SYMMETRY OF C^1 SOLUTIONS OF p-LAPLACE EQUATIONS IN $\mathbb{R}^N$

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Abstract. In this paper we consider positive C^1 solutions of the equation $-\operatorname{div}(|Du|^{p-2}Du) = f(u)$ in $\mathbb{R}^N$, vanishing at infinity, in the case $1 < p \leq 2$, f locally Lipschitz continuous in $(0, \infty)$. We prove that the solutions are radially symmetric under two different sets of assumptions on the behaviour of f near zero. In the first case we assume that f is nonincreasing in $(0, s_0)$, $s_0 > 0$, and improve a previous result of the authors and Pacella [7] where the symmetry was proved under the hypothesis $u \in C^1(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)$. In the second case, previously studied when $p = 2$, we assume that $f(u) = O(u^{\alpha+1})$ $(u \to 0)$, and prove the radial symmetry of the solution u, provided $u = O\left(\frac{1}{|x|^m}\right)$, $Du(x) = O\left(\frac{1}{|x|^{m+1}}\right)$ at infinity with $m(\alpha+2-p) > p$. These results extend to p-Laplace equations, $1 < p < 2$, analogous symmetry results previously known in the case of strictly elliptic equations. The proofs exploit some Poincaré and Hardy-Sobolev type inequalities.

1. **Introduction**

In this paper, we study symmetry properties of positive solutions of p-Laplace equations in $\mathbb{R}^N$, $N \geq 2$, with the ground state condition at infinity, namely

$$\begin{cases} -\Delta_p u(x) = f(u(x)), & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases} \tag{1.1}$$

for $1 < p \leq 2$, with $\Delta_p u = \operatorname{div}(|Du|^{p-2}Du)$.

In the case $p = 2$, this problem for C^2 solutions was first considered by Gidas, Ni and Nirenberg in [9], using the method of moving planes of Alexandrov and Serrin [16]. For a C^1 function f, satisfying $f(0) = 0$, $f'(0) < 0$ and $f \in C^{1+\nu}$ near zero, the radial symmetry of positive solutions is proved there. Furthermore, under another set of assumptions involving the decay rate of u at infinity and the growth rate of f near 0, symmetry results were derived. Later, these results were extended by Li (see [13]) to fully nonlinear elliptic equations under weaker conditions and by Li and Ni (see [14]) when f is C^1 and $f'(s) \leq 0$ for s small.

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For the p-Laplace operator, the coefficient $(|Du|^{p-2})$ is vanishing or singular at the critical points of u for $p > 2$ or $p < 2$ respectively and hence strict comparison principles are available only in a restricted form outside the set of the common critical points of the functions being compared. That is why many earlier symmetry results for (1.1) were proved under some sort of assumptions on Z, the critical set of u. Symmetry results for positive solutions of p-Laplace equations, without any assumptions on Z, were derived in [5], [6] for bounded domains and in [7] for $\mathbb{R}^N$ by the moving plane method, as well as in [2], [3] with a different method.

Positive weak solutions in $W^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ of (1.1) are proved in [7] to be radially symmetric under the assumption that f is locally Lipschitz on $(0, \infty)$ and f is nonincreasing near 0. In [17] the authors study the symmetry of nonnegative C^1 ground states of a class of quasilinear elliptic equations, which includes the p-Laplacian, for $1 < p < 2$, as a special case. Under the same assumptions on f as above, radial symmetry of nonnegative ground states of (1.1) is proved in [17], under the extra assumption that the set where $|Du(x)| > 0$ is connected, but requiring u to belong only to $C^1(\mathbb{R}^N)$. The authors also consider the case $p > 2$ under the assumption that the solution has only one critical point.

We use here the moving plane method to prove that C^1 weak solutions u of (1.1) are radially symmetric about a point in $\mathbb{R}^N$, under two different sets of hypotheses on the nonlinearity f, without requiring any conditions on the set where $Du(x) = 0$. To be more precise, we assume that

(H1) f is locally Lipschitz continuous in $(0, \infty)$.

and either one of the following conditions holds:

(H2) there exists $s_0 > 0$ such that f is nonincreasing on $(0, s_0)$.
(H3) There exists $s_0 > 0$ and $\alpha > p-2$ such that for $0 < u < v < s_0$,

$$\frac{f(v) - f(u)}{(v-u)} \leq \begin{cases} Cv^\alpha & \text{if} \quad \alpha \geq 0 \\ Cu^\alpha & \text{if} \quad \alpha < 0 \end{cases} .$$

Our main results are the following theorems:

THEOREM 1.1. *Under the assumptions (H1) and (H2), if* $u \in C^1(\mathbb{R}^N)$ *is a weak solution of (1.1), for* $1 < p < 2$, *then* u *is radially symmetric about some point* $x_0 \in \mathbb{R}^N$, *i.e.* $u = u(r)$, *with* $r = |x - x_0|$, *and strictly radially decreasing, i.e.* $u'(r) < 0$ *for all* $r > 0$.

Note that this theorem differs from the result in [7] because it does not require the solution u to be in $W^{1,p}(\mathbb{R}^N)$. The main new ingredient here is a Poincaré type inequality for functions on half balls, vanishing on the boundary hyperplane.

Using a Hardy-Sobolev 's inequality for functions vanishing at infinity, together with the previous Poincaré 's inequality, we can treat the more difficult case when the nonlinearity f is not necessarily decreasing near zero, and we prove the following theorem.

THEOREM 1.2. *Under the assumptions (H1) and (H3) let* $u \in C^1(\mathbb{R}^N)$ *be a weak solution of (1.1) for* $1 < p < 2$ *satisfying, for some* $m > 0$,

$$\begin{array}{rrcll} & u(x) & = & 0(\frac{1}{|x|^m}) & \text{as} \quad |x| \to \infty, \\ (& u(x) & \geq & \frac{C}{|x|^m} & \text{as} \quad |x| \to \infty \quad \text{when } \alpha < 0 \quad), \\ & Du(x) & = & 0(\frac{1}{|x|^{m+1}}) & \text{as} \quad |x| \to \infty \quad . \end{array} \tag{1.2}$$

If

$$m(\alpha + 2 - p) > p, \tag{1.3}$$

then u is radially symmetric about some point $x_0 \in \mathbb{R}^N$ *and strictly radially decreasing.*

The hypothesis (H3), with $p = 2$, was used in [13], for a Lipschitz function f to derive symmetry results for positive solutions of quasilinear or fully nonlinear strictly elliptic equations. Here we extend in Theorem 1.2 in a "natural way " the condition in [13] to the degenerate case $1 < p < 2$.

More precisely C. Li derived the symmetry results in [13], in particular, for the case of the Laplacian, under the assumptions

$$\frac{f(v) - f(u)}{v - u} \quad \leq \quad C(|u| + |v|)^\alpha$$

for small u, v and some $\alpha > 0$, with $m\alpha > 2$, requiring the decay assumptions only on u (see Theorem 3 in [13]). Clearly this condition on f is equivalent to (H3) when $\alpha > 0$ and (1.3) reduces to $m\alpha > 2$ when $p = 2$. In particular, if $f(0) = 0$, Li's conditions imply that

$$\frac{f(u)}{u} \quad \leq \quad C|u|^\alpha \quad \text{and} \quad m\alpha > 2\,.$$

For the same nonlinearity f, our conditions can be rewritten as

$$\frac{f(u)}{u^{p-1}} \leq Cu^{\alpha_1} \quad \text{for} \quad u > 0 \quad \text{and} \quad m\alpha_1 > p\,,$$

where $\alpha_1 = \alpha + 2 - p$.

Thus our conditions are a natural generalizations of Li's conditions and reduce to them in the case $p = 2$. Moreover in this case we can drop our decay assumptions on $|Du|$ in (1.2), hence we also give in Theorem 2.1 below an alternative new proof of the symmetry theorem proved in [13] in the case of the laplacian.

In the singular case $1 < p < 2$ instead the use of the Hardy-Sobolev inequality in Theorem 1.2 is essential in order to prove the symmetry result with the natural hypotheses on the decay rate of u.

Example. Some model nonlinearities that we can treat are the functions that behave like the powers $f(s) = s^{\alpha+1}$, $p-1 < \alpha+1$, near zero. More precisely if $\alpha > p-2$, $f \in C^1(0,\infty)$ and $f'(s) = O(s^\alpha)$ as $s \to 0$, then using the mean value theorem it is easy to see that (H3) is satisfied.

Remark 1.1. Our theorems apply to positive solutions. However the conditions on f which force a nonnegative ground state to be positive can be derived using the strong maximum principle in Vazquez [18]. Let us also remark that when f satisfies (H1) and (H2) and $f(0) = 0$, the strict maximum principle has been studied in [15] for general quasilinear operators. For the case of p-Laplacian, a result is that if u is a nonnegative ground state of (1.1) under assumptions (H1) and (H2) and $f(0) = 0$, then u is positive on $\mathbb{R}^N$ if

$$\int_0 \frac{dt}{|F(t)|^{1/p}} \quad = \quad \infty, \tag{1.4}$$

where F is the primitive of f. For the sake of reader's convenience we give in Appendix 2 some conditions, including (1.4), which ensure the positivity of the nonnegative solutions when (H1) and either (H2) or (H3) hold.

The existence and uniqueness of radial ground state solutions of general quasilinear elliptic equations have been studied in [8]. For a continuous function f, with $f(0) = 0$ and f negative near 0 and positive later on, existence and uniqueness of nonnegative radial ground states have been proved under some additional hypotheses (see Corollary 1 in [8]). For those cases, where nonnegative radial ground states are necessarily positive and unique in the radial class, our result (Theorem 1.1) shows that there is a unique nonnegative ground state for (1.1).

In [12] and [1], positive radial solutions of (1.1) with $f(u) = u^{\alpha+1}$ are considered, and it is proved that for $p-1 < \alpha+1 < p^*-1$, the problem

$$\begin{cases} -(r^{N-1}|u'(r)|^{p-2}u'(r))' \quad = \quad r^{N-1}|u|^\alpha u, \ 0 < r < \infty, \\ \qquad u'(0) = 0, \quad u(r) \geq 0 \ , \end{cases} \tag{1.5}$$

has only trivial solutions (see corollary 3.4 in [1]). A nonnegative solution of (1.1) with $f(u) = u^{\alpha+1}$ for the above range of α's, is necessarily positive everywhere because (1.4) holds for f here (see Proposition A.1 (b) below). Then Theorem 1.2 implies that (1.1) with $f(u) = u^{\alpha+1}$, $p \in (1,2)$, $p-1 < \alpha+1 < p^*-1$, has only trivial solutions in the class of C^1 functions which decay like $(\frac{1}{|x|^m})$ at ∞ with $m > p/(\alpha+2-p)$. This extends partially the well known result of Gidas-Spruck [10] regarding subcritical power nonlinearities, to p-Laplace equation. Let us remark that to our knowledge the precise decay rate of the solutions of (1.1) at infinity is not known.

