Monatsh. Math. 143, 35–59 (2004)
DOI 10.1007/s00605-003-0041-6

Monatshefte für
Mathematik
Printed in Austria

Generalized Hardy-Sobolev Inequalities and Exponential Decay of the Entropy of $g(x)\dot{u} = \Delta u$

By

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Communicated by P. Markowich

Received February 13, 2003; in revised form March 26, 2003

Abstract. Provided the non-negative function $g \in L^1_{\mathrm{loc}}(\Omega)$ allows for a generalized Hardy-Sobolev inequality, existence and uniqueness of global weak solutions of the possibly degenerate parabolic PDE $g(x)\dot{u} = \Delta u$, subject to homogeneous Dirichlet boundary conditions, is proved. The maximum/minimum principle holds. The associated entropy decays exponentially as $t \uparrow \infty$ with a rate not exceeding $2/C$, where C is the constant arising in the generalized Hardy-Sobolev inequality.

2000 Mathematics Subject Classification: 46E35, 35K65, 35B05, 35B40, 35B50
Key words: Hardy-Sobolev inequality, degenerate parabolic PDE, existence and uniqueness of global solutions, maximum principle, minimum principle, exponential decay of entropy

1. Introduction

In the sequel, $\Omega \subset \mathbb{R}^d$ is a smooth, bounded, nonvoid domain. We want to investigate the interplay of generalized Hardy-Sobolev inequalities

$$\forall v \in H^1_0(\Omega) : \int_\Omega v^2 g \, dx \leqslant C \int_\Omega |\nabla_x v|^2 \, dx, \tag{1}$$

with (possibly degenerated) parabolic PDE

$$g\dot{u} = \Delta u, \qquad u(t=0) = u_0 \in L^2(g\,dx), \quad u(t,.) \in H^1_0(\Omega). \tag{2}$$

In (1), the constant $C \in \mathbb{R}^+$ only depends on Ω, “dx” is integration with respect to the d-dimensional Lebesgue measure on Ω. A crucial role, both in (1) and (2) is played by the measurable, non-negative function $g : \Omega \to \mathbb{R}^+_0$. All subsequent investigations are based on the assumption

$$0 \leqslant g \in L^1_{\mathrm{loc}}(\Omega).$$

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[*] A.U. acknowledges support from the DFG Forschungszentrum “Mathematics for Key Technologies”, project D10 (Berlin) and from the EU Research Network HYKE.

[†] M.R. acknowledges the hospitality of the mathematical department, Universität Kaiserslautern, where this work was carried out.

Let us explain on a formal level, how the generalized Hardy-Sobolev inequality (1) and the evolution problem (2) are correlated. If we multiply the PDE of (2) for a fixed time $\tau \in \mathbb{R}^+$ by $u(\tau)$, integrate over Ω, interchange differentation with integration and make use of Gauss' integration by parts formula in $H_0^1(\Omega)$, we obtain

$$\frac{d}{dt}E(\tau) = \frac{d}{dt}\left(\frac{1}{2}\int_\Omega u^2(\tau) g\, dx\right) = -\int_\Omega |\nabla_x u(\tau)|^2\, dx, \tag{3}$$

where the "entropy $E(\cdot)$ of (2)" is the function

$$E : \mathbb{R}_0^+ \to \mathbb{R}, \quad E(t) = \frac{1}{2}\int_\Omega u^2(t) g\, dx.$$

Due to the generalized Hardy-Sobolev inequality (1), we deduce

$$\forall \tau \in \mathbb{R}_0^+ : \frac{d}{dt}E(\tau) \leqslant -\frac{2}{C}\int_\Omega u^2(t) g\, dx = -\frac{2}{C}E(\tau),$$

such that

$$\forall \tau \in \mathbb{R}_0^+ : E(\tau) \leqslant E(0) e^{-2\tau/C}, \tag{4}$$

i.e. the entropy decays at an exponential rate. Furthermore, if we integrate (3) from τ to ∞ and if we make use of $\lim_{t\uparrow\infty} E(t) = 0$, we obtain via (4),

$$\forall \tau \in \mathbb{R}_0^+ : \int_\tau^\infty \|\nabla_x u(t)\|^2_{L^2(\Omega:\mathbb{R}^d)}\, dt \leqslant E(0) e^{-2\tau/C},$$

i.e. the integrated (squared) gradient norm of u decays exponentially as $\tau \uparrow \infty$.

In this paper we justify these formal calculations, prove existence and uniqueness of the solution of the degenerate equation (2), when the inequality (1) holds and show the entropy decay and the integrated gradient norm decay in this set up.

Remark 1. An additional difficulty for the analysis is the lack of existence and uniqueness results for weak solutions of (2). Hence, we have to develop such a theory in advance before entering the entropy estimates.

