\lambda^\nu \setminus K$, necessary to apply Theorem 3.1, is satisfied for any $\lambda > \lambda_0(\nu_0) - d$ and any ν close to ν_0). In particular, $u \geqq u_\lambda^\nu$ for ν_n and $\lambda_0(\nu_n) - \gamma$, for some n large and γ small, contradicting the definition of $\lambda_0(\nu_n)$. Thus it follows that for each $\varepsilon > 0$, there is $\eta > 0$ such that if $\nu \in I_\eta(\nu_0)$, then

$$\lambda_0(\nu_0) - \varepsilon < \lambda_0(\nu) < \lambda_0(\nu_0) + \varepsilon.$$

Step 2. There exists a direction ν_1 near ν_0 and an index $i_1 \in \{1, \ldots, n_0\}$ such that for any ν in a small neighbourhood of ν_1, the set $\tilde{B}_{i_1}^{\nu_0} \in \tilde{\mathscr{B}}_\nu$.

Proof. We first prove that for each $\nu \in I_{\eta(\varepsilon_0)}(\nu_0)$, there exists $i \in \{1, 2, \ldots, n_0\}$ such that $\tilde{B}_i^{\nu_0} \in \tilde{\mathscr{B}}_\nu$. Let us fix $\nu \in I_{\eta(\varepsilon_0)}(\nu_0)$. Either

(i) There is some $x_i \in S_i$, and some i with $1 \leqq i \leqq n_0$ such that $u(x_i) = u_{\lambda_0(\nu)}^\nu(x_i)$, or

(ii) $u(x) > u_{\lambda_0(\nu)}^\nu(x)$ for all $x \in \cup_{i=1}^{n_0} S_i$.

In case (i), since $Du(x_i) \neq 0$, by Theorem 3.2, $u \equiv u^{\nu}_{\lambda_0(\nu)}$ in the component C^{ν} of $\Sigma^{\nu}_{\lambda_0(\nu)} \backslash Z^{\nu}_{\lambda_0(\nu)}$ containing x_i. Then $x_i \in S_i \cap \tilde{B}^{\nu} \subset \tilde{B}^{\nu} \cap \tilde{B}^{\nu_0}_i$ and hence $\tilde{B}^{\nu} \equiv \tilde{B}^{\nu_0}_i \in \tilde{\mathscr{B}}_{\nu}$.

In case (ii), $u > u^{\nu}_{\lambda_0(\nu)}$ holds in $S \backslash O$,

$$S = \cup_{i=0}^{n_0} S_i, \quad |U^{\nu}_{\lambda_0(\nu)} \backslash S| < \delta, \quad M_0^{\nu,\lambda} < M,$$

with δ and M as in the previous step. Then once again applying Theorem 3.1 (as in Step 2 of Theorem 4.1) with $\Omega = \Sigma^{\nu}_{\lambda}$ $\Omega' = \Sigma^{\nu}_{\lambda} \backslash (S \backslash O)$, we get $u \geqq u^{\nu}_{\lambda}$ in Σ^{ν}_{λ} for $\lambda < \lambda_0(\nu)$, $\lambda_0(\nu) - \lambda$ small, contradicting the definition of $\lambda_0(\nu)$. Thus case (ii) cannot arise and our claim is proved.

Next we claim that there exist a direction $\nu_1 \in I_{\eta(\varepsilon_0)}(\nu_0)$, a neighbourhood $I_{\eta_1}(\nu_1)$ and an index $i_1 \in \{1, \ldots, n_0\}$ such that for any $\nu \in I_{\eta_1}(\nu_1)$, the set $\tilde{B}^{\nu_0}_i \in \tilde{\mathscr{B}}_{\nu}$. The proof of this assertion goes through exactly as in [5] (see Step 2 in the proof of Theorem 1.1 in [5]).

Step 3. Let ν_1, i_1, η_1 be as in Step 2. Then $(\partial C^{\nu_0}_{i_1})' \cap (\Sigma^{\nu_1}_{\lambda_0(\nu_1)})'$ contains a subset Γ on which u is constant and $Du = 0$ and whose projection on $T^{\nu_1}_{\lambda_0(\nu_1)}$ contains an open subset of that hyperplane. Here $(C^{\nu_0}_{i_1})' = R^{\nu_1}_{\lambda_0(\nu_1)}(C^{\nu_0}_{i_1})$.

