GLOBAL BIFURCATION FOR SEMILINEAR ELLIPTIC PROBLEMS

MARCELLO LUCIA
Universität zu Köln, Mathematisches Institut
50931 Köln, Germany, mlucia@math.uni-koeln.de

MYTHILY RAMASWAMY
TIFR Center, IISc. Campus, Post Box No. 1234
Bangalore 560012, India, mythily@math.tifrbng.res.in

Abstract: We study the existence of a global branch of solutions for the semilinear elliptic problem

$$-\Delta u = \lambda (a(x)u + b(x)r(u)), \quad u \in D^{1,2}_0(\Omega).$$

We work in a general domain $\Omega$ of $\mathbb{R}^n$, with indefinite weights $a, b$ belonging to some Lorentz spaces, and the function $r$ is either asymptotically linear or superlinear at infinity. To derive our result we first prove existence, uniqueness and simplicity of a principal eigenvalue for linear problems with weight in Lorentz spaces.

1. Introduction

The present paper deals with the semilinear elliptic problem

$$-\Delta u = \lambda (a(x)u + b(x)r(u)), \quad u \in D^{1,2}_0(\Omega), \quad (1.1)$$

where $\lambda$ is a real parameter, $r$ a known nonlinearity, $a, b$ are given weights in a domain (connected open set) $\Omega$ of $\mathbb{R}^N$ and the space $D^{1,2}_0(\Omega)$ is defined as the closure of $C^\infty_0(\Omega)$, set of smooth functions having compact support, with respect to the norm

$$\|u\| := \left( \int_{\Omega} |\nabla u|^2 \right)^{1/2}.$$

When $N \geq 3$, or when the domain is bounded, the functional space $D^{1,2}_0(\Omega)$ can be identified with a space of functions (see [10]). Due to the particularity of the dimensions $N = 1, 2$, for the sake of simplicity we will henceforth assume that $N \geq 3$. 

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When the function \( r \) is such that \( r(0) = 0 \), then \( u \equiv 0 \) solves (1.1) and we are interested in finding non-trivial solutions. In such a case the question of existence can be handled by using tools provided by the bifurcation theory like for example the Rabinowitz Theorem [17]. In order to apply such results one needs first to understand the linearized problem which, under suitable assumptions, will be a weighted eigenvalue problem of the type

\[-\Delta u = \lambda w(x)u, \quad u \in D_0^{1,2}(\Omega). \tag{1.2}\]

Both problems (1.1) and (1.2) have attracted a lot of attention because they naturally arise in mathematical physics or biology (combustion, population dynamics ...). Since in atomic physics the relevant weights are attractive-repulsive potentials with singularities of the type \( \frac{1}{|x|^s} \), it is desirable to study the above problems with coefficients allowed to be singular and sign changing. Furthermore it is known that Hardy's weight \( w(x) = \frac{1}{|x|^2} \) arise as “critical” cases when dealing with existence of eigenfunction for linear problems. Therefore it is of interest to understand how far “small perturbation” of such weights are allowed in Problems (1.1) and (1.2).

An important feature of the linear Problem (1.2) is the existence and simplicity of a principal eigenvalue. These are the values \( \lambda \) for which (1.2) admits a non-negative associated eigenfunction, and by simplicity one means that the associated eigenspace is of dimension one. When \( w \equiv 1 \) in a bounded domain of \( \mathbb{R}^N \), a proof of these properties can be found in [8]. In [16], it has been shown that the smallest positive eigenvalue for (1.2) exists and is simple whenever \( w^+ \not\equiv 0 \) and \( w \in L^r(\Omega) \) with \( r > \frac{N}{2} \) in a bounded domain. Sufficient conditions for the existence of principal eigenvalues for the weighted eigenvalue problem in \( \mathbb{R}^N \) have been given by many authors. Brown, Cosner, Fleckinger introduced in [6], a sufficient condition for the existence of a positive principal eigenvalue, namely the weight function \( w \) be negative and bounded away from 0 at infinity. They also used another sufficient condition for dimensions \( N \geq 3 \) requiring that \( w \) has a positive integral and it decays at infinity faster than \( \frac{1}{|x|^s} \). Brown and Tertikas (see [7]) relaxed the first condition in [6], by asking the positive part \( w^+ \) to be of compact support. Allegretto proved in [1] the existence of a principal eigenvalue and also infinitely many eigenvalues when \( w^+ \) lies in \( L^{\frac{N}{2}}(\Omega) \). Szulkin and Willem in [19] studied the problem with \( w \) in \( L^{\frac{N}{2}}(\Omega) \) or having faster decay than \( \frac{1}{|x|^2} \) at infinity or at any finite point. An important aspect of these previous works concerning the simplicity of the principal eigenvalue is that they rely on Harnack’s inequality in order to ensure the associated eigenfunctions to be continuous and strictly positive.
(or negative) in the domain.

In the present paper we extend all the previous results by considering weights \( w \) belonging to some Lorentz space \( L(p, q) \) which are described in Section 2. It allows us to include some classes of functions not satisfying the earlier conditions described above. Furthermore, by going to Lorentz spaces, the importance of the comparison of decay with \( \frac{1}{|x|^2} \) becomes clearer. More precisely the first main result of our paper is the following:

**Proposition 1.1.** Let \( w \in L\left(\frac{N}{2}, q_0\right) \) with \( q_0 \in (1, \infty) \) and such that \( w^+ \not\equiv 0 \). Then

\[
\lambda_+^1 (w) := \inf \left\{ \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} w u^2} : u \in D_0^{1,2}(\Omega), \int_{\Omega} w u^2 > 0 \right\},
\]

is a principal eigenvalue and this is the unique positive principal eigenvalue. Furthermore the associated eigenfunction \( \Phi \) satisfies \( \Phi > 0 \) a.e. or \( \Phi < 0 \) a.e. and is unique up to a constant multiple factor.

If \( w^- \not\equiv 0 \) above proposition applied to \(-w\) provides also existence of a negative principal eigenvalue \( \lambda_-^1 (w) \), which will be simple and the unique negative principal eigenvalue. Let us emphasize that with our assumptions the Harnack’s inequality does not apply and we may have \( \inf_B |\Phi| = 0 \) on some \( B \subset \subset \Omega \). Nevertheless thanks to a strong maximum principle as established by Ancona [4] and Brezis, Ponce [5] the set of zeroes of a non-negative eigenfunction has Lebesgue measure zero. This property will be sufficient for proving that the eigenspace associated to \( \lambda_+^1 (w) \) has dimension one if \( w^+ \not\equiv 0 \).