For the critical case $\alpha = p^* - 2$, Guedda and Veron [12] prove that the only positive radial solutions are

$$y_a(x) = (Na(\frac{N-p}{p-1})^{p-1})^{\frac{N-p}{p^2}}(a + |x|^{p/p-1})^{\frac{p-N}{p}}$$

where a is any positive number. Then Theorem 1.2 with $\alpha = p^* - 2$ and $m = (N-p)/(p-1)$ gives the uniqueness of positive solution which decay like $|x|^{\frac{p-N}{p-1}}$ at ∞, as it happens for the y_a, and more generally with decay like $\frac{1}{|x|^m}$ with $m > \frac{p}{\alpha+2-p} = \frac{p}{p^*-p} = \frac{N-p}{p}$.

As mentioned before, F. Brock (see [2], [3]) has obtained recently symmetry results by an entirely different method using continuous Steiner symmetrization for nonnegative solutions of quasilinear equations, which include p-Laplacian for $1 < p < \infty$, in balls as well as in $\mathbb{R}^N$. He assumes $u \in W^{1,p}(\mathbb{R}^N)$ and $f(u(.)) \in L^1(\mathbb{R}^N)$ but he needs less regularity on f, namely continuity and a certain growth condition in the neighbourhood of the zeros of f. His symmetry results concern either the case when f satisfy (H2) or the case of solutions with prescribed decay rate at infinity, but different from (1.2) (see Theorem 2, (i) and (ii) in [2]).

The paper is organized as follows. In section 2 we prove the Poincaré and weighted Hardy-Sobolev type inequalities needed for the sequel and give a new proof of some known results for the case $p = 2$ using these inequalities.

In section 3 we prove some new weak comparison theorems for solutions of p-Laplace equations in unbounded domains.

In section 4 we begin the proof of Theorem 1.1 and 1.2 by proving a partial (local) symmetry theorem, as in [7] but with weaker and different hypotheses, exploiting the more powerful comparison theorems of the preceeding section. Then the conclusion of the proof of Theorem 1.1 and Theorem 1.2 is very similar (but slightly simpler) to the proof of Theorem 1.1 in [7], Sect.5, and is therefore deferred to Appendix 1.

Finally in Appendix 2 we recall some sufficient conditions which guarantees that nonnegative solutions of (1.1) are actually positive.

2. **Some Inequalities and the case** $p = 2$

We start with some notations. If ν is a direction in $\mathbb{R}^N$, i.e $|\nu| = 1$, we define for $\lambda \in \mathbb{R}$, the half space

$$\Sigma_\lambda^\nu = \{x \in \mathbb{R}^N : x \cdot \nu < \lambda\},$$

and the hyperplane

$$T_\lambda^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = \lambda\}.$$

Let R_λ^ν be the reflection through T_λ^ν , i.e.

$$R_\lambda^\nu(x) = x_\lambda^\nu = x + 2(\lambda - x \cdot \nu)\nu ,$$

for any $x \in \mathbb{R}^N$. We also set

$$(\Sigma_\lambda^\nu)' = R_\lambda^\nu(\Sigma_\lambda^\nu).$$

For a function $u \in C^1(\mathbb{R}^N)$, we define the reflected function

$$u_\lambda^\nu(x) = u(x_\lambda^\nu) \quad \forall\, x \in \mathbb{R}^N.$$

We need weak comparison principles in unbounded domains, for positive weak solution $u \in C^1(\mathbb{R}^N)$ of (1.1). These will rely on the Poincaré and Hardy-Sobolev type inequalities proved below.

Let us consider a ball $B(P,R)$, centered at $P \in T_\lambda^\nu$ having radius R. Define

$$B_\lambda^\nu = B(P,R) \cap \Sigma_\lambda^\nu.$$

For any subset A of Σ_λ^ν, define

$$A' = R_\lambda^\nu(A) \quad \text{and} \quad \tilde{A} = A \cup A' \cup (T_\lambda^\nu \cap \overline{A}).$$

The following lemma applies to functions on B_λ^ν, vanishing on T_λ^ν.

LEMMA 2.1 (Poincaré type inequality). *Let $w \in W^{1,q}_{loc}(\mathbb{R}^N)$, $1 \le q < \infty$, vanish on T_λ^ν and let (supp w) $\cap B_\lambda^\nu = A_1 \cup A_2$, for some disjoint measurable sets A_i. Then there exists a constant $C = C(N)$ such that*

$$\|w\|_{q,B_\lambda^\nu} \le C|B_\lambda^\nu \cap \operatorname{supp} w|^{\frac{1}{Nq}}\left(|A_1|^{\frac{1}{Nq'}}\|Dw\|_{q,A_1} + |A_2|^{\frac{1}{Nq'}}\|Dw\|_{q,A_2}\right). \tag{2.1}$$

Proof. The idea is to take an odd extension $\hat{w}$ of w to $\tilde{B}_\lambda^\nu$, and exploit the fact that $\hat{w}$ has zero average over $\tilde{B}_\lambda^\nu$ to derive the Poincaré inequality. Define

$$\begin{aligned} \hat{w}(x) &= w(x) \quad \text{on} \quad B_\lambda^\nu \\ &= -w(2\lambda - x_1, x_2, \dots x_n) \quad \text{for} \quad x = (x_1 \dots x_n) \in (B_\lambda^\nu)', \end{aligned}$$

so that $\int_{\tilde{B}_\lambda^\nu} \hat{w}(x)dx = 0$.

We now verify that $\hat{w} \in W^{1,1}(\tilde{B}_\lambda^\nu)$. Since $w \in W^{1,q}_{\rm loc}$, it follows that $\hat{w} \in L^1(\tilde{B}_\lambda^\nu)$. Further one can check easily that the distributional derivatives of $\hat{w}$ for $2 \le i \le N$ are

$$\frac{\partial \hat{w}}{\partial x_i}(x) = \begin{cases} \frac{\partial w}{\partial x_i}(x) & \text{on} \quad B_\lambda^\nu \\ -\frac{\partial w}{\partial x_i}(2\lambda - x_1, x_2, \dots x_n) & \text{for} \quad x \in (B_\lambda^\nu)' \end{cases}$$

and

$$\frac{\partial \hat{w}}{\partial x_1}(x) = \begin{cases} \frac{\partial w}{\partial x_1}(x) & \text{on} \quad B_\lambda^\nu \\ \frac{\partial w}{\partial x_1}(2\lambda - x_1, x_2, \dots x_n) & \text{for} \quad x \in (B_\lambda^\nu)'\,. \end{cases}$$

Hence we have for $1 \le q < \infty$,

$$\|\hat{w}\|_{q,\tilde{B}_\lambda^\nu} = 2^{\frac{1}{q}}\|w\|_{q,B_\lambda^\nu}, \quad \|D\hat{w}\|_{q,\tilde{B}_\lambda^\nu} = 2^{\frac{1}{q}}\|Dw\|_{q,B_\lambda^\nu}. \tag{2.2}$$

Since $\hat{w}$ has zero mean in $\tilde{B}^\nu_\lambda = B(P,R)$, applying Lemma 7.16 from [11] we have

$$|\hat{w}(x)| \leq C(N) \int_{\tilde{B}^\nu_\lambda} \frac{|D\hat{w}(y)|}{|x-y|^{N-1}} dy \qquad \text{a.e.}$$

where $C(N) = \frac{2^N}{N w_N}$. Since $D\hat{w}(y) = 0$ a.e. in $\tilde{B}^\nu_\lambda \setminus$ supp $\hat{w}$,

$$|\hat{w}(x)| \leq C(N) \left(V_{\tilde{A}_1}(|D\hat{w}|(x)) + V_{\tilde{A}_2}(|D\hat{w}|(x)) \right), \tag{2.3}$$

where

$$V_A f(x) \; = \; \int_A \frac{f(y)}{|x-y|^{N-1}} \, dy, \quad x \in \mathbb{R}^N.$$

Recall that (see [4], Lemma 2.2)

$$\|V_A \, f\|_{q,K} \; \leq \; N \omega_N (\frac{|A|}{\omega_N})^{\frac{1}{Nq'}} \, (\frac{|K|}{\omega_N})^{\frac{1}{Nq}} \, \|f\|_{q,A} \; .$$

Using this and (2.2) in (2.3), the lemma follows. □

Now we derive the weighted Hardy-Sobolev inequality which we shall use later. Let us recall first the Hardy's inequality ([19], Lemma 1.8.11) for a measurable function $h : (0,\infty) \to (0,\infty)$

$$\int_0^T \frac{1}{\rho^{r+1}} \left(\int_0^\rho h(t) dt \right)^q d\rho \leq \left(\frac{q}{r} \right)^q \int_0^T (h(t))^q \; t^{q-r-1} \; dt$$

for $0 \leq T \leq \infty, \quad 1 \leq q < \infty$ and $r > 0$. We need the following modified version of this inequality. For the convenience of the reader, we include its derivation here.

LEMMA 2.2. *Let $h : (0,\infty) \to (0,\infty)$ be a measurable function. Then*

$$\int_T^\infty \rho^{r-1} \left(\int_\rho^\infty h(t) dt \right)^q d\rho \leq \left(\frac{q}{r} \right)^q \int_T^\infty h(t)^q t^{q+r-1} dt, \tag{2.4}$$

for $0 \leq T < \infty, \quad 1 \leq q < \infty$ *and* $r > 0$.

Proof. Rewriting $h(t)$ as $(h(t) t^{1+r/q})(t^{-1-r/q})$ and applying Hölder's inequality with respect to the measure $(t^{-1-\frac{r}{q}} dt)$, we get

$$\left(\int_\rho^\infty h(t) dt \right)^q \leq \left(\int_\rho^\infty h(t)^q t^{q+r-1-\frac{r}{q}} \, dt \right) \left(\frac{q}{r} \right)^{q-1} \rho^{\frac{-r}{q}(q-1)}$$

Now integrating in ρ from T to ∞ and using Fubini's theorem we get

$$\int_T^\infty \rho^{r-1}\left(\int_\rho^\infty h(t)dt\right)^q d\rho \le$$
$$\left(\frac{q}{r}\right)^{q-1}\int_T^\infty \rho^{r-1}\rho^{-r+\frac{r}{q}}\left(\int_\rho^\infty h(t)^q t^{q+r-1-\frac{r}{q}}\,dt\right)d\rho$$
$$=\left(\frac{q}{r}\right)^{q-1}\int_T^\infty (h(t))^q\, t^{q+r-1-\frac{r}{q}}\left(\int_T^t \rho^{-1+\frac{r}{q}}\,d\rho\right)dt$$
$$\le\left(\frac{q}{r}\right)^{q-1}\int_T^\infty (h(t))^q t^{q+r-1-r/q}\left(\int_0^t \rho^{-1+r/q}d\rho\right)dt$$
$$=\left(\frac{q}{r}\right)^{q}\int_T^\infty h(t)^q t^{q+r-1}dt$$

and (2.4) is proved. □

LEMMA 2.3 (weighted Hardy-Sobolev inequality)**.** *Let $u \in C_0^1(\mathbb{R}^N) = \{u \in C^1(\mathbb{R}^N) : u(x) \to 0$ as $|x| \to \infty\}$ and let $q \ge 1$. Then for any $s > q-N$ and $T \ge 0$ we have*

$$\int_{\mathbb{R}^N\setminus B_T(0)} |u|^q|x|^{s-q}dx \le \left(\frac{q}{N-q+s}\right)^q\int_{\mathbb{R}^N\setminus B_T(0)} |Du|^q|x|^s dx \quad (2.5)$$

The same inequality holds if we substitute u with its positive (negative) part.