Let us collect some examples of functions g in Hardy-Sobolev inequalities (2).

Example 1. If $g \in L^\infty(\Omega)$, then (1) holds and is equivalent to Poincaré's inequality.

Example 2. If $d \geqslant 3$, then $H^1(\Omega)$ is continuously embedded in $L^{\frac{2d}{d-2}}(\Omega)$. Hence, for each $v \in H^1(\Omega)$, we have $v^2 \in L^{\frac{d}{d-2}}(\Omega)$ and by the continuous embedding of $H^1(\Omega)$ in $L^{\frac{2d}{d-2}}(\Omega)$ the generalized Hardy-Sobolev inequality (1) holds whenever $g \in L^{d/2}(\Omega)$, with a constant C not exceeding $\|g\|_{L^{d/2}(\Omega)}$. Similiarly for $d = 2$, (1) holds whenever $g \in L^{1+\delta}(\Omega)$ for some $\delta \in \mathbb{R}^+$.

Example 3. A much less trivial example concerns the usual Hardy-Sobolev inequality, see [7]. If $d \geqslant 3$, and if $x_0 \in \Omega$, then (1) holds for

$$g : \Omega \to \mathbb{R}, \qquad g(x) = \begin{cases} \frac{1}{|x-x_0|^2}, & x \neq x_0 \\ 0, & x = x_0. \end{cases}$$

Note that for $d = 3$, g is *not* in $L^{d/2}(\Omega)$.

Example 4. For the improved versions of Hardy-Sobolev inequality, (see [4] and [1]) (1) holds for $d \geqslant 3$,

$$g : \Omega \to \mathbb{R}, \quad g(x) = \begin{cases} \frac{1}{|x-x_0|^2} + \frac{1}{(|x-x_0|^2|\log|x-x_0||^2)}, & x \neq x_0 \\ 0, & x = x_0. \end{cases}$$

For $d = 2$, (1) holds with $g = (|x-x_0|^2|\log|x-x_0||^2)^{-1}$ which is not in $L^{1+\delta}(\Omega)$ for any $\delta \in \mathbb{R}^+$.

Example 5. $q(x) = (\mathrm{dist}(x, \partial\Omega))^2$, $x \in \Omega$ (see [3]).

Example 6. More generally, for a k co-dimensional piecewise smooth surface K and the distance function $d(x) = \mathrm{dist}\{x, K\}$, under suitable assumptions the inequality (1) holds with $q(x) = d(x)$. (see [2], also for series improvement of the inequality). Note that this includes the examples 3, 4 and 5 as special cases.

Example 7. In all the previous examples, the functions g belong to $L^1(\Omega)$. If $d = 1$, then there are examples for (1) with functions g which *are not in* $L^1(\Omega)$. E.g., we easily prove

Proposition 2. *If* $\Omega =]a, b[\subset \mathbb{R}$ *is an open, nonvoid, bounded interval, if* $x_0 \in \mathbb{R}$, *and if*

$$x \mapsto |x-a|g(x) \in L^1(\Omega), \quad \text{or} \quad x \mapsto |x-b|g(x) \in L^1(\Omega),$$

then (1) *holds.*

Proof. We consider $v \in C_c^\infty(\Omega)$. The case $v \in H_0^1(\Omega)$ follows from a standard density argument.

Furthermore, we assume $x \mapsto |x-a|g(x) \in L^1(\Omega)$. (The other case follows in a quite similar way.) We have due to $v(a) = 0$,

$$\forall x \in]a, b[: v(x) = \int_a^x v'(z)\,dz \leqslant \sqrt{x-a}\sqrt{\int_a^x (v'(z))^2\,dz},$$

and therefore,

$$\begin{aligned} \int_\Omega v^2(z)g(z)\,dz &= \int_a^b v^2(x)g(x)\,dx \\ &\leqslant \int_a^b (x-a)\left(\int_a^x (v'(z))^2\,dz\right)g(x)\,dx \\ &= \int_a^b \left((v'(z))^2 \int_z^b (x-a)g(x)\,dx\right)dz \\ &\leqslant \left(\int_a^b |x-a|g(x)\,dx\right)\int_a^b (v'(x))^2\,dx. \end{aligned}$$

An example for g which is not in $L^1(]a, b[)$ but satisfies the assumptions of Proposition 1 is

$$g(x) = (x-a)^{-2+\delta}, \quad x \in]a, b[,$$

for some $\delta \in]0, 1[$.

2. Weak Solutions – A Priori Estimates

We are concerned with weak solutions of (2) on a fixed time interval $[0,T]$, where $T\in\mathbb{R}^+$.