Proof. Let $\tilde{B}^{\nu_0}_{i_1}$ be the set found in Step 2 and let C be the component of $\Sigma^{\nu_0}_{\lambda_0(\nu_0)} \backslash Z^{\nu_0}_{\lambda_0(\nu_0)}$ such that $B^{\nu_0}_{i_1} = C \cap K_r$. By the strong comparison principle, $\tilde{C} \in \tilde{\mathscr{F}}_{\nu}$, for any $\nu \in I_{\eta_1}(\nu_1)$. In particular,

$$u \equiv u^{\nu}_{\lambda_0(\nu)} \quad \text{in } \tilde{C} \quad \forall \, \nu \in I_{\eta_1}(\nu_1). \tag{5.6}$$

Then we take a point $\bar{x} \in \partial C \cap \Sigma^{\nu_1}_{\lambda_0(\nu_1)}$ and consider the points

$$A(\nu) = \bar{x} + 2(\lambda_0(\nu) - \bar{x}.\nu)\nu \tag{5.7}$$

obtained by reflecting $\bar{x}$ through the hyperplanes $T^{\nu}_{\lambda_0(\nu)}$ for $\nu \in I_{\eta_1}(\nu_1)$. By (5.6) the set $\Gamma = \{A(\nu) : \nu \in I_{\eta_1}(\nu_1)\}$ is such that u is constant on it and $Du = 0$ on Γ. Moreover, proceeding exactly as in the proof of Step 3 of Theorem 1.1 of [5] we get that the projection of Γ on the hyperplane $T^{\nu_1}_{\lambda_0(\nu_1)}$ contains an open subset of $T^{\nu_1}_{\lambda_0(\nu_1)}$.

Step 4. u is radially symmetric about some point $x_0 \in \mathbb{R}^N$ and is radially strictly decreasing.

Proof. By the three previous steps we have reached a contradiction, since a set Γ like that constructed in Step 3 cannot exist, because of Proposition 5.1. Therefore the only possibility is that $u \equiv u^{\nu}_{\lambda_0(\nu)}$ in the half space $\Sigma^{\nu}_{\lambda_0(\nu)}$ for any direction ν. Now, if u were strictly decreasing in the ν-direction up to the hyperplane $T^{\nu}_{\lambda_0(\nu)}$, for any direction ν, we would get the assertion by considering N linearly independent directions in $\mathbb{R}^N$. But by (4.4) of Theorem 4.1 we have the strict monotonicity of u only outside the critical set Z of u. Therefore an extra argument is needed. This comes again from Proposition 5.1, which actually allows us to prove that $Du(x) \neq 0$ for any $x \in \Sigma^{\nu}_{\lambda_0(\nu)}$ for all directions ν.

In fact, if $\bar{x} \in \Sigma^{\nu}_{\lambda_0(\nu_1)} \cap Z$ for a certain direction ν_1, then, using the symmetry of u in any direction and arguing as in Step 3, we get that the set

$$\Gamma = \{y = \bar{x} + 2(\lambda_0(\nu) - \bar{x}.\nu)\nu, \quad \nu \in I(\nu_1)\},$$

for a suitable neighbourhood of ν_1 has the properties :

$$Du(x) = 0 \quad \text{for any} \quad x \in \Gamma, \tag{5.8}$$

$$u \quad \text{is constant on } \Gamma, \tag{5.9}$$

$$\text{the projection of } \Gamma \text{ on } T^{\nu_1}_{\lambda_0(\nu_1)} \text{ contains an open subset of } T^{\nu_1}_{\lambda_0(\nu_1)}. \tag{5.10}$$

By Proposition 5.1 such a set Γ cannot exist. Hence $Z \cap \Sigma^{\nu}_{\lambda_0(\nu)} = \emptyset$ for any direction ν. Thus u has only one critical point in $\mathbb{R}^N$ and the assertion follows. □

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