Proof. Let $u \in C^1(\mathbb{R}^N)$. We have for $w \in S^{N-1}$ and $\rho \ge 0$,

$$\begin{aligned}|u(\rho w)| &= |-\int_\rho^\infty \frac{d}{dt}(u(tw))dt| \\ &= |\int_\rho^\infty Du(tw).wdt| \le \int_\rho^\infty |Du(tw)|dt.\end{aligned}$$

Calling $h(t) = |Du(tw)|$ and $r = N-q+s > 0$ and using (2.4), we get

$$\begin{aligned}\int_T^\infty \rho^{N-1}|u(\rho w)|^q\rho^{s-q}d\rho &\le \int_T^\infty \rho^{N-1+s-q}\left(\int_\rho^\infty |Du(tw)|dt\right)^q d\rho \\ &\le \left(\frac{q}{N-q+s}\right)^q\int_T^\infty |Du(tw)|^q t^{N-1}t^s dt\end{aligned}$$

Now integrating with respect to w in S^{N-1} yields (2.5).

To prove that the inequality holds for the positive part u^+ of u, we consider the functions $G_n : \mathbb{R} \to \mathbb{R}$ with $G_n(s) = 0$ if $s < 0$, $G_n(s) = \left(s^2 + \frac{1}{n^2}\right)^{\frac{1}{2}} - \frac{1}{n}$ if $s \ge 0$, $n \in \mathbb{N}$ and define $u_n(x) = G_n(u(x))$. We have that $u_n \in C_0^1(\mathbb{R}^N)$ for each $n \in \mathbb{N}$ and for any $x \in \mathbb{R}^N$

$u_n(x) \to u^+(x)$ as $n \to \infty$ and $|Du_n(x)| \le |Du(x)|$, so that

$$\begin{aligned}\int_{\mathbb{R}^N \setminus B_T(0)} |u|^q |x|^{s-q}\, dx &\le \liminf_{n\to\infty} \int_{\mathbb{R}^N \setminus B_T(0)} |u_n|^q |x|^{s-q}\, dx \\ &\le \left(\frac{q}{N-q+s}\right)^q \limsup_{n\to\infty} \int_{\mathbb{R}^N \setminus B_T(0)} |Du_n|^q |x|^s\, dx \\ &\le \left(\frac{q}{N-q+s}\right)^q \int_{\mathbb{R}^N \setminus B_T(0)} |Du|^q |x|^s\, dx\end{aligned}$$

□

Remark 2.1. Of course the inequality (2.5) holds for a larger class of measurable functions. In fact suppose that $u \in W^{1,1}_{loc}(\mathbb{R}^N)$ with $|Du| \in L^q(|x|^s dx; \mathbb{R}^N)$ (otherwise there is nothing to prove). Proceeding as in the last part of the previous proof it is immediate to see that a sufficient condition for the validity of (2.5) is the existence of a sequence u_n of functions in $C^1_0(\mathbb{R}^N)$ such that $u_n \to u$ a. e. and $|Du_n| \to |Du|$ in $L^q(|x|^s dx; \mathbb{R}^N)$.

In particular if $s = 0$, $q < N$ and $u \in L^r(\mathbb{R}^N)$ for some $r \ge 1$, then the usual mollifiers $u_\varepsilon = \rho_\varepsilon * u$ tend to zero at infinity and for a sequence $\varepsilon_n \to 0$ the sequence $u_n = u_{\varepsilon_n}$ satisfies the requirements, so that (2.5) holds.

This happens also for every $u \in W^{1,q}(\mathbb{R}^N)$, $1 < q < N$. □

In the next section these inequalities will help us in proving some comparison theorems which are crucial in the proofs of the main results. Since there will be some technical complications due to the degenerate character of the p-laplacian operator, in order to clarify the main ideas in the simpler context of $p = 2$, we present now a new proof of Theorem 3 in [13] that exploits the previous inequalities (see also the remark after the proof).

Let $u \in C^1(\mathbb{R}^N)$ be a (weak) solution of the problem

$$\begin{cases} -\Delta u(x) = f(u(x)), & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases} \tag{2.6}$$

where $N \ge 3$ and f is locally Lipschitz in $(0, \infty)$.

THEOREM 2.1. *Suppose there exist $s_0, \alpha > 0$ such that for all u, v such that $0 < u < v < s_0$ we have $\frac{f(v)-f(u)}{v-u} \le C(u+v)^\alpha$. Let u be a solution of (2.6) satisfying $u(x) = O(\frac{1}{|x|^m})$ as $|x| \to \infty$, with $m\,\alpha > 2$. Then u is radially symmetric and strictly decreasing about some point.*

Proof. Let ν be an arbitrary direction in $\mathbb{R}^N$ and let us define the set

$$\Lambda(\nu) := \{\lambda \in \mathbb{R} \ : \ \forall \mu > \lambda, \ u \geq u_\mu \ \text{ in } \Sigma_\mu^\nu\}.$$

In the first step we will show that $\Lambda(\nu)$ is nonempty and bounded from below. The second step will consist in showing that if $\lambda_0(\nu) := \inf \Lambda(\nu)$, then $u \equiv u^\nu_{\lambda_0(\nu)}$ in $\Sigma^\nu_{\lambda_0(\nu)}$. In the third step the radial symmetry and monotonicity of u will follow with standard arguments.

Step 1 Let us put $v = u^\nu_\lambda$ and observe that v satisfies the same equation as u in Σ^ν_λ. We now take in the equations for v and u the test function $[(v-u-\varepsilon)^+]^t$, where $t \geq 1$, to be chosen precisely later on and $\varepsilon > 0$ (it has compact support in Σ^ν_λ since $u \to 0$ when $x \to \infty$ and $u = v$ on T^ν_λ). Subtracting the equations we get

$$\begin{aligned} t \int_{\Sigma^\nu_\lambda \cap [v \geq u+\varepsilon]} & [(v-u-\varepsilon)^+]^{t-1} |D(v-u)|^2 \\ &= \int_{\Sigma^\nu_\lambda \cap [v \geq u+\varepsilon]} [f(v)-f(u)][(v-u-\varepsilon)^+]^t \\ &\leq \int_{\Sigma^\nu_\lambda \cap [v \geq u+\varepsilon]} [f(v)-f(u)]^+[(v-u-\varepsilon)^+]^t \end{aligned} \tag{2.7}$$

Here we have used the fact that $D(v-u-\varepsilon)^+ = D(v-u)^+ \chi_{\{v \geq u+\varepsilon\}}$.

Let us now fix t so that $m\,(\alpha + t + 1) > N$. Then

$$\int_{\Sigma^\nu_\lambda} [f(v)-f(u)]^+[(v-u)^+]^t < \infty.$$

In fact the integrand is bounded in compact sets and by hypothesis outside a compact set we have $[f(v)-f(u)]^+[(v-u)^+]^t \leq C\, v^{\alpha+t+1}$ with $\int_{\Sigma^\nu_\lambda} v^{\alpha+t+1} \leq \int_{\mathbb{R}^N} u^{\alpha+t+1} < \infty$ (since $u^{\alpha+t+1} = O(\frac{1}{|x|^{m(\alpha+t+1)}})$ at infinity).

If $\varepsilon \to 0$ we get by monotone convergence

$$\int_{\Sigma^\nu_\lambda} [(v-u)^+]^{t-1}\, |D(v-u)^+|^2 \leq C \int_{\Sigma^\nu_\lambda} [f(v)-f(u)]^+[(v-u)^+]^t < \infty\,.$$

Defining $w := [(v-u)^+]^{\frac{t+1}{2}}$, this can be written as

$$\int_{\Sigma^\nu_\lambda} |Dw|^2 \leq C_0 \int_{\Sigma^\nu_\lambda} [f(v)-f(u)]^+[(v-u)^+]^t < \infty \tag{2.8}$$

If λ is large then $v = u_\lambda < s_0$ and $\frac{f(v)-f(u)}{v-u} \leq C\, v^\alpha$, so we get

$$\int_{\Sigma^\nu_\lambda} |Dw|^2 \leq C \int_{\Sigma^\nu_\lambda} v^\alpha\, w^2 \leq C \int_{\Sigma^\nu_\lambda} \frac{1}{|x^\nu_\lambda|^{m\,\alpha}}\, w^2 \tag{2.9}$$

On the other hand if $\lambda > 0$ then $|x_\lambda^\nu| > |x|$ and by Hardy's inequality (2.5), with $q = 2$, $s = 0$, we get

$$\int_{\Sigma_\lambda^\nu} |Dw|^2 \le C \int_{\Sigma_\lambda^\nu} \frac{1}{|x_\lambda^\nu|^{m\,\alpha-2}} \frac{w^2}{|x|^2} \le C_1 \,(\sup_{\Sigma_\lambda^\nu} \frac{1}{|x_\lambda^\nu|^{m\alpha-2}}) \int_{\Sigma_\lambda^\nu} |Dw|^2 \tag{2.10}$$

Since $\sup_{\Sigma_\lambda^\nu} \frac{1}{|x_\lambda^\nu|^{m\alpha-2}} \to 0$ as $\lambda \to \infty$ (because $m\,\alpha > 2$) we obtain that there exists λ' such that if $\lambda \ge \lambda'$ then $\int_{\Sigma_\lambda^\nu} |Dw|^2 = 0$ i.e. w is constant. This implies that $w = 0$, since $w = 0$ on T_λ^ν, i.e. $u \ge u_\lambda^\nu$ in Σ_λ^ν. This means that $\Lambda(\nu)$ is non empty and, since $\Lambda_{-\nu}$ is also nonempty, it is also bounded below.

Step 2 Let us put $\lambda_0(\nu) := \inf \Lambda(\nu)$. By continuity $u \ge u_{\lambda_0(\nu)}^\nu$ in $\Sigma_{\lambda_0(\nu)}^\nu$. Suppose by contradiction that $u \not\equiv u_{\lambda_0(\nu)}^\nu$ in $\Sigma_{\lambda_0(\nu)}^\nu$.

Since $z = z_{\lambda_0(\nu)} = u - u_{\lambda_0(\nu)}^\nu$ satisfies in $\Sigma_{\lambda_0(\nu)}^\nu$ the linear equation $-\Delta\, z = c_{\lambda_0}(x)\, z$, where $c_{\lambda_0}(x) = \frac{f(u(x)) - f(u_{\lambda_0(\nu)}^\nu(x))}{u(x) - u_{\lambda_0(\nu)}^\nu(x)} \in L_{loc}^\infty(\mathbb{R}^N)$, by the strong maximum principle we get $u > u_{\lambda_0(\nu)}^\nu$ in $\Sigma_{\lambda_0(\nu)}^\nu$. We will show that this implies that the inequality $u \ge u_\lambda^\nu$ in Σ_λ^ν continues to hold when $\lambda < \lambda_0(\nu)$ is close to $\lambda_0(\nu)$, contradicting the definition of $\lambda_0(\nu)$.