We put

$$\forall t\in[0,T]: Q=\mathbb{R}\times\Omega, \qquad Q_t=]0,t[\times\Omega, \qquad Q_{\bar t}=[0,t]\times\Omega$$

and we introduce

$$C_c^\infty(Q_{\overline{T}})=\{\phi\downharpoonright Q_{\overline{T}}:\phi\in C_c^\infty(Q)\}$$

Definition 3. u is a weak solution of (2) iff

ws.1 $u\in L^\infty(0,T;L^2(g\,dx))\cap L^2(0,T;H_0^1(\Omega))$,

ws.2 for all $\eta\in C_c^\infty(Q_{\overline{T}})$, and for almost all $t\in[0,T]$,

$$\int_\Omega u(t)\eta(t)g\,dx+\int_{Q_t}(\nabla_x u\cdot\nabla_x\eta)\,d(\tau,x)=\int_\Omega u_0\eta(0)g\,dx+\int_{Q_t}u\dot\eta g\,d(\tau,x). \quad (5)$$

Note that the definition is similar to the one in [5] but the main difference is the coefficient g for $\dot u$.

2.1. Continuity with respect to time.

Lemma 4. *If u is a weak solution of* (2), *then*

$$u\in C(0,T;L^2(g\,dx)) \qquad \textit{and} \qquad \lim_{t\downarrow 0}\|u(t)-u_0\|_{L^2(g\,dx)}=0.$$

Proof. By definition, $u\in L^\infty(0,T;L^2(g\,dx))$. Redefining u, if necessary, on a set of measure zero, we have

$$\exists K\in\mathbb{R}^+:\forall t\in]0,T[:\|u(t)\|_{L^2(g\,dx)}\leqslant K.$$

We recall $Q=\mathbb{R}\times\Omega$ and we introduce

$$u^*:Q\to\mathbb{R}, \qquad u^*(t,x)=\begin{cases}u(t,x), & (t,x)\in Q_T\\ u(-t,x), & (t,x)\in]-T,0[\times\Omega\\ 0, & \text{else.}\end{cases}$$

We observe $u^*\in L^2(\mathbb{R};H_0^1(\Omega))$ and since u is a weak solution of (2), we obtain for each $\eta\in C_c^\infty(Q)$ with $\operatorname{supp}(\eta)\subset\,]-T,T[\times\Omega$, after some elementary manipulations from (5),

$$\begin{aligned}\int_{Q_T}(\nabla_x u^*\cdot\nabla_x\eta)\,d(\tau,x)&=\int_{Q_T}(\nabla_x u\cdot\nabla_x\eta)\,d(\tau,x)\\&=\int_\Omega u_0\eta(0)g\,dx+\int_{Q_T}u\dot\eta g\,d(\tau,x)\\&=\int_\Omega u_0\eta(0)g\,dx+\int_{Q_T}u^*\dot\eta g\,d(\tau,x),\end{aligned}$$

and if we set $\bar{\eta}(\tau,x) = \eta(-\tau,x)$, $(\tau,x) \in Q$, then

$$\begin{aligned}\int_{]-T,0[\,\times\,\Omega} (\nabla_x u^* \cdot \nabla_x \eta)\, d(\tau,x) &= \int_{Q_T} (\nabla_x u \cdot \nabla_x \bar{\eta})\, d(\tau,x) \\ &= \int_\Omega u_0 \bar{\eta}(0) g\, dx + \int_{Q_T} u \dot{\bar{\eta}} g\, d(\tau,x) \\ &= \int_\Omega u_0 \eta(0) g\, dx \\ &\quad + \int_{]-T,0[\,\times\,Q} u(-\tau,x) \dot{\bar{\eta}}(-\tau,x) g\, d(\tau,x) \\ &= \int_\Omega u_0 \eta(0) g\, dx - \int_{]-T,0[\,\times\,Q} u^* \dot{\eta} g\, d(\tau,x).\end{aligned}$$

Introducing

$$\text{sign} : \mathbb{R} \to \mathbb{R}, \qquad \text{sign}(\tau) = \begin{cases} +1, & 0 \leqslant \tau \\ -1, & 0 > \tau \end{cases}$$

we deduce

$$\int_Q u^* \dot{\eta} g\, d(\tau,x) = \sum_{j=1}^{d} \int_Q (\partial_j u^*) \cdot (\partial_j \eta) \cdot \text{sign}(\tau)\, d(\tau,x). \tag{6}$$

If we set

$$\begin{aligned} V = \Big\{ &\eta \in L^2(\mathbb{R} : H_0^1(\Omega)) : \dot{\eta} \in L^2(\mathbb{R} : L^2(g\, dx)) \text{ and} \\ &\text{there is a sequence } (\phi_n)_{n\in\mathbb{N}} \text{ in } C_c^\infty(Q_{\overline{T}}) \text{ such that} \\ &\lim_{n\uparrow\infty} \|\eta - \phi_n\|_{L^2(\mathbb{R}:H_0^1(\Omega))} = 0 \text{ and } \lim_{n\uparrow\infty} \|\dot{\eta} - \dot{\phi}_n\|_{L^2(\mathbb{R}:L^2(g\,dx))} = 0 \Big\}, \end{aligned}$$

then one verifies by a standard density argument that (6) holds for all $\eta \in V$.