Proposition 1.1 opens the possibility of proving existence of global branch of solutions for Problem (1.1) when the coefficients \( a, b \) have very less regularity. Positive solution branches have been studied by many authors for nonlinearities which are asymptotically linear (see [3]) and also for superlinear cases as in [11]. In [12] both asymptotically linear and superlinear nonlinearities have been considered with the restriction that the coefficients are bounded. But that is not necessary for the existence of a branch of solutions. Indeed in the present paper we derive existence of at least one global branch under the only following requirements:

\begin{itemize}
  \item[(H1)] \( r \in C^0(\mathbb{R}), \lim_{s \to 0} r(s) = 0, \limsup_{s \to \infty} \frac{|r(s)|}{|s|^\gamma} < \infty \) for some \( \gamma \in [1, 2^* - 1); \)
  \item[(H2)] \( a \in L\left(\frac{N}{2}, q_0\right), b \in L\left(\frac{N}{2}, q_0\right) \cap L(p_0, q_0) \) with \( p_0 = \frac{2^*}{2^* - \gamma - 1} \) and for some \( q_0 \in (1, \infty) \).
\end{itemize}
As in Proposition 1.1, \( L(p, q) \) denotes the Lorentz space. Since \( \gamma \in [1, 2^*-1) \), it follows that \( p_0 \in \left[ \frac{N}{2}, \infty \right) \) and in particular for bounded domain the assumption on \( b \) reduces to \( b \in L(p_0, q_0) \) as given in (H2). But for unbounded domain one needs to work on a smaller space. Note also that (H1) implies \( r(0) = 0 \) and we are in fact interested in existence of non-trivial solutions for Problem (1.1). By defining

\[ S := \{ (\lambda, u) \in \mathbb{R} \times D_{0}^{1,2}(\Omega) : (\lambda, u) \text{ solves (1.1), } u \not\equiv 0 \}, \]

our main result is as follows:

**Proposition 1.2.** Assume (H1), (H2) and \( a^+ \not\equiv 0 \) (resp. \( a^- \not\equiv 0 \)). Then, there exists a set \( C^+ \) connected in \( S \) bifurcating from \( \lambda_1^+(a) \) (resp. \( C^- \) bifurcating from \( \lambda_1^-(a) \)). Moreover, \( C^+ \) (resp. \( C^- \)) is

(i) either unbounded,

(ii) or contains a point \( (\lambda, 0) \) with \( \lambda \neq \lambda_1^+(a) \) (resp. \( \lambda \neq \lambda_1^-(a) \)).

Our results improve considerably the result of [12] since we are working with weights belonging only to some Lorentz space and we work in any domain of \( \mathbb{R}^N \). But in the present paper we shall not discuss the positivity of the solution of the branch. This issue will be discussed in more details in a coming work, where we will also see how our assumptions can further be relaxed.

The paper is organized as follows. Section 2 provides the definitions and basic properties of the Lorentz \( L(p, q) \) spaces. For weights \( w \in L(\frac{N}{2}, q_0) \) with \( w^+ \not\equiv 0 \), we prove in Section 3 that the positive principal eigenvalue of the linear problem is unique and its associated eigenspace has dimension one. In Section 4, we establish existence of at least one global branch of solutions for the nonlinear problem (1.1) as stated in Proposition 1.2.

### 2. Prerequisites on Lorentz spaces

The Lorentz \( L(p, q) \) spaces, introduced by Lorentz in [14], are generalization of the Lebesgue \( L^p \) spaces. We collect here their main properties and refer to [13], [18], [20] for more detailed discussions. To introduce their definitions, we start by recalling that the distribution function and nonincreasing rearrangement of a measurable function \( f : \Omega \to \mathbb{R} \) are respectively defined as:

\[
\alpha_f(s) := \left| \{ x \in \Omega : |f(x)| > s \} \right|, \quad f^*(t) := \inf \{ s > 0 : \alpha_f(s) \leq t \}. \quad (2.1)
\]

Then \( \alpha_f : \mathbb{R} \to \mathbb{R} \) is nonnegative, nonincreasing, continuous from the right and we easily verify that
(i) \( f^* \geq 0 \), nonincreasing and continuous from the right;
(ii) \( f^* (\alpha f(s)) \leq s, \alpha f (f^*(t)) \leq t; \)
(iii) if \( \alpha f \) is continuous and strictly decreasing then \( f^* = \alpha^{-1} f^* \);

\( \alpha f^* = \alpha f \) and therefore \( \int_{\Omega} |f| \, dx = \int_{0}^{\infty} |f^*(t)|^p \, dt. \)

Observe that for \( f \in L^p(\Omega), \)
\[
||f||_p = \left( \int_{0}^{\infty} \left[ \frac{t}{t^p} f^*(t) \right]^q \frac{dt}{t} \right)^\frac{1}{q},
\]

(2.2)

This can be used to motivate the definition of Lorentz spaces:

**Definition 2.1.** We define
\[
||f||_{p,q}^* = \begin{cases} 
\left( \frac{q}{p} \int_{0}^{\infty} \left[ \frac{t}{t^p} f^*(t) \right]^q \frac{dt}{t} \right)^\frac{1}{q}, & \text{if } 1 \leq p, q < \infty, \\
\sup_{t>0} \left\{ \frac{t}{t^p} f^*(t) \right\}, & \text{if } 1 \leq p \leq \infty, q = \infty,
\end{cases}
\]

\( L(p, q) = \{ f : \Omega \to \mathbb{R} : f \text{ measurable, } ||f||_{p,q} < \infty \}. \)

Notice that with this definition we have
\[ L(p, p) = L^p(\Omega), \quad 1 \leq p \leq \infty. \]

The space \( L(p, \infty) \) is known as the "weak \( L^p \) space" and coincides with the Marcinkiewicz space \( M^p \), defined as follows:

\[ M^p := \{ f : \Omega \to \mathbb{R} : f \text{ measurable, } \int_{\Omega} |f| \, dx \leq C |\omega|^{1 - \frac{1}{p}} \forall \omega \subset \Omega \}. \]

When \( 1 \leq q_1 \leq q_2 \leq \infty \), one can prove (see [13], Section 1):
\[ ||f||_{p,q_2} \leq ||f||_{p,q_1}^* \quad \text{and} \quad L(p, q_1) \subset L(p, q_2). \]

(2.3)

In general \( ||f||_{p,q}^* \) need not be a norm as Minkowski inequality may fail. In spite of this, it can be used to define a norm on \( L(p, q) \). Indeed let us set
\[ f^{**}(t) := \frac{1}{t} \int_{t}^{\infty} f^*(r) \, dr, \quad ||f||_{p,q} := ||f^{**}||_{p,q}^*. \]

We easily see that
\[
||f||_{p,q} = \begin{cases} 
\left( \int_{0}^{\infty} \left[ \frac{t}{t^p} f^{**}(t) \right]^q \frac{dt}{t} \right)^\frac{1}{q}, & \text{if } 1 \leq p, q < \infty, \\
\sup_{t>0} \left\{ \frac{t}{t^p} f^{**}(t) \right\}, & \text{if } 1 \leq p \leq \infty, q = \infty.
\end{cases}
\]
This is a norm which is “equivalent” to \( \| \cdot \|_{p,q}^\ast \) (see Chapter V, Theorem 3.2, [18]):

\[
\| f \|_{p,q}^\ast \leq \| f \|_{p,q} \leq \frac{p}{p-1} \| f \|_{p,q}^\ast ,
\]

and enjoys of the following monotonicity property:

\[
f, g \in L(p, q) \text{ with } |f| \leq |g| \implies \| f \|_{p,q} \leq \| g \|_{p,q} .
\]