To this end, we consider again (2.8), with t fixed so that $m\,(\alpha + t + 1) > N$ and λ close to $\lambda_0(\nu)$. If $K = K_\lambda^\nu = B(P_\lambda^\nu, R)$ is a ball centered at P_λ^ν, projection of the origin onto T_λ^ν and radius R to be fixed later on, we consider the integral in the RHS of (2.8) as the sum of two integrals, one over $\Sigma_\lambda^\nu \setminus K$ and the other over $B_\lambda^\nu = \Sigma_\lambda^\nu \cap K$. We then use Hardy's inequality (which requires only the functions to vanish at infinity) in the former and Poincaré's inequality (which requires only the functions to vanish on the hyperplane T_λ^ν) in the latter. More precisely if we shift the origin to P_λ^ν, then $|x| = |x_\lambda^\nu|$ and we can find $R_0 > 0$, $C > 0$, such that for all λ in a neighborhood of $\lambda_0(\nu)$ we have $u(x) \le \frac{C}{|x|^m}$ if $|x| > R_0$ (i.e. C does not depend on the origin P_λ^ν). Then if $R \ge R_0$ is big enough, proceeding as in the deduction of (2.10) we obtain for the integral over $\Sigma_\lambda^\nu \setminus K$

$$\begin{aligned} C_0 \int_{\Sigma_\lambda^\nu \setminus K} [f(v) - f(u)]^+ [(v-u)^+]^t &\le C_1 \,(\sup_{\Sigma_\lambda^\nu \setminus K} \frac{1}{|x_\lambda^\nu|^{m\alpha-2}}) \int_{\Sigma_\lambda^\nu \setminus K} |Dw|^2 \\ &\le \frac{1}{2} \int_{\Sigma_\lambda^\nu \setminus K} |Dw|^2 \end{aligned} \tag{2.11}$$

for all λ in a neighborhood of $\lambda_0(\nu)$.

On the other hand having fixed R we can bound in K the ratio $\frac{f(u) - f(u_\lambda^\nu)}{u - u_\lambda^\nu}$ by the Lipschitz constant of f in $[\inf_K u, \sup_{\mathbb{R}^N} u]$ and use the Poincaré's inequality (2.1) (with $q = 2$ and $A_2 = \emptyset$). Since $u - u_{\lambda_0(\nu)}^\nu$ is positive in any compact subset of $B_{\lambda_0(\nu)}^\nu$ we have that $|B_\lambda^\nu \cap$ supp $(u_\lambda^\nu - u)^+|$ is small if λ is close to $\lambda_0(\nu)$ and for each λ in a neighborhood of $\lambda_0(\nu)$ we have

$$C_0 \int_{B_\lambda^\nu} [f(v) - f(u)]^+ [(v-u)^+]^t \leq C \int_{B_\lambda^\nu} |w|^2 \leq$$
$$C\, |B_\lambda^\nu \cap \text{ supp } (u_\lambda^\nu - u)^+|^{\frac{2}{N}} \int_{B_\lambda^\nu} |Dw|^2 \leq \tfrac{1}{2} \int_{B_\lambda^\nu} |Dw|^2 \tag{2.12}$$

By (2.8), (2.11) and (2.12) we get that if λ is close to $\lambda_0(\nu)$ we have $\int_{\Sigma_\lambda^\nu} |Dw|^2 \leq \frac{1}{2} \int_{\Sigma_\lambda^\nu} |Dw|^2$, i.e. $Dw = 0$ and $u \geq u_\lambda^\nu$. This contradicts the definition of $\lambda_0(\nu)$ and show that $u \equiv u_{\lambda_0(\nu)}^\nu$ in $\Sigma_{\lambda_0(\nu)}^\nu$

Step 3 By step 2 for every direction ν we have $u \equiv u_{\lambda_0(\nu)}^\nu$ in $\Sigma_{\lambda_0(\nu)}^\nu$. Moreover, using the strong maximum principle as in the beginning of step 2 we get $u > u_\lambda^\nu$ in Σ_λ^ν for every $\lambda > \lambda_0(\nu)$. This means that u is strictly monotone decreasing in the ν-direction in the reflected cap $(\Sigma_{\lambda_0(\nu)}^\nu)'$. So if we take the point P_0, intersection of the hyperplanes $T_{\lambda_0(\nu_i)}^{\nu_i}$ for N orthogonal directions $\nu_1, \ldots, \nu_N$, we have that P_0 is the only maximum point of u and belongs to the hyperplanes $T_{\lambda_0(\nu)}^\nu$ for all the directions ν. Therefore u is radial and strictly radially decreasing about P_0, i.e. there exists $U \in C^1(0, \infty)$ such that $u(x) = U(|x - P_0|)$ with $U'(r) < 0$, $r > 0$.

□

Remark 2.2. In previous proofs, the strong comparison principle and a pointwise analysis of the behaviour of $u - u_\lambda$ were used. On the contrary, in the case of $1 < p < 2$, such pointwise arguments and the use of strict comparison principle do not work well due to the possible presence of critical points of u and u_λ. That is why the method of comparing the integrals was introduced in [4] to prove such weak comparison principles for bounded domains. In [7], this method was extended to unbounded domains by splitting the integrals to be compared into a sum of two integrals, one over a bounded domain and the other over unbounded domain. If (H2) holds, the integral over unbounded domain can be ignored, but when (H3) holds, we can no longer ignore it and we have to estimate it very carefully using Hardy-Sobolev inequality, with suitable weights when $p \neq 2$ as we shall see.

3. Some Comparison Theorems

In the proof of the weak comparison principles we will use the following standard estimate whose proof can be found, for example, in [4], Lemma 2.1:

$$(|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta') \cdot (\eta - \eta') \ \geq \ c(|\eta| + |\eta'|)^{p-2}\, |\eta - \eta'|^2 \tag{3.1}$$

for any $\eta, \eta' \in \mathbb{R}^N$, where c is a constant depending on N and p.

Now we prove a weak comparison principle, when u solves (1.1) weakly and f satisfies (H1) and (H2). Since $u(x) \to 0$ as $|x| \to \infty$ we can find a ball B_{R_0}, centered at the origin with radius R_0 such that

$$\max_{R^N \setminus B_{R_0}} u(x) < s_0. \tag{3.2}$$

Let us put $v = u_\lambda^\nu$ and, for a set T, $M_T = M_T^{\nu,\lambda}(u) = \sup_T(|Du|+|Dv|)$.

PROPOSITION 3.1. *Let $N \geq 2$, $1 < p < 2$ and let u in $C^1(\mathbb{R}^N)$ be a weak solution of (1.1), with f satisfying (H1) and (H2). Let R_0 be such that (3.2) holds for u. Then*

(a) *If $\Sigma_\lambda^\nu \cap R_\lambda^\nu(B_{R_0}) = \phi$, then $v \leq u$ on Σ_λ^ν.*

(b) *Let $\Sigma_\lambda^\nu \cap R_\lambda^\nu(B_{R_0})$ be nonempty and let $P \in T_\lambda^\nu$ and $R > 0$ be such that for the ball $B(P,R)$, with center P and radius R, we have*

$$\Sigma_\lambda^\nu \cap R_\lambda^\nu(B_{R_0}) \subset B(P,R).$$

Then for any $\hat{R} \geq R$, there exist positive numbers δ and $M > 0$ depending on $p, N, \hat{R}, f$ and the L^∞ norms of $u, v, |Du|$ and $|Dv|$ on $K := B(P,\hat{R})$ such that whenever

$$\text{supp } (v-u)^+ \cap B_\lambda^\nu = A \cup B$$

with $B_\lambda^\nu = K \cap \Sigma_\lambda^\nu$, $|A \cap B| = 0\,, M_B < M$ and $|A| < \delta$, it follows that

$$v \leq u \quad \text{in } \Sigma_\lambda^\nu.$$

Proof. As in Theorem 2.1 we take $(v-u-\varepsilon)^+$, $\varepsilon > 0$, as a test function in the equation satisfied by u and v. Subtracting the equations we get, by (3.1)

$$\begin{aligned} c \int_{\Sigma_\lambda^\nu \cap \{v \geq u+\varepsilon\}} (|Du| + |Dv|)^{p-2} \, |D(v-u)^+|^2 \, dx \\ \leq \int_{\Sigma_\lambda^\nu \cap \{v \geq u+\varepsilon\}} (f(v) - f(u))(v-u-\varepsilon)^+ \, dx \end{aligned} \tag{3.3}$$

(a) Assume that

$$\Sigma_\lambda^\nu \cap R_\lambda^\nu(B_{R_0}) = \phi.$$

Then $v(x) < s_0$ on Σ_λ^ν and by (H2) we have $f(v) \leq f(u)$ on $\{v \geq u+\varepsilon\}$ since on this set $u < v < s_0$. So we have

$$c \int_{\Sigma_\lambda^\nu \cap \{v \geq u+\varepsilon\}} (|Dv| + |Du|)^{p-2} |D(v-u)^+|^2 \, dx \leq 0$$

and by monotone convergence as $\varepsilon \to 0$ we get

$$\int_{\Sigma_\lambda^\nu} (|Dv| + |Du|)^{p-2} |D(v-u)^+|^2 \, dx \leq 0. \tag{3.4}$$

Hence $D(v-u)^+ = 0$ a.e. on Σ_λ^ν and since $(v-u)^+ = 0$ on $\partial\Sigma_\lambda^\nu$, $(v-u)^+ = 0$ on Σ_λ^ν.

(b) Let L denote the Lipschitz constant of f on $[\inf_K u\,, \sup_K v]$ and as before let us put $B^\nu_\lambda = K \cap \Sigma^\nu_\lambda$. Since outside K we have $v > u \Rightarrow f(v) - f(u) \leq 0$, we get by (3.3),

$$c\int_{\Sigma^\nu_\lambda \cap \{v \geq u+\varepsilon\}} (|Du|+|Dv|)^{p-2}|D(v-u)^+|^2 dx \leq L\int_{B^\nu_\lambda \cap \{v\geq u+\varepsilon\}} ((v-u)^+)^2 dx$$

As $\varepsilon \to 0$, we get

$$c\int_{\Sigma^\nu_\lambda} (|Dv| + |Du|)^{p-2}|D(v-u)^+|^2\, dx \leq L\int_{B^\nu_\lambda} |(v-u)^+|^2\, dx$$

and the integrands being nonnegative we have

$$c\int_{B^\nu_\lambda} (|Dv| + |Du|)^{p-2}|D(v-u)^+|^2\, dx \leq L\int_{B^\nu_\lambda} |(v-u)^+|^2\, dx$$

Since $(v-u)^+$ vanishes on T^ν_λ and is in $W^{1,2}_{\text{loc}}(\mathbb{R}^N)$, Lemma 2.1 can be used to get

$$\begin{aligned} & c\int_{A\cup B} (|Dv| + |Du|)^{p-2}|D(v-u)^+|^2 dx \\ \leq \quad & LC|B^\nu_\lambda|^{\frac{1}{N}} \left(|A|^{\frac{1}{N}} \int_A |D(v-u)^+|^2 dx + |B|^{\frac{1}{N}} \int_B |D(v-u)^+|^2 dx \right). \end{aligned}$$