In particular, if $\omega \in C_c^\infty(\mathbb{R})$ with $\text{supp}(\omega) \subset \,]-T,T[$, then for any $\Phi \in V$, taking $\eta = \omega\Phi$ in (6)

$$\int_Q u^* \omega \dot{\Phi} g\, d(\tau,x) = \sum_{j=1}^{d} \int_Q \omega \cdot (\partial_j u^*) \cdot (\partial_j \Phi)\, \text{sign}(\tau)\, d(\tau,x) - \int_Q u^* \dot{\omega} \Phi g\, d(\tau,x). \tag{7}$$

Given $(h,\eta) \in \mathbb{R}^+ \times L^2(\mathbb{R}; L^2(\Omega))$ we introduce

$$\begin{aligned} I_h^-(\eta) &: Q \to \mathbb{R}, \qquad I_h^-(\eta)(t,x) = \frac{1}{h} \int_{t-h}^{t} \eta(s,x)\, ds, \\ I_h^+(\eta) &: Q \to \mathbb{R}, \qquad I_h^+(\eta)(t,x) = \frac{1}{h} \int_{t}^{t+h} \eta(s,x)\, ds. \end{aligned}$$

Then for $(h,\eta) \in \mathbb{R}^+ \times L^2(\mathbb{R}; L^2(\Omega))$,

(I1) $I_h^{\pm}(\eta) \in L^2(\mathbb{R}; L^2(\Omega))$,
and if additionally $\eta \in L^2(\mathbb{R}; H_0^1(\Omega))$, then $I_h^{\pm}(\eta) \in L^2(\mathbb{R}; H_0^1(\Omega))$ and for $j = 1, \dots, d$,

$$\partial_j(I_h^{\pm}(\eta)) = I_h^{\pm}(\partial_j \eta) \in L^2(\mathbb{R}; L^2(\Omega)),$$

and if $\eta_1 \in L^2(\mathbb{R}; L^2(\Omega))$, then

$$\int_Q \eta_1 \cdot I_h^+(\eta)\, d(\tau, x) = \int_Q I_h^-(\eta_1) \cdot \eta\, d(\tau, x),$$

(I2) $\lim_{h\downarrow 0} \|I_h^{\pm}(\eta) - \eta\|_{L^2(\mathbb{R}; L^2(\Omega))} = 0$ (see [LSU88]),
(I3) if $\eta \in L^2(\mathbb{R}; H_0^1(\Omega))$, then $I_h^{\pm}(\eta) \in L^2(\mathbb{R}; L^2(g\, dx))$ (here we use (1) for the first time),
(I4) $I_h^{\pm}(\eta)$ are absolutely continuous in t and since for all $\alpha \in C_c^{\infty}(Q)$,

$$\int_Q I_h^-(\eta)\dot{\alpha}\, d(\tau, x) = -\int_Q \frac{\eta(s, x) - \eta(s - h, x)}{h} \alpha\, d(\tau, x),$$
$$\int_Q I_h^+(\eta)\dot{\alpha}\, d(\tau, x) = -\int_Q \frac{\eta(s + h, x) - \eta(s, x)}{h} \alpha\, d(\tau, x),$$

we have in the sense of distributions

$$\partial_t(I_h^-(\eta)) = D_h^- \eta : Q \to \mathbb{R}, \qquad (D_h^- \eta)(t, x) = \frac{\eta(t, x) - \eta(t - h, x)}{h},$$
$$\partial_t(I_h^+(\eta)) = D_h^+ \eta : Q \to \mathbb{R}, \qquad (D_h^+ \eta)(t, x) = \frac{\eta(t + h, x) - \eta(t, x)}{h},$$

(I5) if $\eta \in L^2(\mathbb{R}; H_0^1(\Omega))$, we have $\partial_t(I_h^{\pm}(\eta)) = (D_h^{\pm}\eta) \in L^2(\mathbb{R}; H_0^1(\Omega))$ as well, and therefore $I_h^{\pm}(\eta) \in V$ (here we use (1) for the second time).