Under this norm, \( L(p, q) \) is a Banach space and the following Hölder’s inequality holds (see [13]):

\[
\| fg \|_{p,q} \leq C \| f \|_{p_1,q_1} \| g \|_{p_2,q_2}, \quad \forall (f, g) \in L(p_1, q_1) \times L(p_2, q_2),
\]

whenever

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \quad p_i, q_i \in [1, \infty].
\]

This inequality applied with \( p = q = 1 \) helps to identify the topological dual of the Lorentz spaces:

**Proposition 2.1.**

(a) Let \((p, q) \in (1, \infty) \times [1, \infty) \) or \( p = q = 1 \). Then the dual space of \( L(p, q) \) is isomorphic to \( L(p', q') \) where \( 1/p + 1/p' = 1 \) and \( 1/q + 1/q' = 1 \).

(b) The spaces \( L(p, q) \) are reflexive when \( 1 < p, q < \infty \).

A main feature of the Lorentz space is that they allow to improve the usual Sobolev embedding \( D^{1,2}_0(\Omega) \hookrightarrow L^{2^*}(\Omega) \simeq L(2^*, 2^*) \) as follows:

**Proposition 2.2. (Sobolev-Lorentz embedding).**

\[
D^{1,2}_0(\Omega) \hookrightarrow L(2^*, 2).
\]

Observe that \( 2 < 2^* \) for \( N \geq 3 \) (see for example, appendix in [2]).

3. Existence and uniqueness of principal eigenvalue

For the existence of principal eigenvalue for the problem (1.2), a sufficient condition given by Allegretto in [1] is that the weight function is in \( L^N(\Omega) \). Szulkin and Willem relaxed this in [19] by introducing the following conditions:
\( V \in L^1_{\text{loc}}(\Omega), \ V^+ = V_1 + V_2 \neq 0, \ V_1 \in L^\frac{N}{2}(\Omega), \)

\[
\begin{align*}
\lim_{|x| \to \infty} |x|^2 V_2(x) &= 0, \quad \lim_{x \to x_0} |x - x_0|^2 V_2(x) = 0 \quad \forall x_0 \in \Omega.
\end{align*}
\]

Under this condition, the existence of a positive principal eigenvalue and a sequence of eigenvalues were proved in [19]. Further, they give examples of weight functions

\[
W_1(x) = \frac{1}{1 + |x|^2}, \quad W_2(x) = \frac{1}{|x|^2(1 + |x|^2)},
\]

which do not satisfy \((H)\) and the eigenvalue problems do not possess an eigenvalue (see also [21]). But the modified versions of these functions,

\[
\bar{W}_1(x) = \frac{W_1(x)}{(\log(2 + |x|^2))^{\frac{N}{2^N}}}, \quad \bar{W}_2(x) = \frac{W_2(x)}{(\log(2 + \frac{1}{|x|^2}))^{\frac{N}{2^N}}},
\]

satisfy \((H)\) and the eigenvalue problems possess infinitely many eigenvalues. Observe that none of these functions lie in \(L^\frac{N}{2}(\Omega)\).

By using Lorentz spaces, we will try to relax further the condition \((H)\). In this section we are doing a first step in this direction by showing that the classical results concerning the principal eigenvalue for linear weighted eigenvalue problem holds when

\[
w \in L \left( \frac{N}{2}, q_0 \right) \text{ for some } 1 < q_0 < \infty. \quad (3.1)
\]

First, let us see some examples to compare different conditions.

**Example 1**: The function \(W_1\) does not satisfy the condition (3.1). Denoting by \(\omega_N\) the volume of the unit sphere in \(\mathbb{R}^N\) we have for \(w = W_1\),

\[
\alpha_w(s) = |\{ x : w(x) > s \}| = \omega_N |x_0|^N, \quad s = \frac{1}{1 + |x_0|^2}, \quad 0 < s < 1,
\]

\[
w^*(t) = \frac{1}{1 + \left( \frac{t}{\omega_N} \right)^\frac{1}{N}}, \quad t \in [0, \infty).
\]

Now it is easy to check that \(w \notin L \left( \frac{N}{2}, q \right)\) for any \(q \in (1, \infty)\):

\[
(||w||_{\frac{N}{2}, q})^q = \int_0^\infty \left\{ t^\frac{1}{N} \frac{1}{1 + \left( \frac{t}{\omega_N} \right)^\frac{1}{N}} \right\}^q \frac{dt}{t} = \infty \quad \text{for } q \geq 1.
\]
But \( w \in L \left( \frac{N}{2}, \infty \right) \) since

\[
|||w|||_{\frac{N}{2}, \infty}^* = \sup_{t \in [0, \infty)} \left\{ t^\frac{N}{2} \frac{1}{1 + \left( \frac{t}{\omega_N} \right)^\frac{N}{2}} \right\} = \omega_N^\frac{2}{N}.
\]

Thus condition (3.1) is not satisfied by the function \( W_1 \) and similarly for \( W_2 \).

**Example 2:** The functions \( \tilde{W}_1 \) and \( \tilde{W}_2 \) satisfy condition (3.1). We do the calculation only for \( w = \tilde{W}_1 \). We have

\[
\alpha_w(s) = \left| \left\{ x : w(x) > s \right\} \right| = \omega_N |x_0|^N,
\]

where

\[
s = \frac{1}{(1 + |x_0|^2)(\log(2 + |x_0|))^{\frac{N}{2}}}, \quad 0 < s < \frac{1}{(\log 2)^{\frac{N}{2}}},
\]

\[
w^*(t) = \frac{1}{(1 + (\frac{t}{\omega_N})^{\frac{N}{2}})(\log(2 + (\frac{t}{\omega_N})^{\frac{N}{2}}))^{\frac{N}{2}}}, \quad t \in (0, \infty).
\]

Now we check if \( w \in L \left( \frac{N}{2}, q_0 \right) \) for some \( q_0 \). A direct calculation shows

\[
(|||w|||_{\frac{N}{2}, q}^*)^q = \int_0^\infty \left\{ t^{\frac{N}{2}} \frac{1}{(1 + (\frac{t}{\omega_N})^{\frac{N}{2}})(\log(2 + (\frac{t}{\omega_N})^{\frac{N}{2}}))^{\frac{N}{2}}} \right\}^q \frac{dt}{t},
\]

which is finite if \( q > \frac{N}{2} \). Thus \( \tilde{W}_1 \) satisfies condition (3.1) for any \( q_0 \in \left( \frac{N}{2}, \infty \right) \).

The following example shows that there are functions failing the condition (H) but satisfying condition (3.1).