It then follows that

$$\begin{aligned} & \frac{c}{M_K^{2-p}}\int_A |D(v-u)^+|^2\, dx + \frac{c}{M_B^{2-p}}\int_B |D(v-u)^+|^2\, dx \qquad (3.5) \\ \leq \quad & LC|B^\nu_\lambda|^{\frac{1}{N}} \left(|A|^{\frac{1}{N}} \int_A |D(v-u)^+|^2\, dx + |B^\nu_\lambda|^{\frac{1}{N}} \int_B |D(v-u)^+|^2\, dx \right). \end{aligned}$$

Now choose $\delta = \frac{1}{|B^\nu_\lambda|}\left(\frac{c}{LCM_K^{2-p}}\right)^N$ and $M^{p-2} = \frac{LC|B^\nu_\lambda|^{(2/N)}}{c}$. With these choices of δ and M, it is easy to see from (3.5) that under the assumptions as in (b), $|D(v-u)^+| = 0$ a.e in Σ^ν_λ and hence $v \leq u$ in Σ^ν_λ. □

Now we prove a weak comparison principle, when $u \in C^1$ is a weak solution of (1.1) satifying (1.2), f satisfies (H1), (H3), and (1.3) holds. We choose a ball $B_{\hat{R}_0}$, centered at the origin and of radius $\hat{R}_0$ such that for $x \notin B_{\hat{R}_0}$, we have

$$\begin{aligned} |u(x)| \quad &\leq \quad \frac{C_0}{|x|^m} \leq s_0, \\ \left(\, \frac{C_0'}{|x|^m} \right. \quad &\leq \quad \left. |u(x)| \leq s_0 \text{ if } \alpha < 0 \,\right), \\ |Du(x)| \quad &\leq \quad \frac{C_0}{|x|^{m+1}}\,. \end{aligned}$$

PROPOSITION 3.2. *Let $N \geq 2$, $1 < p < 2$ and u in $C^1(\mathbb{R}^N)$ be a weak solution of (1.1) satisfying (1.2) . Let f satisfy (H1) and (H3) and assume that (1.3) holds. Let v be u_λ^ν.*

(a) *There exists $R_0 > \hat{R}_0$, R_0 depending on $N, p, f, \hat{R}_0$, such that if*

$$\Sigma_\lambda^\nu \cap R_\lambda^\nu(B_{R_0}) = \phi,$$

then $v \leq u$ on Σ_λ^ν.

(b) *Let $\Sigma_\lambda^\nu \cap R_\lambda^\nu(B_{R_0})$ be nonempty and let R_1, depending on R_0, λ and ν, be such that $\Sigma_\lambda^\nu \cap R_\lambda^\nu(B_{R_0}) \subset B(P_\lambda^\nu, R_1)$, where P_λ^ν is the projection of the origin on T_λ^ν, and that after shifting the origin to P_λ^ν, we have*

$$\left.\begin{array}{lclcl} |u(x)| & \leq & \frac{C(\lambda,\nu)}{|x|^m} & \leq & s_o\,, \\ \left(\ \frac{C_1(\lambda,\nu)}{|x|^m}\right. & \leq & |u(x)| & \leq & s_0 \quad \text{if } \alpha < 0\ \Big), \\ |Du(x)| & \leq & \frac{C(\lambda,\nu)}{|x|^{m+1}} & & . \end{array}\right\} \tag{3.6}$$

for $x \notin B(P_\lambda^\nu,\ R_1)$. Then there exists $R > R_1$, depending on R_1, such that for any $\hat{R} \geq R$ there exist positive numbers δ and M, depending on $\hat{R}, \lambda, p, N, f$ and the L^∞ norms of $u, v, |Du|$ and $|Dv|$ on $B(P_\lambda^\nu, \hat{R})$ with the property that whenever

$$\text{supp } (v-u)^+ \cap B(P_\lambda^\nu, \hat{R}) = A \cup B$$

with $|A \cap B| = 0\,, M_B < M$ and $|A| < \delta$, we have

$$v \leq u \quad \text{in } \Sigma_\lambda^\nu.$$

Proof. Let us take $[(v-u-\varepsilon)^+]^t$, $t \geq 1$, $\varepsilon > 0$, as a test function. Proceeding as in the deduction of (3.3) we get

$$\begin{aligned} c\int_{\Sigma_\lambda^\nu \cap \{v \geq u+\varepsilon\}} & (|Du| + |Dv|)^{p-2}\, t\, [(v-u-\varepsilon)^+]^{t-1} |D(v-u)^+|^2 dx \\ \leq \ \int_{\Sigma_\lambda^\nu \cap \{v \geq u+\varepsilon\}} & (f(v) - f(u))[(v-u-\varepsilon)^+]^t\, dx \end{aligned}$$

and passing to the limit as $\varepsilon \to 0$ we have

$$\begin{aligned} & c\int_{\Sigma_\lambda^\nu \cap \{v \geq u\}} (|Du| + |Dv|)^{p-2}\, t[(v-u)^+]^{t-1} |D(v-u)^+|^2\, dx \\ \leq & \ \liminf_{\varepsilon \to 0} \int_{\Sigma_\lambda^\nu \cap \{v \geq u+\varepsilon\}} (f(v) - f(u))[(v-u-\varepsilon)^+]^t dx \end{aligned} \tag{3.7}$$

(a) Let $R > \hat{R}_0$, to be fixed later. There exists $R_0 = R_0(R) > R$ such that $\inf_{B_R} u \geq \sup_{\mathbb{R}^N \setminus B_{R_0}} u$.

If $\Sigma_\lambda^\nu \cap R_\lambda^\nu(B_{R_0}) = \phi$ we have then that $u(x) \geq v(x) = u(x_\lambda^\nu)$ in B_R, so we get

$$\text{supp } (v-u)^+ \subset \Sigma_\lambda^\nu \setminus B_R, \quad u(x) \leq v(x) \leq s_0 \text{ in supp } (v-u)^+ .$$

Then by (H3) and (1.2),

$$\int_{\Sigma_\lambda^\nu} (f(v)-f(u))[(v-u-\varepsilon)^+]^t\,dx \le C\int_{\Sigma_\lambda^\nu\setminus B_R} z^\alpha |(v-u)^+|^{t+1}\,dx$$

where $z = u$ if $\alpha < 0$, while $z = v$ if $\alpha \ge 0$.
In the first case $u(x) \ge \frac{C_0'}{|x|^m}$ in $\mathbb{R}^N \setminus B_R$ by hypothesis, while in the second case we have

$$v(x) = u(x_\lambda^\nu) \le \frac{C_0}{|x_\lambda^\nu|^m} \le \frac{C_0}{|x|^m}$$

because the hypothesis $\Sigma_\lambda^\nu \cap R_\lambda^\nu(B_{R_0}) = \phi$ implies that $\lambda > 0$, so that $|x| \le |x_\lambda^\nu|$.

In any case we have that if $\Sigma_\lambda^\nu \cap R_\lambda^\nu(B_{R_0}) = \phi$, $R_0 = R_0(R)$, then

$$\int_{\Sigma_\lambda^\nu} (f(v)-f(u))[(v-u-\varepsilon)^+]^t\,dx \le C\int_{\Sigma_\lambda^\nu\setminus B_R} \frac{1}{|x|^{m\alpha}}|(v-u)^+|^{t+1}\,dx.$$

Then by (3.7) we obtain, with $w = w(\lambda,\nu,t) = [(v-u)^+]^{\frac{t+1}{2}}$,

$$\int_{\Sigma_\lambda^\nu} (|Du|+|Dv|)^{p-2}|Dw|^2\,dx \le C\int_{\Sigma_\lambda^\nu\setminus B_R} \frac{1}{|x|^{m\alpha}}\,|w|^2\,dx\,. \tag{3.8}$$

Let us now fix $t \ge 1$ so that the integrals in (3.8) are finite and observe that by (1.3)

$$(|Du(x)| + |Du_\lambda^\nu(x)|) \le C\left(\frac{1}{|x|^{m+1}} + \frac{1}{|x_\lambda^\nu|^{m+1}}\right) \le \frac{2C}{|x|^{m+1}}$$

and hence

$$(|Du(x) + |Du_\lambda^\nu(x)|)^{p-2} \ge C_1|x|^{(m+1)(2-p)}.$$

This yields

$$C\int_{\Sigma_\lambda^\nu} |x|^{(m+1)(2-p)}|Dw|^2\,dx \le \int_{\Sigma_\lambda^\nu} (|Du|+|Dv|)^{p-2}|Dw|^2\,dx \tag{3.9}$$

On the other hand, using Lemma 2.3, with $q = 2$, $T = 0$ and $s = (m+1)(2-p)$, after extending $(v-u)^+ = 0$ outside Σ_λ^ν, we get

$$\int_{\Sigma_\lambda^\nu\setminus B_R} \frac{|w|^2}{|x|^{m\alpha}}\,dx = \int_{\Sigma_\lambda^\nu\setminus B_R} \frac{|w|^2|x|^{(m+1)(2-p)-2}}{|x|^{m\alpha-2+(m+1)(2-p)}}\,dx \tag{3.10}$$

$$\le C \sup_{\Sigma_\lambda^\nu\setminus B_R}\left(\frac{1}{|x|^{m\alpha-2+(m+1)(2-p)}}\right)\int_{\Sigma_\lambda^\nu} |Dw|^2\,|x|^{(m+1)(2-p)}\,dx.$$

Notice that $m(\alpha+2-p) > p$ if and only if $m\alpha > 2-(m+1)(2-p)$. Then

$$\sup_{\Sigma_\lambda^\nu\setminus B_R}\left(\frac{1}{|x|^{m\alpha-2+(m+1)(2-p)}}\right) \le \frac{1}{R^{m\alpha-2+(m+1)(2-p)}} := G(R)$$

with $G(R) \to 0$ as $R \to \infty$.

In view of (3.9) and (3.10), (3.8) reduces to

$$\int_{\Sigma_\lambda^\nu} |x|^{(m+1)(2-p)}|Dw|^2\,dx \le C\,G(R)\int_{\Sigma_\lambda^\nu} |x|^{(m+1)(2-p)}\,|Dw|^2\,dx \quad (3.11)$$

We can now choose $R > \hat{R}_0$, such that $G(R)\,C < 1$, since $G(R) \to 0$ as $R \to \infty$ and then take $R_0 = R_0(R)$. Then (3.11) will imply that $|Dw| = 0$ a.e and hence $w = 0$ a.e on Σ_λ^ν i. e. $u \ge v$ on Σ_λ^ν.

(b) Now suppose that $\Sigma_\lambda^\nu \cap R_\lambda^\nu(B_{R_0}) \ne \phi$ and let $R > R_1$, to be fixed later. For $\hat{R} \ge R$ define $K = B(P_\lambda^\nu, \hat{R})$. Let $P_\lambda^\nu \in T_\lambda^\nu$ be the new origin. Observe that in $\Sigma_\lambda^\nu \cap K^c \cap \text{supp}\,(v-u)^+$ we have $u \le v \le s_0$ and the estimate (3.6) holds. Further $|x| = |x_\lambda^\nu|$. Hence we can proceed as in (a) to obtain estimates analogous to (3.9) and (3.10) outside K, namely

$$C\int_{\Sigma_\lambda^\nu\setminus K} |x|^{(m+1)(2-p)}|Dw|^2\,dx \le \int_{\Sigma_\lambda^\nu\setminus K} (|Du|+|Dv|)^{p-2}|Dw|^2\,dx$$

and, using Lemma 2.3 with $q=2$, $s=(m+1)(2-p)$ and $T=\hat{R}$,

$$\begin{aligned}&\int_{\Sigma_\lambda^\nu\setminus K} (f(v)-f(u))\left[(v-u-\varepsilon)^+\right]^t\,dx\\ &\le \int_{\Sigma_\lambda^\nu\setminus K} \frac{1}{|x|^{m\alpha}}\,|w|^2\,dx \le C\,G(R)\int_{\Sigma_\lambda^\nu\setminus K} |Dw|^2\,|x|^{(m+1)(2-p)}\,dx\end{aligned}$$

where $w = [(v-u)^+]^{\frac{t+1}{2}}$, $G(R) \to 0$ as $R \to \infty$ and the constants now depend on λ and ν.