As a consequence, by (7), for all $\eta \in L^2(\mathbb{R}; H_0^1(\Omega))$,

$$\int_Q (u^*\omega)\partial_t(I_h^-(\eta)) g\, d(\tau, x)$$
$$= \sum_{j=1}^{d} \int_Q (\partial_j \eta) I_h^+((\partial_j u^*)\omega)\, \mathrm{sign}(\tau)\, d(\tau, x) - \int_Q \eta I_h^+(u^*\dot{\omega}) g\, d(\tau, x), \quad (8)$$

where we made use of (I1), (I3), (I5).

Now, if we make the particular choice $\eta = \chi(\cdot)\phi(\cdot)$, where $\chi \in C_c^{\infty}(\mathbb{R})$ and $\phi \in H_0^1(\Omega)$, then the left hand side of (8) is

$$\begin{aligned}\int_Q (u^*\omega)\partial_t(I_h^-(\chi\phi)) g\, d(\tau, x) &= \int_Q (u^*\omega)\phi \frac{\chi(s) - \chi(s - h)}{h} g\, d(\tau, x) \\ &= \int_Q (u^*\omega)\phi I_h^-(\dot{\chi}) g\, d(\tau, x) \\ &= \int_Q \dot{\chi} I_h^+(u^*\omega)\phi g\, d(\tau, x),\end{aligned}$$

and (8) leads to

$$\int_{\mathbb{R}} \dot{\chi}\left[\int_{\Omega} I_h^+(u^*\omega)\phi g\,dx\right] d\tau$$
$$= \int_{\mathbb{R}} \chi\left[\operatorname{sign}(\tau)\int_{\Omega}\sum_{j=1}^{d}(\partial_j\phi)I_h^+((\partial_j u^*)\omega)\,dx - \int_{\Omega}\phi I_h^+(u^*\dot{\omega})g\,dx\right] d\tau,$$

As a consequence, in the sense of distributions,

$$\partial_t\left(\int_{\Omega} I_h^+(u^*\omega)\phi g\,dx\right)$$
$$= -\operatorname{sign}(\cdot)\sum_{j=1}^{d}\int_{\Omega}(\partial_j\phi)I_h^+((\partial_j u^*)\omega)\,dx + \int_{\Omega}\phi I_h^+(u^*\dot{\omega})g\,dx \in L^1(\mathbb{R}),$$

such that via (I5) for all $(h,H)\in\mathbb{R}^+\times\mathbb{R}^+$ and for all $\phi\in H_0^1(\Omega)$,

$$\begin{aligned} J(\phi)(t) &:= \int_{\Omega}\phi\cdot(D_h^+(u^*\omega) - D_H^+(u^*\omega))g\,dx \\ &= -\operatorname{sign}(\cdot)\sum_{j=1}^{d}\int_{\Omega}(\partial_j\phi)\cdot(I_h^+((\partial_j u^*)\omega) - I_H^+((\partial_j u^*)\omega))\,dx \\ &\quad + \int_{\Omega}\phi\cdot(I_h^+(u^*\dot{\omega}) - I_H^+(u^*\dot{\omega}))g\,dx \\ &=: S(\phi)(t), \quad \text{almost everywhere on } \mathbb{R}. \end{aligned} \tag{9}$$

The function $S(\phi)(t)$ is uniformly continuous on $\mathbb{R}^+$ and on $\mathbb{R}^-$, respectively and the limits $S(\phi)(0\pm)$ exist (with $S(\phi)(0) = S(\phi)(0+)$).

Concerning (9) we observe that the set where equality holds depends on ϕ a priori, i.e. there is for each $\phi\in H_0^1(\Omega)$ a set $N(\phi)\subset\mathbb{R}$, such that $\lambda_1(N(\phi)) = 0$ (where λ_1 is the one dimensional Lebesgue measure) and $J(\phi) = S(\phi)$ for all $t\in\mathbb{R}\backslash N(\phi)$.

We prove:

There is a one-dimensional Lebesgue nullset N,

$$\text{such that for all } (t,\phi)\in(\mathbb{R}\backslash N)\times H_0^1(\Omega)\text{: } J(\phi)(t) = S(\phi)(t). \tag{10}$$

Proof. We recall (see, e.g., [6]): $t\in\mathbb{R}$ is a Lebesgue point of $J(\phi)$ iff

$$\lim_{r\downarrow 0}\frac{1}{2r}\int_{t-r}^{t+r}|J(\phi)(s) - J(\phi)(t)|\,ds = 0.$$