**Example 3:** Consider the following function which is singular all along the \( x_2 \)-axis in a square \( \Omega = \left\{ (x_1, x_2) : |x_1| < R \right\} \) in \( \mathbb{R}^2 \) with \( R < 1 \),

\[
W_3(x_1, x_2) = \frac{1}{|x_1 \log(|x_1|)|} \quad \text{in} \ \Omega, \quad x_1 \neq 0.
\]

As in Example 1, we have for \( w = W_3 \),

\[
\alpha_w(s) = \left| \left\{ x : w(x) > s \right\} \right| = R |x_1|^N
\]

where

\[
s = \frac{1}{|x_1^N \log(|x_1^N|)|}, \quad 0 < s < \frac{1}{R \log R},
\]
\[w^*(t) = \frac{1}{(\frac{t}{R})|\log(\frac{t}{R})|^q}, \quad t \in (0, R^2).\]

We now check that this function satisfies the condition (3.1) (with \(N = 2\)):
\[
(\|w\|_{\infty, q})^q = \int_0^{R^2} \left\{ \frac{1}{(\frac{t}{R})|\log(\frac{t}{R})|^q} \right\}^q \frac{dt}{t} = C \int_0^{R^2} \frac{dt}{t|\log(\frac{t}{R})|^q}
\]
\[= C \int_{|\log R|}^{\infty} \frac{dy}{y^q} < \infty \text{ for } q > 1.
\]

But \(W_3\) does not satisfy (H). Indeed the limit of \(|x|^2W_3(x)\), as \(x\) tends to 0 along the curve \(x_2 = \sqrt{x_1}\), tends to infinity, while along the \(x_1\) axis it tends to 0. Thus the limit does not exist. The same holds for the function in a cube \(\Omega = \{(x_1, \ldots, x_N) : |x_i| < R < 1\} \) in \(\mathbb{R}^N\),
\[W_3(x) = \frac{1}{|x_1|^{\frac{N}{2}}|\log(|x_1|)|^{\frac{N}{2}}} \text{ in } \Omega.
\]

Thus \(W_3\) does not satisfy the condition (H) but satisfies condition (3.1).

There are also functions failing condition (3.1) but satisfying condition (H). Thus the two conditions are independent. Work is in progress towards a unifying condition, which will be a relaxation of both these conditions.

**Example 4:** Consider the function
\[W_4(x) = \frac{W_1(x)}{\log \log (2 + |x|^2)}.
\]
For the function \(w = W_4\), we check that
\[w^*(t) = \frac{1}{\left(1 + \left(\frac{1}{\frac{1}{x^N}}\right)^{\frac{N}{2}}\right) \log \log \left(2 + \left(\frac{1}{x^N}\right)^{\frac{N}{2}}\right)}, \quad t \in (0, \infty).
\]

Now we check if \(w \in L \left(\frac{N}{2}, q_0\right)\) for some \(q_0\). A direct calculation shows
\[
(\|w\|_{\frac{N}{2}, q_0})^q = \int_0^{\infty} \left\{ \frac{1}{1 + \left(\frac{1}{x^N}\right)^{\frac{N}{2}}} \log \log \left(2 + \left(\frac{1}{x^N}\right)^{\frac{N}{2}}\right) \right\}^q \frac{dt}{t}
\]
\[\leq C + \int_M^{\infty} \frac{dt}{t(\log \log (2 + \left(\frac{1}{x^N}\right)^{\frac{N}{2}}))^q}
\]
\[\leq C + \int_{\log M}^{\infty} \frac{dy}{y^q}.
\]

Since \(\log y < y^\alpha\) for large \(y\) and any \(\alpha > 0\), we see that the last integral is divergent. for any \(q\). But this function does satisfy (H) as \(|x|\) tends to infinity.
In our setting the existence of a principal eigenvalue will mainly be a consequence of the following result:

**Proposition 3.1.** Let \( w \in L(\frac{N}{2}, q_0) \) with \( 1 < q_0 < \infty \). Then the mapping

\[
D_0^{1,2}(\Omega) \to \mathbb{R}, \quad u \mapsto \int_{\Omega} w(x) \, u^2(x) \, dx.
\] (3.2)

is weakly continuous.

**Proof.** Note first that for \( 2 \leq p, q < \infty \) the mapping

\[
L(p, q) \to L\left(\frac{p}{2}, \frac{q}{2}\right), \quad v \mapsto v^2,
\] (3.3)

is well-defined and continuous. Indeed, the inequality \( ||v^2||_{\left(\frac{p}{2}, \frac{q}{2}\right)} \leq ||v||_{p,q} ||v||_{p,q} \) holds for any \( v \in L(p, q) \) (see (2.6)) and for any sequence \( u_n \to u \) in \( L(p, q) \), we have

\[
||u_n^2 - u^2||_{\left(\frac{2}{2}, \frac{q_0}{2}\right)} \leq ||u_n - u||_{p,q} ||u_n + u||_{p,q}.
\] (3.4)

Let \( u_n \) be a sequence bounded in \( D_0^{1,2}(\Omega) \). Then

(i) \( u_n \) is bounded in \( L(2^*, 2) \) (by Lorentz-Sobolev embedding);
(ii) \( u_n^2 \) is bounded in \( L\left(\frac{2}{2}, 1\right) \) (by continuity of (3.3)) and so \( u_n^2 \) is bounded in \( L(\frac{2}{2}, q_0') \).

Since both spaces \( D_0^{1,2}(\Omega) \) and \( L\left(\frac{2}{2}, q_0'\right) \) are reflexive (note that \( 1 < q_0' < \infty \)), we deduce:

\[ u_n \to f \text{ weakly in } D_0^{1,2}(\Omega) \quad \text{and} \quad u_n^2 \to g \text{ weakly in } L\left(\frac{2}{2}, q_0'\right). \]

We claim \( g = f^2 \). Indeed for each \( \varphi \in C_0^\infty \) (fixed) we have

\[
\int_{\Omega} |u_n^2 - f^2| ||\varphi|| \leq \int_{\Omega} |u_n - f||u_n|| ||\varphi|| + \int_{\Omega} |u_n - f||f|| ||\varphi||.
\] (3.5)

We claim that the right hand-side of (3.5) tends to zero as \( n \to \infty \). In order to estimate \( A_n \), denote by \( \Omega_0 \) the interior of the support of \( \varphi \) and apply Hölder inequality (2.6) to derive

\[
A_n := ||u_n - f||u_n||_{1,1} \leq ||u_n - f||\chi_{\Omega_0}||_{2,2} ||u_n||_{2^*,2} ||\varphi||_{N,\infty}.
\]

Since there is a compact embedding of \( D_0^{1,2}(\Omega) \) in \( L_{loc}^{2^*(\infty)}(\Omega) \) (because \( 2 < 2^* \)) and \( ||u_n||_{2^*,2} \) is bounded (uniformly in \( n \)) we deduce that \( A_n \to 0 \).
To estimate $B_n$, we simply observe that $(f, \varphi) \in L(2^*, 2) \times L(2^*, \infty)$ and therefore $f\varphi \in L([2^*]'', 2)$. Hence by duality we get $\lim_{n \to \infty} B_n = 0$. So (3.5) tends to zero when the test function $\varphi \in C^\infty_0(\Omega)$. Since on the other hand we have also

$$\int_\Omega |u_n^2 - g||\varphi| \to 0, \quad \forall \varphi \in L([\frac{2^*}{2}'', q_0]),$$

we easily deduce that $g = f^2$.