Let L be the Lipschitz constant of f on $[\inf_K u(x),\ ||u||_\infty]$ and as before $B_\lambda^\nu = K \cap \Sigma_\lambda^\nu$. Then

$$\begin{aligned}&\int_{B_\lambda^\nu} (f(v)-f(u))[(v-u-\varepsilon)^+]^t\,dx \le L\int_{B_\lambda^\nu} |w|^2\,dx \qquad (3.12)\\ &\le LC|B_\lambda^\nu|^{1/N}\left(|A|^{1/N}\int_A |Dw|^2\,dx + |B|^{1/N}\int_B |Dw|^2\,dx\right)\end{aligned}$$

by Lemma 2.1, since w vanishes on T_λ^ν and belongs to $W^{1,2}_{loc}(\mathbb{R}^N)$.

The L.H.S of (3.7) can be bounded below as follows:

$$\begin{aligned}&\int_{\Sigma_\lambda^\nu} (|Du|+|Dv|)^{p-2}|Dw|^2\,dx \quad \ge \qquad (3.13)\\ &\frac{1}{M_K^{2-p}}\int_A |Dw|^2\,dx + \frac{1}{M_B^{2-p}}\int_B |Dw|^2\,dx + C\int_{\Sigma_\lambda^\nu\setminus K} |x|^{(m+1)(2-p)}\,|Dw|^2\,dx\,.\end{aligned}$$

Combining this inequality with (3.12) and the one for the integral on $\Sigma_\lambda^\nu \setminus K$ we get

$$\begin{aligned}
&\frac{c}{M_K^{2-p}}\int_A |Dw|^2\,dx + \frac{c}{M_B^{2-p}}\int_B |Dw|^2\,dx + C_2\int_{\Sigma_\lambda^\nu\setminus K} |x|^{(m+1)(2-p)}\,|Dw|^2\,dx \\
&\quad\le\ (LC|B_\lambda^\nu|^{1/N})\left(|A|^{1/N}\int_A |Dw|^2\,dx + |B_\lambda^\nu|^{1/N}\int_B |Dw|^2\,dx\right) \qquad (3.14)\\
&\quad\quad + C_3\,G(R)\int_{\Sigma_\lambda^\nu\setminus K} |x|^{(m+1)(2-p)}|Dw|^2\,dx.
\end{aligned}$$

where C_2 and C_3 depend on λ, ν. We take now R so that $G(\hat{R}) \le G(R) < \frac{C_2(\lambda,\nu)}{C_3(\lambda,\nu)}$ for each $\hat{R} \ge R$. Then it is enough to choose

$$\delta = \frac{1}{|B_\lambda^\nu|}\left(\frac{c}{LCM_k^{2-p}}\right)^N \quad \text{and } M^{p-2} = \left(\frac{LC|B_\lambda^\nu|^{2/N}}{c}\right)$$

With these choices of δ and M, it is easy to see that whenever

$$\text{supp }(v-u)^+\cap K = A\cup B\,,\quad |A\cap B| = 0\,,\, |A| < \delta\,,\, M_B < M\,,$$

(3.14) implies that $|D(v-u)^+| = 0$ a.e and hence $v \le u$ on Σ_λ^ν. □

Now we introduce the set

$$\Lambda(\nu) = \{\lambda\in\mathbb{R}\,:\, u\,\ge\, u_\mu^\nu \quad\text{in}\quad \Sigma_\mu^\nu \quad \forall\,\mu>\lambda\}\,. \qquad (3.15)$$

Then if $\Lambda(\nu)\ne\phi$, we consider

$$\lambda_0(\nu) = \inf\Lambda(\nu). \qquad (3.16)$$

We further define

$$Z\ =\ \{x\in\mathbb{R}^N\,:\, Du(x) = 0\}\,, \qquad (3.17)$$

$$Z_\lambda^\nu\ =\ \{x\in\Sigma_\lambda^\nu\,:\, Du(x) = Du_\lambda^\nu(x) = 0\}\,. \qquad (3.18)$$

We will denote by P_λ^ν the projection of the origin on T_λ^ν.

Remark 3.1. Observe that for a given λ_0 and ν_0, $\hat{R}$ can be chosen so large that for all λ in a small neighbourhood of λ_0, say $[\lambda_0-\theta_0, \lambda_0+\theta_0]$, and for all ν in $\overline{I_{\theta_0}(\nu_0)} := \{\nu\,:\,|\nu| = 1\,,\,|\nu-\nu_0|\le\theta_0\}$ we have

$$\hat{R}\ge R = R(\lambda,\nu) \quad\text{and}\quad B(\,P_\lambda^\nu\,,R)\,\subset\,B(\,P_{\lambda_0}^{\nu_0},\hat{R}\,)$$

where $R = R(\lambda,\nu)$ is the number given by Proposition 3.1 (b) or 3.2 (b), corresponding to (H2) or (H3) respectively. Further δ and M can also be chosen uniformly for all these λ and ν, by replacing $|B_\lambda^\nu|$ by $|B(\,P_{\lambda_0}^{\nu_0},\hat{R}\,)|$ on the R.H.S of (3.5) or (3.14). □

The next lemma gives some sufficient conditions under which the procedure of moving the hyperplanes can be continued, using the comparison principles.

LEMMA 3.1. *Let $u \in C^1(\mathbb{R}^N)$ be a weak solution of (1.1) for $1 < p < 2$. Assume that either (H1) and (H2) or (H1), (H3), (1.2) and (1.3) hold. Let $\lambda_0 \in \Lambda(\nu_0)$. Let $\hat{R}$ be as in Remark 3.1 corresponding to a θ_0-neighbourhood of (λ_0, ν_0).*

If we have

$$u > u_{\lambda_0}^{\nu_0} \quad \text{in} \quad (\Sigma_{\lambda_0}^{\nu_0} \setminus Z_{\lambda_0}^{\nu_0}) \cap B(P_{\lambda_0}^{\nu_0}, \hat{R}),$$

then there exists $\theta_1 \leq \theta_0$ such that

(a) $u \geq u_\lambda^{\nu_0}$ *in* $\Sigma_\lambda^{\nu_0}$ $\quad \forall\, \lambda \in (\lambda_0 - \theta_1\,,\, \lambda_0 + \theta_1)$,

(b) $u \geq u_\lambda^{\nu}$ *in* Σ_λ^{ν} $\quad \forall\, \lambda \in (\lambda_0 - \theta_1\,,\, \lambda_0 + \theta_1),\ \forall\, \nu \in I_{\theta_1}(\nu_0)$,

where $I_{\theta_1}(\nu_0) := \{\nu \,:\, |\nu| = 1\,,\, |\nu - \nu_0| < \theta_1\}$.

Proof. Let $K = B(P_{\lambda_0}^{\nu_0}, \hat{R})$ and let δ and M be chosen as in Remark 3.1 uniformly for all (λ, ν), in a θ_0-neighbourhood of (λ_0, ν_0). Now choose a small neighbourhood A of $\overline{K} \cap T_{\lambda_0}^{\nu_0}$ with $|A| < \delta/2$ and another neighbourhood O of $Z_{\lambda_0}^{\nu_0} \cap K$ with $M_O^{\nu_0,\lambda_0} < M/2$ where

$$M_O^{\nu,\lambda} := \sup_O\{|Du| + |Du_\lambda^\nu|\}.$$

Now $K_0 = \overline{K \cap \Sigma_{\lambda_0}^{\nu_0}} \setminus (A \cup O)$ is a compact set and by our assumption

$$u - u_{\lambda_0}^{\nu_0} > m > 0 \quad \text{on } K_0.$$

Then continuity with respect to λ and ν imply that there exists θ_1 depending on λ_0, ν_0 and K with $\theta_1 \leq \theta_0$, such that

$$\begin{aligned} u - u_\lambda^\nu &> \frac{m}{2} \quad \text{on} \quad K_0, \\ M_{O \cap \Sigma_\lambda^\nu}^{\lambda,\nu} &< M \\ |A \cap B(P_\lambda^\nu, R) \cap \Sigma_\lambda^\nu| &< \delta\,, \\ B(P_\lambda^\nu, R) \cap \Sigma_\lambda^\nu &\subset (B(P_{\lambda_0}^{\nu_0}, R) \cap \Sigma_{\lambda_0}^{\nu_0}) \cup A\,, \end{aligned}$$

for all $\lambda \in (\lambda_0 - \theta_1, \lambda_0 + \theta_1)$ and $\nu \in I_{\theta_1}(\nu_0)$. Here $R = R(\lambda)$ is the number given by Proposition 3.1 (b) or 3.2 (b), corresponding to (H2) or (H3) respectively. Then clearly for such λ's near λ_0

$$\text{supp } (\chi_{K_\lambda}(u_\lambda^{\nu_0} - u)^+) \subset (A \cup O) \cap K_\lambda,$$

where $K_\lambda = B(P_\lambda^{\nu_0}, R)$. Further K_λ and

$$\text{supp } (\chi_{K_\lambda}(u_\lambda^{\nu_0} - u)^+) = ((K_\lambda \cap A) \cup (K_\lambda \cap O)) \cap \Sigma_\lambda^{\nu_0}$$

satisfy the conditions of Proposition 3.1 (b) or 3.2 (b). Hence we can conclude that $u_\lambda^{\nu_0} \leq u$ on $\Sigma_\lambda^{\nu_0}$ for all $\lambda \in (\lambda_0 - \theta_1, \lambda_0 + \theta_1)$ and (a) is proved. The proof of (b) is analogue. □

The following corollary can be deduced easily by using a similar reasoning as in the above theorem.

COROLLARY 3.1. *Under the same assumptions on u and f as in Lemma 3.1, suppose that for some $\lambda_0 \in \Lambda(\nu_0)$ there exist open neighbourhoods A and O of the sets $T^{\nu_0}_{\lambda_0} \cap B(P^{\nu_0}_{\lambda_0}, \hat{R})$ and $Z^{\nu_0}_{\lambda_0}$ respectively such that*

$$u > u^{\nu_0}_{\lambda_0} \quad \text{in} \quad \Sigma^{\nu_0}_{\lambda_0} \cap \Big(B(P^{\nu_0}_{\lambda_0}, \hat{R}) \setminus (A \cup O)\Big)$$

with $|A| < \delta/2$ and $M^{\nu_0,\lambda_0}_O < \frac{M}{2}$, where δ and M are chosen as in Remark 3.1 uniformly for all (λ, ν) in a θ_0-neighbourhood of (λ_0, ν_0). Then there exists θ_1 small enough such that (a) and (b) of lemma 3.1 hold.