Since $J(\phi)(t) = S(\phi)(t)$ for almost all $t\in\mathbb{R}$, and since $S(\phi)$ is continuous on $\mathbb{R}\backslash\{0\}$, we have for all $t\in\mathbb{R}\backslash\{0\}$: if $J(\phi)(t) = S(\phi)(t)$, then t is a Lebesgue point of $J(\phi)$, because due to continuity of $S(\phi)$,

$$\begin{aligned} 0 &\leqslant \limsup_{r\downarrow 0}\frac{1}{2r}\int_{t-r}^{t+r}|J(\phi)(s)-J(\phi)(t)|\,ds \\ &= \limsup_{r\downarrow 0}\frac{1}{2r}\int_{t-r}^{t+r}|J(\phi)(s)-S(\phi)(t)|\,ds \\ &= \limsup_{r\downarrow 0}\frac{1}{2r}\int_{t-r}^{t+r}|S(\phi)(s)-S(\phi)(t)|\,ds = 0. \end{aligned}$$

On the other hand, if $t\in\mathbb{R}\backslash\{0\}$ is a Lebesgue point of $J(\phi)$, then for all $r\in\mathbb{R}^+$,

$$\begin{aligned} 0 &\leqslant |S(\phi)(t)-J(\phi)(t)| \\ &\leqslant \left|S(\phi)(t)-\frac{1}{2r}\int_{t-r}^{t+r}S(\phi)(s)\,ds\right| + \left|J(\phi)(t)-\frac{1}{2r}\int_{t-r}^{t+r}S(\phi)(s)\,ds\right| \\ &= \left|S(\phi)(t)-\frac{1}{2r}\int_{t-r}^{t+r}S(\phi)(s)\,ds\right| + \left|J(\phi)(t)-\frac{1}{2r}\int_{t-r}^{t+r}J(\phi)(s)\,ds\right| \\ &= \left|\frac{1}{2r}\int_{t-r}^{t+r}(S(\phi)(t)-S(\phi)(s))\,ds\right| + \left|\frac{1}{2r}\int_{t-r}^{t+r}(J(\phi)(t)-J(\phi)(s))\,ds\right| \\ &\leqslant \frac{1}{2r}\int_{t-r}^{t+r}|S(\phi)(s)-S(\phi)(t)|\,ds + \frac{1}{2r}\int_{t-r}^{t+r}|J(\phi)(s)-J(\phi)(t)|\,ds, \end{aligned}$$

where both terms of the right-hand side of this inequalities tend to zero as $r\downarrow 0$.

We conclude:

$$\forall t\in\mathbb{R}\backslash\{0\} : J(\phi)(t) = S(\phi)(t) \text{ if and only if } t \text{ is a Lebesgue point of } J(\phi).$$

It remains to be shown that there is a one-dimensional nullset N such that for all $\phi\in H_0^1(\Omega)$, $\mathbb{R}\backslash N$ is contained in the set of all Lebesgue points of $J(\phi)$. $H_0^1(\Omega)$ is separable. Let $\{\phi_n : n\in\mathbb{N}\}$ be a dense subset of $H_0^1(\Omega)$. By (9), for each $n\in\mathbb{N}$, there is a one-dimensional Lebesgue null set $N(\phi_n)$ such that $J(\phi_n)(t) = S(\phi_n)(t)$ if and only if $t\in\mathbb{R}\backslash N(\phi_n)$, i.e. the set of all Lebesgue points of $J(\phi_n)$ is $\mathbb{R}\backslash N(\phi_n)$.

We set $N = \bigcup_{n\in\mathbb{N}} N(\phi_n)$. Since N is a countable union of one-dimensional Lebesgue null sets, N is a one-dimensional null set. We have to prove: If $(t,\phi)\in(\mathbb{R}\backslash N)\times H_0^1(\Omega)$, then t is a Lebesgue point of $J(\phi)(t)$.

Let us fix $\phi\in H_0^1(\Omega)$. If $\epsilon\in\mathbb{R}^+$, then there is $n(\epsilon)\in\mathbb{N}$ such that

$$\|\phi-\phi(n(\epsilon))\|_{H^1(\Omega)} \leqslant \epsilon.$$

Using (1) for the third time we deduce for all $t\in\mathbb{R}$,

$$\begin{aligned} &|J(\phi)(t)-J(\phi(n(\epsilon)))(t)| \\ &\qquad\leqslant \|\phi-\phi(n(\epsilon))\|_{L^2(g\,dx)}\|D_h^+(u^*\omega)(t)-D_H^+(u^*\omega)(t)\|_{L^2(g\,dx)} \\ &\qquad\leqslant \sqrt{C}\|\phi-\phi(n(\epsilon))\|_{H^1(\Omega)}\|D_h^+(u^*\omega)(t)-D_H^+(u^*\omega)(t)\|_{L^2(g\,dx)} \\ &\qquad\leqslant \sqrt{C}\epsilon\cdot\frac{\|(u^*\omega)(t+h)\|_{L^2(g\,dx)}+\|(u^*\omega)(t)\|_{L^2(g\,dx)}}{h} \end{aligned}$$

$$
\begin{aligned}
&+ \sqrt{C}\epsilon \cdot \frac{\|(u^*\omega)(t+h)\|_{L^2(g\,dx)} + \|(u^*\omega)(t)\|_{L^2(g\,dx)}}{H} \\
&\leqslant 2K\sqrt{C}\epsilon \cdot \left(\frac{1}{h} + \frac{1}{H}\right),
\end{aligned}
$$

where K is a uniform bound on $\|u^*(t)\|_{L^2(g\,dx)}$.