To summarize, we have $u_n^2 \to u^2$ weakly in $L(\frac{N}{2}, q_0)$ and $w \in L(\frac{N}{2}, q_0) = [L(\frac{2^*}{2}, q_0)]'$. Hence the definition of weak convergence implies

$$\int_\Omega w(u_n^2 - u^2) \to 0.$$

Thus the proposition is proved.

Based on the above proposition, standard arguments imply that (1.3) is a principal eigenvalue. With the aim of proving that this is the unique positive principal eigenvalue and that its associated eigenspace is of dimension one, we need the following weak version of the Strong Maximum Principle due to Ancona [4] and Brezis-Ponce [5], that in our setting can be stated as follows:

**Theorem 3.1. (Strong Maximum Principle).** Let $V \in L(\frac{N}{2}, q_0)(\Omega)$ with $V \geq 0$. Assume that the following holds in the sense of distribution:

$$-\Delta u + V(x)u \geq 0, \quad u \in D^{1,2}_0(\Omega) \setminus \{0\}, \quad u \geq 0,$$

and consider the “precise representative” $\tilde{u}$ of $u$. Then, $\{x \in \Omega : \tilde{u}(x) = 0\}$ is a set of $H^1$-capacity zero and in particular of Lebesgue measure zero.

The result holds in much wider generality and we refer to [5] for a more detailed statement. Note that if in the above theorem we make the stronger assumption $V \in L^p_{loc}(\Omega)$ for some $p > N/2$, the usual Strong Maximum Principle would imply $\inf_B u > 0$ for any $B \subset \subset \Omega$. To avoid any confusion, we make the following definition:

**Definition 3.1.** We say that $\lambda \in \mathbb{R}$ is a “principal eigenvalue” and $u \in D^{1,2}_0(\Omega) \setminus \{0\}$ a “principal eigenfunction” for Problem (1.2) if $(\lambda, u)$ solves (1.2) and $u \geq 0$. 

In our setting a principal eigenfunction is not necessarily locally bounded and may vanish. But note that if \( \lambda > 0 \) is a principal eigenvalue and \( u \) an associated principal eigenfunction we have
\[
-\Delta u + \lambda w^+ u = \lambda w^+ u \geq 0, \quad u \geq 0, \quad u \not\equiv 0,
\]
and so Theorem 3.1 implies that the set \( u^{-1}(0) \) is of \( H^1 \)-capacity zero. A similar argument holds if the principal eigenvalue is negative. We can now prove

**Proof of Proposition 1.1:**

**Existence:** As remarked in [19, Remark 2.4(b)], the weak continuity of the mapping (3.2) is enough to ensure the existence of \( \Phi \geq 0 \) solving the minimizing Problem (1.3). Therefore
\[
-\Delta \Phi = \lambda^+_1(w) \Phi, \quad \Phi \geq 0, \quad \int_{\Omega} w \Phi^2 > 0,
\]
and so \( \lambda^+_1(w) \) is a principal eigenvalue.

**Uniqueness:** To prove that \( \lambda^+_1(w) \) is the unique positive principal eigenvalue, let us assume the existence of another pair \( (\lambda, \varphi) \in (0, \infty) \times D_0^{1,2}(\Omega) \) satisfying
\[
-\Delta \varphi = \lambda w \varphi, \quad \varphi \geq 0, \quad \varphi \not\equiv 0.
\]
From the definition (1.3), we immediately see that
\[
\lambda \geq \lambda^+_1(w).
\]
To show that equality holds we modify slightly some of the arguments in [Prop. 4.1, [9]], by taking into account that our eigenfunctions may be unbounded. Let us define for each \( k \geq 0 \) the following truncated function:
\[
\Phi_k(x) := \begin{cases} k & \text{if } \Phi(x) \geq k, \\ \Phi(x) & \text{if } \Phi(x) \in [0, k). \end{cases}
\]
Clearly \( \Phi_k \in L^\infty(\Omega) \) and it is well-known that \( \Phi_k \in D_0^{1,2}(\Omega) \). Hence, both \( \Phi_k \) and \( \frac{\Phi_k^2}{\varphi + \epsilon} \) are legitimate trial functions in (3.7), respectively (3.8) and therefore
\[
\int_{\Omega} |\nabla \Phi_k|^2 - \int_{\Omega} \nabla \Phi_k \nabla \left( \frac{\Phi_k^2}{\varphi + \epsilon} \right) = \int_{\Omega} \lambda^+_1(w) w \Phi_k^2 - \int_{\Omega} \lambda w \varphi \frac{\Phi_k^2}{\varphi + \epsilon},
\]
which is equivalent to
\[
\int_{\Omega} \left\{ |\nabla \Phi_k|^2 - \nabla \Phi_k \nabla \left( \frac{\Phi_k^2}{\varphi + \epsilon} \right) \right\} = \int_{\Omega} \left\{ \lambda^+_1(w) w \Phi_k^2 - \lambda w \varphi \frac{\Phi_k^2}{\varphi + \epsilon} \right\}. \tag{3.10}
\]
But, a direct calculation shows that the following “Picone’s identity” holds:

\[
|\nabla \Phi_k|^2 - \nabla \varphi \nabla \left( \frac{\Phi_k^2}{\varphi + \epsilon} \right) = \left| \nabla \Phi_k - \left( \frac{\Phi_k}{\varphi + \epsilon} \right) \nabla \varphi \right|^2.
\]  
(3.11)

By plugging (3.11) in (3.10), we get

\[
0 \leq \int_\Omega \left| \nabla \Phi_k - \left( \frac{\Phi_k}{\varphi + \epsilon} \right) \nabla \varphi \right|^2 = \int_\Omega \left\{ \lambda_1^+(w)w\Phi_k - \lambda w\varphi \frac{\Phi_k^2}{\varphi + \epsilon} \right\}.
\]  
(3.12)

Since by Theorem 3.1 the set \{\varphi = 0\} is of measure zero, (3.12) is equivalent to

\[
0 \leq \int_{\{\varphi > 0\}} \left| \nabla \Phi_k - \left( \frac{\Phi_k}{\varphi + \epsilon} \right) \nabla \varphi \right|^2 = \int_{\{\varphi > 0\}} \left\{ \lambda_1^+(w)w\Phi_k - \lambda w\varphi \frac{\Phi_k^2}{\varphi + \epsilon} \right\}.
\]  
(3.13)

Now, letting \(\epsilon \to 0\) and \(k \to \infty\) in (3.13) and applying Lebesgue dominated Theorem to the right handside, we get

\[
0 \leq \left( \lambda_1^+(w) - \lambda \right) \int_\Omega w\Phi^2.
\]  
(3.14)

Using (3.9) and the fact that \(\int_\Omega w\Phi^2 > 0\), (3.14) implies \(\lambda = \lambda_1^+(w)\).