4. **Proof of the Symmetry Results**

As in [5] and [7], the comparison theorems are crucial for the proof of the symmetry results and we exploit the theorems proved in the previous section to derive first a partial (local) symmetry result.

LEMMA 4.1 (Partial Symmetry)**.** *Let $u \in C^1(\mathbb{R}^N)$ be a weak solution of (1.1) with $1 < p < 2$ and let either (H1) and (H2) or (H1), (H3), (1.2), (1.3) hold.*

(a) *For any direction ν we have $\Lambda(\nu) \neq \phi$ and*

$$u > u^\nu_\lambda \quad \text{in} \quad \Sigma^\nu_\lambda \setminus Z^\nu_\lambda, \quad \forall \quad \lambda,\ \lambda_0(\nu) < \lambda\,, \tag{4.1}$$

$$\frac{\partial u}{\partial \nu}(x) < 0 \quad \forall \quad x \in (\Sigma^\nu_{\lambda_0(\nu)})' \setminus Z\,. \tag{4.2}$$

where $\Lambda(\nu)$ and $\lambda_0(\nu)$ are defined in (3.15) and (3.16).

(b) *Let $K = B(P^\nu_{\lambda_0(\nu)}, \hat{R})$, be a ball centered at $P^\nu_{\lambda_0(\nu)} \in T^\nu_{\lambda_0(\nu)}$, having radius $\hat{R}$, where $\hat{R}$ is chosen as in Remark 3.1 corresponding to $\lambda_0(\nu)$ and ν. Then either $u \equiv u^\nu_{\lambda_0(\nu)}$ on $\Sigma^\nu_{\lambda_0(\nu)}$ or there exists at least one connected component C^ν of $\Sigma^\nu_{\lambda_0(\nu)} \setminus Z^\nu_{\lambda_0(\nu)}$ such that $C^\nu \cap K \neq \phi$ and $u \equiv u^\nu_{\lambda_0(\nu)}$ in C^ν. In the latter case, we also have*

$$Du(x) \neq 0 \quad \forall\, x \in C^\nu \text{ and } Du(x) = 0 \quad \forall\, x \in \partial C^\nu \setminus T^\nu_{\lambda_0(\nu)}\,. \tag{4.3}$$

Proof. (a) We prove that $\Lambda(\nu)$ actually contains an interval $(R_0, +\infty)$ where R_0 is defined as in Proposition 3.1 (a) or 3.2 (a) corresponding to f satisfying (H2) or (H3) respectively. If $\lambda \geq R_0$,

$$\Sigma^\nu_\lambda \cap R^\nu_\lambda(B_{R_0}) = \phi.$$

Hence applying the weak comparison principle, (part (a) of Proposition 3.1 or 3.2) we get $u \geq v$ in Σ^ν_λ so that $\lambda \in \Lambda(\nu)$ for any $\lambda \geq R_0$. The other assertions follow as in [7] (see the proof of Theorem 4.1 in [7]).

(b) At the minimal position $\lambda_0(\nu)$, either $u \equiv u^\nu_{\lambda_0(\nu)}$ in $\Sigma^\nu_{\lambda_0(\nu)}$ or there exists a point $x_0 \in \Sigma^\nu_{\lambda_0(\nu)}$ such that $u(x_0) > u^\nu_{\lambda_0(\nu)}(x_0)$. Let us assume that the latter happens.

Arguing by contradiction, we suppose that there does not exist any component C^ν of $\Sigma^\nu_{\lambda_0(\nu)} \backslash Z^\nu_{\lambda_0(\nu)}$ such that $C^\nu \cap K \neq \phi$ and $u \equiv u^\nu_{\lambda_0(\nu)}$ on C^ν. By the strong comparison principle (Theorem 1.4 in [4]) we get $u > u^\nu_{\lambda_0(\nu)}$ on $B^\nu_{\lambda_0(\nu)} \backslash Z^\nu_{\lambda_0(\nu)}$ with $B^\nu_{\lambda_0(\nu)} = K \cap \Sigma^\nu_{\lambda_0(\nu)}$. Then by Lemma 3.1, we see that $u \geq u^\nu_\lambda$ holds for $\lambda \in (\lambda_0(\nu) - \theta_1, \lambda_0(\nu))$, contradicting the minimality of $\lambda_0(\nu)$. Thus the first part of the assertion is proved. The rest of the assertions can be proved as in [7] Theorem 4.1.

□

Once we have this partial symmetry theorem the end of the proof of Theorem 1.1 and 1.2 proceeds very much like the proof of Theorem 1.1 in [7], Sect.5, and consists roughly speaking in showing that partial symmetry without symmetry is not possible.

For the reader's convenience we include in Appendix 1 the proof.

Appendix 1

In this appendix we recall the technique used in [5] and [7] to conclude the proof of the symmetry theorems. We continue the numeration of formulas of section 4.

The idea of the proof is to show that if $u \not\equiv u^\nu_{\lambda_0(\nu)}$ for some ν, then, using Lemma 4.1, we can construct a subset Γ of Z on which $u \equiv$ constant and whose projection on $T^\nu_{\lambda_0(\nu)}$ contains an open subset of that hyperplane. The fact that such a set cannot exist has been proved already in [5] for bounded domain and in [7] for unbounded domain. This contradiction will help us to conclude.

As in [7], we remark here that if the critical set Z of u were regular, then u would be constant on the boundary ∂C^ν of the component C^ν constructed above. This would give us the "bad" set Γ indicated in the beginning of the proof. But in general Z need not be regular. (See the discussion in Remark 3.1 in [5]). So we construct such a set Γ in the next step, by moving simultaneously the hyperplanes orthogonal to the directions ν in a neighbourhood of ν_0 and picking a set which is symmetric with respect to $T^\nu_{\lambda_0(\nu)}$ for all these ν's simultaneously.

Let us now fix a direction ν_0 such that $u \not\equiv u^{\nu_0}_{\lambda_0(\nu_0)}$ in $\Sigma^{\nu_0}_{\lambda_0(\nu_0)}$ and choose θ_0, $\hat{R}$, as in Remark 3.1 corresponding to $\lambda_0(\nu_0)$, ν_0. Let $\mathcal{F}_{\nu_0}$ be the collection of the connected components C^{ν_0} of $(\Sigma^{\nu_0}_{\lambda_0\,(\nu_0)} \setminus Z^{\nu_0}_{\lambda_0\,(\nu_0)})$ such that $C^{\nu_0} \cap K \neq \phi$ and $u \equiv u^{\nu_0}_{\lambda_0\,(\nu_0)}$ in C^{ν_0}, and let $\{C^{\nu_0}_i\}_{i \in I \subset \mathbb{N}}$ be an enumeration of the sets in $\mathcal{F}_{\nu_0}$.

Step - 1: *Let* $K = B(P^{\nu_0}_{\lambda_0(\nu_0)}, \hat{R})$ *and let us put*

$$\tilde{\mathcal{F}}_{\nu_0} = \{\tilde{C}_i \,:\, C_i \in \mathcal{F}_{\nu_0}\}.$$

where

$$\tilde{C}_i = C_i \cup R^{\nu_0}_{\lambda_0\,(\nu_0)}(C_i) \cup (\partial\, C_i \cap T^{\nu_0}_{\lambda_0\,(\nu_0)} \setminus Z).$$

Then there exists a direction ν_1 near ν_0 and a set $\tilde{C}_i^{\nu_1} \in \tilde{\mathcal{F}}_{\nu_1}$ such that the set $\tilde{C}_i^{\nu_1} \in \tilde{\mathcal{F}}_{\nu}$, for every ν in a small neighbourhood of ν_1 .

Proof : We first prove that $\lambda_0(\nu)$ is a continuous function of ν at ν_0.

Let us fix $\varepsilon > 0$, $\varepsilon < \theta_0$. By the definition of $\lambda_0(\nu_0)$, there exist $\lambda \in (\lambda_0(\nu_0) - \varepsilon, \lambda_0(\nu_0))$ and $x \in \Sigma_\lambda^{\nu_0}$ such that $u(x) < u_\lambda^{\nu_0}(x)$. By continuity of u with respect to ν, there exists $\delta_1 > 0$ such that $x \in \Sigma_\lambda^\nu$ and $u(x) < u_\lambda^\nu(x)$ for every $\nu \in I_{\delta_1}(\nu_0)$. Hence for every $\nu \in I_{\delta_1}(\nu_0)$ we have

$$\lambda_0(\nu_0) - \varepsilon < \lambda < \lambda_0(\nu).$$

Now we claim that there exists $\delta_2 > 0$ such that $\lambda_0(\nu) < \lambda_0(\nu_0) + \varepsilon$ for any $\nu \in I_{\delta_2}(\nu_0)$.

If this is not true, then there exists a sequence $\{\nu_n\}$ of directions such that $\nu_n \to \nu_0$ and $\lambda_0(\nu_n) \geq \lambda_0(\nu_0) + \varepsilon\ \forall\ n$. By Lemma 4.1 (a) we have that $\lambda_0(\nu) < R_0$ for any direction ν. Thus the sequence $\lambda_0(\nu_n)$ is bounded and hence, up to a subsequence, it converges to a number $\overline{\lambda} \geq \lambda_0(\nu_0) + \varepsilon$. Then by (4.1) we have

$$u > u_{\overline{\lambda}}^{\nu_0} \quad \text{in} \quad \Sigma_{\overline{\lambda}}^{\nu_0} \setminus Z_{\overline{\lambda}}^{\nu_0}.$$

Now by Lemma 3.1 we have that $u \geq u_\lambda^\nu$ in Ω_λ^ν for any λ close to $\overline{\lambda}$ and any ν close to ν_0. In particular, this will hold for ν_n and $\lambda_0(\nu_n) - \gamma$, for some n large and γ small, contradicting the definition of $\lambda_0(\nu_n)$.

Thus it follows that for each $\varepsilon > 0$ there exists $\eta = \eta(\varepsilon) > 0$ such that if $\nu \in I_\eta(\nu_0)$,

$$\lambda_0(\nu_0) - \varepsilon < \lambda_0(\nu) < \lambda_0(\nu_0) + \varepsilon.$$

Let M and δ corresponding to K be chosen as in Remark 3.1, uniformly for all (λ, ν) in a θ_0–neighbourhood of (λ_0, ν_0). Choose a bounded neighbourhood O of $Z_{\lambda_0(\nu_0)}^{\nu_0} \cap K$ such that

$$M_O^{\nu_0, \lambda_0(\nu_0)} < \frac{M}{4}\,.$$

Choose a neighbourhood A of $T_{\lambda_0(\nu_0)}^{\nu_0} \cap \overline{K}$ such that

$$|A \cap K| < \frac{\delta}{4}\,.$$

Notice that $u - u_{\lambda_0(\nu_0)}^{\nu_0} \geq \gamma > 0$ in $\overline{K \cap \Sigma_{\lambda_0(\nu_0)}^{\nu_0}} \setminus \left(A \cup O \cup \bigcup_{i \in I} C_i^{\nu_0}\right)$.