Now let $(t,r) \in (\mathbb{R}\backslash N) \times \mathbb{R}^+$. Then t is a Lebesgue point of $J(\phi(n(\epsilon)))$, and we calculate

$$
\begin{aligned}
0 &\leqslant \frac{1}{2r}\int_{t-r}^{t+r} |J(\phi)(s) - J(\phi)(t)|\,ds \\
&\leqslant \frac{1}{2r}\int_{t-r}^{t+r} |J(\phi)(s) - J(\phi(n(\epsilon)))(s)|\,ds \\
&\quad + \frac{1}{2r}\int_{t-r}^{t+r} |J(\phi(n(\epsilon)))(s) - J(\phi(n(\epsilon)))(t)|\,ds \\
&\quad + \frac{1}{2r}\int_{t-r}^{t+r} |J(\phi(n(\epsilon)))(t) - J(\phi)(t)|\,ds \\
&\leqslant 2K\sqrt{C}\epsilon \cdot \left(\frac{1}{h} + \frac{1}{H}\right) + \frac{1}{2r}\int_{t-r}^{t+r} |J(\phi(n(\epsilon)))(s) - J(\phi(n(\epsilon)))(t)|\,ds \\
&\quad + 2K\sqrt{C}\epsilon \cdot \left(\frac{1}{h} + \frac{1}{H}\right) \\
&= 4K\sqrt{C}\epsilon \cdot \left(\frac{1}{h} + \frac{1}{H}\right) + \frac{1}{2r}\int_{t-r}^{t+r} |J(\phi(n(\epsilon)))(s) - J(\phi(n(\epsilon)))(t)|\,ds,
\end{aligned}
$$

hence, since t is a Lebesgue point of $J(\phi(n(\epsilon)))$,

$$
\begin{aligned}
0 &\leqslant \limsup_{r\downarrow 0} \frac{1}{2r}\int_{t-r}^{t+r} |J(\phi)(s) - J(\phi)(t)|\,ds \\
&\leqslant 4K\sqrt{C}\epsilon \cdot \left(\frac{1}{h} + \frac{1}{H}\right) + \lim_{r\downarrow 0} \frac{1}{2r}\int_{t-r}^{t+r} |J(\phi(n(\epsilon)))(s) - J(\phi(n(\epsilon)))(t)|\,ds, \\
&= 4K\sqrt{C}\epsilon \cdot \left(\frac{1}{h} + \frac{1}{H}\right) + 0.
\end{aligned}
$$

This estimate holds for all $\epsilon \in \mathbb{R}^+$. Thus,

$$
0 = \lim_{r\downarrow 0} \frac{1}{2r}\int_{t-r}^{t+r} |J(\phi)(s) - J(\phi)(t)|\,ds.
$$

□

Note that the function $\|I_h^+(u^*\omega) - I_H^+(u^*\omega)\|_{L^2(g\,dx)}^2$ is in $L^1(\mathbb{R})$ thanks to (I3). It is easy to see that in the sense of distributions,

$$
\begin{aligned}
&\frac{1}{2}\partial_t\big(\|I_h^+(u^*\omega) - I_H^+(u^*\omega)\|_{L^2(g\,dx)}^2\big) \\
&\quad = \int_\Omega \big(D_h^+(u^*\omega) - D_H^+(u^*\omega)\big) \cdot \big(I_h^+(u^*\omega) - I_H^+(u^*\omega)\big) g\,dx. \qquad (11)
\end{aligned}
$$