**Simplicity:** let \(V(\lambda_1^+(w))\) be the eigenspace associated to the principal eigenvalue (1.3). Before proving that \(V_1\) has dimension one, we observe that any \(\Phi \in V(\lambda_1^+(w))\) satisfies precisely one of the alternative:

(i) \(\Phi > 0\) a.e. in \(\Omega\),  
(ii) \(\Phi < 0\) a.e. in \(\Omega\),  
(iii) or \(\Phi \equiv 0\).  
(3.15)

Indeed, since

\[
\int_\Omega |\nabla \Phi|^2 = \int_\Omega |\nabla |\Phi||^2 = \lambda_1^+(w) \quad \text{and} \quad \int_\Omega w|\Phi|^2 = \int_\Omega w\Phi^2,
\]

we deduce that both \(\Phi\) and \(|\Phi|\) solve the minimization problem (1.3). As a consequence

\[
\Phi^+ = \Phi + |\Phi| \in V(\lambda_1^+(w)) \quad \text{and} \quad \Phi^- = \Phi - |\Phi| \in V(\lambda_1^+(w)).
\]  
(3.16)

Now by considering the Euler-Lagrange equation satisfied by \(\Phi^+, \Phi^-\) in the form (3.6) and applying Theorem 3.1 we conclude that \(\Phi^+ > 0\) a.e. or \(\Phi^- < 0\) a.e. or else \(\Phi^+ \equiv \Phi^- \equiv 0\). This proves (3.15).

As in [15], we can now modify the “continuation argument” given in Lemma 7 of [16]. More precisely, since \(V(\lambda_1^+(w)) \neq \{0\}\), consider \(\Phi_1, \Phi_2 \in \ldots\)
By \(3.15\) we can assume without loss of generality that \(\Phi_1, \Phi_2 > 0\) a.e.. Let us consider the set

\[ T := \{ t \in \mathbb{R} : \Phi_1 + t\Phi_2 > 0 \text{ a.e.} \}. \]

We shall show that \(\Phi_1 + t_0\Phi_2 \equiv 0\) when \(t_0 = \inf T\).

**Claim 1:** \(T \neq \emptyset\) and \(\inf T > -\infty\).

Since \(0 \in T\), we see that \(T \neq \emptyset\). To prove that this set is bounded from below, let us consider for each \(\delta, M > 0\), the set

\[ A_{\delta, M} := \{ \Phi_1 < M, \Phi_2 > \delta \}. \]

Since \(\Phi_2 > 0\) a.e., there exists \(\tilde{\delta} > 0\) such that \(|\{\Phi_2 > \tilde{\delta}\}| > 0\). Moreover, since \(\cup_{M>0} A_{\tilde{\delta}, M} = \{\Phi_2 > \tilde{\delta}\}\), we deduce the existence of \(\tilde{M}\) such that \(|A_{\tilde{\delta}, \tilde{M}}| > 0\). Now, for any \(x \in A_{\tilde{\delta}, \tilde{M}}\) and \(t < -\frac{\tilde{M}}{\delta}\), we get

\[ (\Phi_1 + t\Phi_2)(x) < \tilde{\delta} - t\tilde{M} < 0. \]

Hence, when \(t < -\frac{\tilde{M}}{\delta}\), the function \(\Phi_1 + t\Phi_2\) is negative on a set of positive measure, which proves \(\inf T > -\infty\).

**Claim 2:** Setting \(t_0 := \inf T\), we claim \(\Phi_1 + t_0\Phi_2 \equiv 0\).

Since \(\Phi_1 + t_0\Phi_2 \in V(\lambda_1^+ (w))\) the alternative \(3.15\) applies and so one of the following case holds:

(a) \(\Phi_1 + t_0\Phi_2 > 0\) a.e. in \(\Omega\), (b) \(\Phi_1 + t_0\Phi_2 < 0\) a.e. in \(\Omega\), (c) \(\Phi_1 + t_0\Phi_2 \equiv 0\).

Assume (a) holds. By setting

\[ E_{\delta, M} := \{ \Phi_1 + t_0\Phi_2 > \delta, \Phi_2 < M \}, \]

and arguing as in claim (1), we prove \(|E_{\tilde{\delta}, \tilde{M}}| > 0\) for some \(\tilde{\delta}, \tilde{M} > 0\). Thus, for any \(x \in E_{\tilde{\delta}, \tilde{M}}\) and \(\varepsilon \in (0, \frac{\delta}{\tilde{M}})\), we get

\[ (\Phi_1 + (t_0 - \varepsilon)\Phi_2)(x) > \tilde{\delta} - \varepsilon\tilde{M} > 0. \]

Alternative \(3.15\) implies then \(\Phi_1 + (t_0 - \varepsilon)\Phi_2 > 0\) a.e., in contradiction with the definition of \(t_0\). A similar contradiction is reached if we assume (b). Therefore, the alternative (c) holds, which concludes the proof of the proposition. \(\square\)
Remark 3.1.

(a) In [15], Proposition 1.1 has been stated for weight belonging to $L^{2\frac{N}{2}}(\Omega)$. Some of the arguments has been repeated for the sake of completeness.

(b) Existence of a principal eigenvalue when the weight belongs to some Lorentz space has already been obtained by [22]. Our arguments are different [22] and here we have completed the result by emphasizing that uniqueness and simplicity of the positive (or negative) principal eigenvalue holds.

(c) Note that in [19] the simplicity of a principal eigenvalue is proved under stronger conditions.

4. Global bifurcation

Existence of global branches for Problem (1.1) will mainly rely on the following result:

Theorem 4.1. (Rabinowitz, [17]) Given a Banach space $(B, ||\cdot||)$, consider a mapping

$$G : \mathbb{R} \times B \to B, \quad (\lambda, u) \mapsto \lambda L(u) + H(\lambda, u),$$

where $L : B \to B$ is a compact linear operator and $H(\lambda, \cdot) : B \to B$ is a continuous compact mapping satisfying

$$\lim_{\|u\| \to 0} \frac{\|H(\lambda, u)\|}{\|u\|} = 0.$$

Denote by

$$r(L) := \{ \mu \in \mathbb{R} : \mu^{-1} \text{ is an eigenvalue of } L \text{ with odd multiplicity} \},$$

$$S := \{ (\lambda, u) \in \mathbb{R} \times B : (\lambda, u) \text{ is solution of } u = G(\lambda, u), u \not\equiv 0 \}.$$

Then, given $\mu \in r(L)$, $S$ has a connected branch $C_\mu$ bifurcating from $(\mu, 0)$ and

(i) either $C_\mu$ is unbounded in $\mathbb{R} \times B$, (ii) or, $C_\mu \ni (\hat{\mu}, 0)$ with $\mu \neq \hat{\mu} \in r(L)$.