By continuity there exists $\eta_0 \leq \theta_0$ such that for each $\nu \in I_{\eta_0}(\nu_0)$ we have

$$\lambda_0(\nu_0) - \theta_0 < \lambda_0(\nu) < \lambda_0(\nu_0) + \theta_0$$

and, putting $B_{\lambda_0(\nu)}^\nu = B(P_{\lambda_0(\nu)}^\nu, \hat{R}) \cap \Sigma_{\lambda_0(\nu)}^\nu$,

$$B_{\lambda_0(\nu)}^\nu \cap Z_{\lambda_0(\nu)}^\nu \subset O\,, \qquad B_{\lambda_0(\nu)}^\nu \subset B_{\lambda_0(\nu_0)}^{\nu_0} \cup A, \tag{4.4}$$

$$u - u_{\lambda_0(\nu)}^\nu > 0 \text{ in } \overline{B_{\lambda_0(\nu)}^\nu} \setminus \left(A \cup O \cup \bigcup_{i \in I} C_i^{\nu_0}\right) \tag{4.5}$$

$$M_{O \cap \Sigma_{\lambda_0(\nu)}^\nu}^{\nu, \lambda_0(\nu)} < M/2\,, \quad |A \cap B_{\lambda_0(\nu)}^\nu| < \delta/2. \tag{4.6}$$

Our second claim is that for each $\nu \in I_{\eta_0}(\nu_0)$, there exists some $\tilde{C}_i \in \tilde{\mathcal{F}}_\nu \cap \tilde{\mathcal{F}}_{\nu_0}$.

Define for each $C_i \in \mathcal{F}_{\nu_0}$,

$$S_i = \overline{C_i \cap K} \setminus (A \cup O).$$

Notice that S_i is compact for each i and there must be some S_i which are nonempty. In fact, if all the S_i were empty, then $C_i \cap K \subset A \cup O$ for each $C_i \in \mathcal{F}_{\nu_0}$ and hence $u > u_{\lambda_0(\nu_0)}^{\nu_0}$ would hold in $K \setminus (A \cup O)$. Then by Corollary 3.1, $\lambda_0(\nu_0)$ would not be the critical position, which is absurd.

Another crucial remark is that if ν_1 and ν_2 are two directions and $C^{\nu_1} \in \mathcal{F}_{\nu_1}$, and $C^{\nu_2} \in \mathcal{F}_{\nu_2}$, then either $\tilde{C^{\nu_1}} \cap \tilde{C^{\nu_2}} = \phi$ or $\tilde{C^{\nu_1}} \equiv \tilde{C^{\nu_2}}$. (For a proof of this, see the arguments in Remark 4.1 in [5]).

Let us fix $\nu \in I_{\eta_0}(\nu_0)$. We can have

(i) either $u \equiv u^{\nu}_{\lambda_0(\nu)}$ on some S_i ,
(ii) or $u > u^{\nu}_{\lambda_0(\nu)}$ on all S_i.

In case (i), by the strong comparison principle, S_i is contained in some component C^ν of $\Sigma^\nu_{\lambda_0(\nu)} \backslash Z^\nu_{\lambda_0(\nu)}$. Hence $u \equiv u^\nu_{\lambda_0(\nu)}$ in C^ν. Then $S_i \subset C^\nu \cap C_i$ and hence $\tilde{C^\nu} = \tilde{C_i} \in \tilde{\mathcal{F}}_\nu \cap \tilde{\mathcal{F}}_{\nu_0}$.

In case (ii) happens, $u > u^\nu_{\lambda_0(\nu)}$ holds in every S_i and hence in $B^\nu_{\lambda_0(\nu)} \setminus (A \cup O)$ because of (4.4) and (4.5). Then using (4.6), the minimality of $\lambda_0(\nu)$ is contradicted by Corollary 3.1. Hence case (ii) cannot arise and our claim is proved.

Next we claim that there exists a direction $\nu_1 \in I_{\eta_0}(\nu_0)$ and a neighbourhood $I_{\eta_1}(\nu_1)$ and a fixed $\tilde{C_i} \in \tilde{\mathcal{F}}_{\nu_1}$ such that for any $\nu \in I_{\eta_1}(\nu_1)$, the set $\tilde{C_i} \in \tilde{\mathcal{F}}_\nu$. Let $\{\tilde{C}^{\nu_0}\}_{i\in I}$ be an enumeration of the sets in $\tilde{\mathcal{F}}_{\nu_0}$.

If $\tilde{C}^{\nu_0}_1 \in \tilde{\mathcal{F}}_\nu$ for all $\nu \in I_{\eta_0}(\nu_0)$, then we stop. If not, there exists $\nu_1 \in I_{\eta_0}(\nu_0)$ such that $\tilde{C}^{\nu_0}_1 \notin \tilde{\mathcal{F}}_{\nu_1}$ i.e. $u > u^{\nu_1}_{\lambda_0(\nu_1)}$ on S_1. Since S_1 is compact we have that $u - u^{\nu_1}_{\lambda_0(\nu_1)} \geq \gamma > 0$ in S_1, so that $u > u^{\nu}_{\lambda_0(\nu)}$ holds in S_1 for the directions close to ν_1 and by the previous argument $C^{\nu_0}_1 \notin \mathcal{F}_\nu$ for all $\nu \in I_{\eta_1}(\nu_1)$ for some $\eta_1 < \eta_0 - |\nu_1 - \nu_0|$. Now we check if $\tilde{C}^{\nu_0}_2 \in \tilde{\mathcal{F}}_\nu$ for all $\nu \in I_{\eta_1}(\nu_1)$. If not, we find a direction $\nu_2 \in I_{\eta_1}(\nu_1)$ and a neighbourhood $I_{\eta_2}(\nu_2)$ such that $\eta_2 < \eta_1 - |\nu_2 - \nu_1|$ and $\tilde{C}^{\nu_0}_2 \notin \tilde{\mathcal{F}}_\nu$ for all $\nu \in I_{\eta_2}(\nu_2)$. Proceeding in this way,

(i) either we stop at some k'th stage where $\tilde{C}^{\nu_0}_{k+1} \in \tilde{\mathcal{F}}_\nu \; \forall \, \nu \in I_{\eta_k}(\nu_k)$
(ii) or the process does not stop at all and $I = \mathbb{N}$.

Now we claim that (ii) cannot arise. In case (ii), we obtain a sequence of nested compact sets $\{\overline{I_{\eta_i}(\nu_i)}\}_{i\in I}$ with the finite intersection property. Then by Cantor's intersection theorem $\cap_{i\in I}\overline{I_{\eta_i}(\nu_i)} \neq \phi$. For the direction ν in this intersection, $\tilde{C}^{\nu_0}_i \notin \tilde{\mathcal{F}}_\nu$ for all i. This contradicts our second claim. Thus step - 1 now follows.

Step - 2: *Let $\tilde{C}^{\nu_1}_i$ be as in step - 1 and let $(C^{\nu_1}_i)' = R^{\nu_1}_{\lambda_0(\nu_1)}(C^{\nu_1}_i)$. Then $\left(\partial C^{\nu_1}_i \cap \Sigma^{\nu_1}_{\lambda_0(\nu_1)}\right)'$ contains a subset Γ on which u is a constant and $Du = 0$ and whose projection on $T^{\nu_1}_{\lambda_0(\nu_1)}$ contains an open subset of that hyperplane.*

This step, can be proved exactly as in [7].(Refer the proof of step -3 in [7]).

Step - 3: *u is radially symmetric about some point $x_0 \in \mathbb{R}^N$ and is radially strictly decreasing.*

Proof : The set Γ constructed in step -2 leads to a contradiction, in view of the following proposition proved mainly using Hopf's lemma in [5] for bounded domain and in [7] for unbounded domain. (See Proposition 5.1 in [7]) :

Suppose that $u \in C^1(\mathbb{R}^N)$ is a weak solution of (1.1) for $1 < p < 2$. For any direction ν, the half space $(\Sigma^\nu_{\lambda_0(\nu)})'$ cannot contain a subset Γ of Z on which $u =$ constant and whose projection on the hyperplane $T^\nu_{\lambda_0(\nu)}$ contains a set, open in $T^\nu_{\lambda_0(\nu)}$.

Using this proposition we conclude that $u \equiv u^\nu_{\lambda_0(\nu)}$ at the critical position for every ν. Further we can conclude using the same proposition again that $Z \cap \Sigma^\nu_{\lambda_0(\nu)} = \phi$ for any direction ν (see the proof of Step-4 in [7]). Thus u is strictly decreasing in every direction. Now by considering N linearly independent directions in $\mathbb{R}^N$, the final symmetry result follows. ∎

Appendix 2

Here we recall some sufficient conditions which will help us to conclude that a nonnegative solution is necessarily a positive solution. In [15] this subject has been treated in a very general and exhaustive way, including also the case of solutions with compact support. For the reader's convenience we give here the proof in our particular case.

Proposition A1 *Let u be a nonnegative ground state solution of (1.1), for $1 < p < 2$. Let f satisfy (H1) and any one of the following set of conditions:*

(a) *either f satisfies (H2), $f(0) \geq 0$ and for $F(s) = \int_0^s [f(0) - f(t)]\, dt$,*

$$\int_0 \frac{ds}{(F(s))^{1/p}} = \infty$$

(b) *or there exists some $c > 0$ and $\delta > 0$ such that*

$$f(u) + cu^{p-1} \geq 0 \quad \forall s \in [0, \delta).$$

Then $u > 0$ in $\mathbb{R}^N$.

Proof : Let us recall the strict maximum principle, proved in [18]:
Let $u \in L^1_{loc}(\Omega)$, $\Delta_p u \in L^1_{loc}(\Omega)$ and $u \geq 0$ a.e. in Ω and

$$-\Delta_p u \geq -\beta(u) \quad \text{a.e.}$$

where $\beta : [0, \infty) \to \mathbb{R}$ is a continuous nondecreasing function with $\beta(0) = 0$ and

- either $\beta(s) = 0$ for some $s > 0$
- or $\beta(s) > 0$ for $s > 0$ and $\int_0 \frac{ds}{\left(\int_0^s \beta(t)dt\right)^{1/p}} = \infty$

Then we have either $u \equiv 0$ or $u > 0$ in Ω.

Hence it is enough to find a function β satisfying the above conditions such that $f(u) \geq -\beta(u)$, so that the strict maximum principle applies.
Case 1: Since f is decreasing in $(0, s_0)$, we can take $\beta(u) = -[f(u) - f(0)]$. Then $\beta(0) = 0$ and β is nondecreasing. Further

$$\begin{aligned} -\Delta_p u = f(u) &= (f(u) - f(0)) + f(0) \\ &= -\beta(u) + f(0) \geq -\beta(u) \end{aligned}$$

since $f(0) \geq 0$.
Case 2: In the second case, by (b)

$$f(u) \geq -cu^{p-1} \quad \forall \quad u \in [0, \delta)$$

Calling $\beta(u) = cu^{p-1}$, we have

$$\int_0^\delta \frac{ds}{\left(\int_0^s \beta(u)du\right)^{1/p}} = \infty.$$

Thus in both the cases, Vazquez strict maximum principle is applicable and hence $u > 0$ on $\mathbb{R}^N$. □

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