Let us recast Problem (1.1) in the framework of Theorem 4.1. To this aim we first note that as a consequence of Riesz Theorem, the mapping

$$D_0^{1,2}(\Omega) \to L([2^*]'', [2]'') = L(N - 2N + 2), \quad u \mapsto -\Delta u,$$  \hspace{1cm} (4.1)

is 1-1 with continuous inverse. Based on our hypotheses we will see that the following mappings are well-defined:

$$L : D_0^{1,2}(\Omega) \to D_0^{1,2}(\Omega), \quad u \mapsto (-\Delta)^{-1}(a(x)u),$$  \hspace{1cm} (4.2)
\[ H : \mathbb{R} \times D^{1,2}_0(\Omega) \to D^{1,2}_0(\Omega), \quad (\lambda, u) \mapsto (\lambda) (\Delta)^{-1}(\lambda b(x)r(u)), \quad (4.3) \]

and so finding a solution to Problem (1.1) is reduced to solving

\[ u = \lambda L(u) + H(\lambda, u), \quad u \in D^{1,2}_0(\Omega). \quad (4.4) \]

In order to prove the continuity and compactness of the mappings \( L, H \) we need the following lemma borrowed from [Lemma 4.2,[22]]:

**Lemma 4.1.** Let \( V \in L^{p, q}(\Omega) \) with \( 1 \leq p, q < \infty \). Then for each \( \varepsilon > 0 \) there exists a measurable set \( \Omega^\varepsilon \subseteq \Omega \) such that \( \Omega^\varepsilon \) is bounded, \( V \chi_{\Omega^\varepsilon} \in L^\infty(\Omega) \), \( \| V \chi_{\Omega^\varepsilon} \|_{p, q} < \varepsilon \).

**Proposition 4.1.** Assume (H1)-(H2) hold. Then the mappings \( L \) and \( H \) defined by (4.2) and (4.3) are continuous, compact and furthermore

\[ \lim_{\| u \| \to 0} \frac{\| H(\lambda, u) \|}{\| u \|} = 0. \quad (4.5) \]

**Proof.** Let us introduce the two following mappings

\[ \tilde{L} : L^{(2^*), 2} \to L^{([2^*])', 2}, \quad \tilde{H} : L^{(2^*), 2} \to L^{([2^*])', 2} \]

\[ u \mapsto a(x)u \quad u \mapsto b(x)r(u) \quad (4.6) \]

In order to justify the continuity and compactness of the mappings \( H, L \) we first prove the following general statement

**Claim 1:** Let \( \bar{u} \in L^{(2^*), 2} \) and \( u_n \in L^{(2^*), 2} \) be a sequence satisfying

(i) for some \( \alpha > \frac{2^*}{2^* - \gamma} \) we have \( u_n \to \bar{u} \) in \( L^\alpha(\Omega \cap B) \) for each ball \( B \subset \mathbb{R}^N \);

(ii) \( u_n \) is bounded in \( L^{(2^*), 2} \);

then we claim

\[ \| a[u_n - \bar{u}] \|_{(2^*), 2} \to 0 \quad \text{and} \quad \| b[r(u_n) - r(\bar{u})] \|_{(2^*), 2} \to 0. \quad (4.7) \]

We only give the arguments for \( \| b[r(u_n) - r(\bar{u})] \|_{(2^*), 2} \). Notice first that (H1), and the fact that the sequence \( u_n \) is bounded in \( L^{(2^*), 2} \) implies the existence of \( C > 0 \) such that

\[ |r(s)| \leq C(|s| + |s|^\gamma), \quad (4.8) \]

\[ \| u_n \|_{2^*, 2} + \| \bar{u} \|_{2^*, 2} + \| u_n \|_{\gamma, 2} + \| \bar{u} \|_{\gamma, 2} \leq C. \]

Using Lemma 4.1, we can choose a measurable set \( \Omega^\varepsilon \subseteq \Omega \) such that \( \Omega^\varepsilon \) is bounded,

\[ b\chi_{\Omega^\varepsilon} \in L^\infty(\Omega), \quad \| b\chi_{\Omega^\varepsilon} \|_{\frac{\alpha}{\alpha - \gamma}, \frac{\alpha}{\alpha - \gamma}} + \| b\chi_{\Omega^\varepsilon} \|_{p_0, q_0} < \frac{\varepsilon}{4C^2}. \quad (4.9) \]
We have
\[
\left\| b[r(u_n) - r(\bar{u})]\right\|_{[2^*],2} \\
\leq \left\| b[r(u_n) - r(\bar{u})]\chi_{\Omega}\right\|_{[2^*],2} + \left\| b[r(u_n) - r(\bar{u})]\chi_{\Omega\setminus\Omega}\right\|_{[2^*],2}. \tag{4.10}
\]

Let us estimate the first term appearing in the right-hand-side of (4.10). By setting
\[
\frac{1}{p} := \frac{1}{[2^*]} - \frac{\gamma}{\alpha} = 1 - \frac{1}{2^*} - \frac{\gamma}{\alpha}, \quad \text{(notice } 0 < \frac{1}{p} < 1),
\]
we get
\[
\left\| b[r(u_n) - r(\bar{u})]\chi_{\Omega}\right\|_{[2^*],2} \\
\leq \left\| b\chi_{\Omega}\right\|_{p,2} \left\| [r(u_n) - r(\bar{u})]\chi_{\Omega}\right\|_{2,\infty} \tag{4.11}
\]
\[
\leq \left\| b\chi_{\Omega}\right\|_{L^\infty(\Omega)} \left\| [r(u_n) - r(\bar{u})]\chi_{\Omega}\right\|_{p,2}. \tag{4.11}
\]

Since \( u_n \to \bar{u} \) in \( L^\infty(\Omega \cap B) \) for any ball \( B \subset \mathbb{R}^N \), by using Vitali’ Theorem we check that \( \|r(u_n) - r(\bar{u})\|_{L^\infty(\Omega \cap B)} \to 0 \). Therefore the expression in (4.11) can be made as small as we wish. Namely there exists \( n_0 \in \mathbb{N} \) such that
\[
\left\| b[r(u_n) - r(\bar{u})]\chi_{\Omega}\right\|_{[2^*],2} \leq \frac{\varepsilon}{2}, \quad \forall n \geq n_0. \tag{4.12}
\]

To estimate the second term in the right-hand-side of (4.10) we use (4.8), (4.9). We get using \( \frac{1}{p_0} = \frac{2^* - 1}{2^*} - \frac{\gamma}{\alpha} \) and \( \gamma \geq 1 \):
\[
\left\| b[r(u_n) - r(\bar{u})]\chi_{\Omega\setminus\Omega}\right\|_{[2^*],2} \\
\leq C \left\| b\chi_{\Omega\setminus\Omega}\right\|_{p_0,\infty} (\|u_n\|_{2^*,2} + \|\bar{u}\|_{2^*,2}) \\
+ C \left\| b\chi_{\Omega\setminus\Omega}\right\|_{p_0,\infty} (\|u_n\|_{\infty}^\gamma \|\bar{u}\|_{2^*,2}) \\
\leq C.M \left( \|u_n\|_{2^*,2} + \|\bar{u}\|_{2^*,2} + \|u_n\|_{2^*,2} + \|\bar{u}\|_{2^*,2} \right) \tag{4.13}
\]
\[
\leq C.M \left( \|u_n\|_{2^*,2} + \|\bar{u}\|_{2^*,2} + \|u_n\|_{2^*,2} + \|\bar{u}\|_{2^*,2} \right) \leq \frac{\varepsilon}{2}.
\]

(In the above formula we have set \( M = (\|b\chi_{\Omega\setminus\Omega}\|_{p_0,\infty} + \|b\chi_{\Omega\setminus\Omega}\|_{p_0,\infty}) \)).

Putting together (4.10), (4.12), (4.13) we deduce
\[
\left\| b[r(u_n) - r(\bar{u})]\right\|_{[2^*],2} \leq \varepsilon, \quad \forall n \geq n_0,
\]
Claim 2: Continuity and compactness.

Due to the Sobolev-Lorentz embedding and the fact that $\Delta^{-1}$ in (4.1) is an isomorphism, it is enough to prove the continuity of the mappings (4.6). This follows immediately from the embedding $L(2^*, 2) \hookrightarrow L(2^*, 2^*)$ and by applying claim 1 with $\alpha = 2^*.$

Concerning compactness. Let $u_n$ be a bounded sequence in $D^{1,2}_0(\Omega).$ Then there is a subsequence, still denoted $u_n,$ converging weakly in $D^{1,2}_0(\Omega)$ to $\bar{u}.$ Then on each open bounded open set $\omega \subset \Omega$ this sequence converges strongly in $L^p(\omega)$ whenever $p \in [1, 2^*).$ Hence choosing $\alpha \in (\frac{2^*}{2-1}, 2^*)$ the sequence $u_n$ satisfies the conditions stated in claim 1. Therefore (4.7) hold, and the continuity of (4.1) allows to conclude.

Claim 3: Differentiability.

To prove (4.5), it is enough to prove that $\tilde{H}$ is differentiable at $u \equiv 0.$ Let us fix $\varepsilon > 0.$ Thanks to (H1), there exist $s_0, C_0 > 0$ depending only on $\varepsilon$ such that

$$\left| \frac{r(s)}{s} \right| \leq \frac{\varepsilon}{3 \|b\|_{\frac{2}{2-q_0}}} \quad \forall 0 < |s| < s_0,$$

and

$$\left| \frac{r(s)}{|s|^\gamma} \right| \leq C_0 \quad \forall |s| \geq s_0.$$  

(4.14)

For each $u \in L(2^*, 2),$ let us introduce the sets

$$E := \{ x \in \Omega : |u(x)| < s_0 \} \quad \text{and} \quad F := \{ x \in \Omega : |u(x)| \geq s_0 \}.$$

Using triangle inequality, Hölder’s inequality and (4.14), we obtain

$$\left\| br(u) \right\|_{[2^*]^{\gamma}, 2} \leq \left\| br(u) \right\|_{[2^*]^{\gamma}, 2} + \left\| br(u) \right\|_{[2^*]^{\gamma}, 2} \leq \left\| b \right\|_{\frac{2}{2-q_0}} \left\| r(u) \right\|_{2^*, 2} + \left\| br(u) \right\|_{[2^*]^{\gamma}, 2} \leq \frac{\varepsilon}{3} \| u \|_{2^*, 2} + C_0 \| b \|_{2^*, 2} \| u \|_{[2^*]^{\gamma}, 2}.$$  

(4.15)

If $\gamma > 1$ we simply use Hölder’s inequality to estimate the second term in (4.15), and get

$$\left\| br(u) \right\|_{[2^*]^{\gamma}, 2} \leq \frac{\varepsilon}{3} + C_0 \| b \|_{p_0, \infty} \left\| \frac{|u|^\gamma}{|u|^\gamma} \right\|_{2^*, 2} = \frac{\varepsilon}{3} + C_0 \| b \|_{p_0, \infty} \left\| u \right\|_{2^*, 2}^{-\frac{\gamma}{2^*}} \leq \frac{\varepsilon}{3} + C_0 \| b \|_{p_0, \infty} \left\| u \right\|_{2^*, 2}^{-1}.$$

Since $\gamma > 1,$ the conclusion follows immediately.
When $\gamma = 1$, this latter estimate is too rough. In this case we first use Lemma 4.1 to get a measurable set $F^\varepsilon$ such that

$$F^\varepsilon \subseteq F, \ b\chi_{F^\varepsilon} \in L^\infty(\Omega), \ \|b\chi_{F^\varepsilon}\|_{\frac{\infty}{q}} \lesssim \frac{\varepsilon}{3C_0}. \ (4.16)$$

And secondly by fixing $1 < \tilde{\gamma} < 2^* - 1$ we note that

$$|s| \leq |s| \left(\frac{|s|}{s_0}\right)^{\tilde{\gamma}-1}, \quad \forall |s| \geq s_0. \ (4.17)$$

Then by using (4.16), (4.17) and setting $p := \frac{2^*}{2^*-2-\tilde{\gamma}}$, the second term in (4.15) (with $\gamma = 1$) is estimated as follows

$$\|bu\chi_{F}\|_{[2,2]^\gamma,2} \leq \|b\chi_{F\setminus F^\varepsilon}\|_{[2,2]^\gamma,2} + \|bu\chi_{F^\varepsilon}\|_{[2,2]^\gamma,2}
\leq \|b\chi_{F\setminus F^\varepsilon}\|_{\frac{\infty}{\tilde{\gamma}},\infty}\|u\|_{2^*,2} + s_0^{\tilde{\gamma}-1}\|b\chi_{F^\varepsilon}\|_{p,\infty}\|u\|^{\tilde{\gamma}}_{2^*,2}
\leq \frac{\varepsilon}{3C_0}\|u\|_{2^*,2} + s_0^{\tilde{\gamma}-1}\|b\chi_{F^\varepsilon}\|_{L^\infty(\Omega)}\|\chi_{F^\varepsilon}\|_{p,\infty}\|u\|^{\tilde{\gamma}}_{2^*,2} \ (4.18)$$

Inequality (4.18) shows that we can find $\delta > 0$ such that

$$\|bu\chi_{F}\|_{[2,2]^\gamma,2} \leq \frac{2\varepsilon}{3C_0}\|u\|_{2^*,2} \quad \forall \|u\|_{2^*,2} < \delta. \ (4.19)$$

By putting together (4.15) and (4.19), we deduce that $\tilde{H}$ is differentiable at $u \equiv 0$, which concludes the proof of (4.5). \hfill \Box

We now have all the elements for proving the existence of at least one global branch of solutions for Problem (1.1):

**Proof of proposition 1.2:** Proposition 1.1 shows that $\lambda_{\gamma}^1(\alpha)$ is an eigenvalue of multiplicity one. Hence Proposition 4.1 together with Theorem 4.1 allow to conclude. \hfill \Box

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**References**