Now let us take $t_1 \in \mathbb{R}\backslash N$, where N is the one dimensional null set in (10). We choose

$$\phi = (I_h^+(u^*\omega) - I_H^+(u^*\omega))(t_1) \in H_0^1(\Omega)$$

in (9) and evaluate (9) at $t = t_1$, to deduce

$$\begin{aligned}
&\frac{1}{2}\partial_t\big(\|I_h^+(u^*\omega)(t_1) - I_H^+(u^*\omega)(t_1)\|^2_{L^2(g\,dx)}\big) \\
&\quad = -\mathrm{sign}(t_1)\sum_{j=1}^{d}\int_\Omega \left|I_h^+((\partial_j u^*)\omega) - I_H^+((\partial_j u^*)\omega)(t_1)\right|^2 dx \\
&\qquad + \int_\Omega \big(I_h^+(u^*\omega) - I_H^+(u^*\omega)\big)(t_1)\cdot\big(I_h^+(u^*\dot\omega) - I_H^+(u^*\dot\omega)\big)(t_1) g\,dx. \qquad (12)
\end{aligned}$$

for almost all $t_1 \in \mathbb{R}$. The right-hand side of (12) is in $C[0,+\infty[$. Now let us take $\omega \in C_c^\infty(\mathbb{R})$ with $\mathrm{supp}(\omega) \subseteq\,]-T,+T[$ and $\omega = 1$ on an interval $]-\tau,+\tau[$, where $\tau \in]0,T[$. Furthermore, let $\theta \in]0,1[$. Then, for all $h,H \in]0,(1-\theta)\tau[$,

$$I_h^+(u^*\dot\omega) = I_H^+(u^*\dot\omega) = 0 \quad \text{on } [-\tau,+\theta\tau].$$

We observe that for each $\omega \in C_c^\infty(\mathbb{R})$ and for each pair $(h,H) \in \mathbb{R}^+ \times \mathbb{R}^+$,

$$\kappa(\omega,h,H,.) = \|I_h^+(u^*\omega)(\cdot) - I_H^+(u^*\omega)(\cdot)\|^2_{L^2(g\,dx)}$$

is a continuous function on $\mathbb{R}$. Furthermore, due to (11), in the sense of distributions,

$$\partial_t\kappa(\omega,h,H,.) \leqslant 0 \quad \text{on }]0,+\theta\tau],$$

and

$$\partial_t\kappa(\omega,h,H,.) \geqslant 0 \quad \text{on } [-\tau,0[,$$

where both derivatives are continuous functions on the respective intervals. Thus,

$$0 \leqslant \kappa(\omega,h,H,t) \leqslant \kappa(\omega,h,H,0) \quad \text{on } [-\tau,+\theta\tau].$$

Since $\lim_{(h,H)\to(0,0)}\kappa(\omega,h,H,0) = 0$, we deduce

$$\lim_{(h,H)\to(0,0)} \sup\big\{\kappa(\omega,h,H,t) : t \in [-\tau,+\theta\tau]\big\} = 0.$$

As a consequence, for each sequence $(h(n))_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ with $\lim_{n\uparrow\infty} h(n) = 0$, the sequence

$$\big(I_{h(n)}^+(u^*\omega) \lfloor [-\tau,+\theta\tau]\big)_{n\in\mathbb{N}}$$

is a Cauchy sequence in $C_B([-\tau,+\theta\tau];L^2(g\,dx))$. Hence, for each such sequence there is an $I_0 \in C_B([-\tau,+\theta\tau];L^2(g\,dx))$ with

$$\lim_{h\downarrow 0}\|I_h^+(u^*\omega) - I_0\|_{C_B([-\tau,+\theta\tau];L^2(g\,dx))} = 0.$$

On the other hand, for almost all $t \in \mathbb{R}$, $\lim_{h\downarrow 0}\|I_h^+(u^*\omega)(t) - (u^*\omega)(t)\|_{L^2(g\,dx)} = 0$, hence $I_0 = u^*\omega$ almost everywhere on Q, i.e. changing, if necessary, $u^*\omega$ on a set of measure zero, $u^*\omega \in C_B([-\tau,+\theta\tau];L^2(g\,dx))$.

Repeating the argument for variable τ,θ, we deduce, possibly after redefining u^* on a set of measure zero: $u^* \lfloor]-T,T[\times\Omega \in C(-T,T;L^2(g\,dx))$. In particular, if we redefine, if necessary, u on a set of measure zero, then $u \in C(0,T;L^2(g\,dx))$ and there is $w \in L^2(g\,dx)$ with $\lim_{t\downarrow 0} \|u(t) - w\|_{L^2(g\,dx)}$. As a consequence, (5) holds for all $t \in]0,T[$ and for all $\eta \in C_c^\infty(\Omega_{\overline{T}})$.

It remains to be shown: $w = u_0$. Let $\theta \in C_c([0,+\infty[)$ with $\operatorname{supp}(\theta) \subset [0,t_0] \subset [0,T[$, $0 \leqslant \theta$, $\dot{\theta} \leqslant 0$ and

$$\int_{[0,T[} \theta(t)\,dt = 1.$$

We introduce

$$\forall k \in \mathbb{N} : \theta_k : [0,T[\to \mathbb{R}, \quad \theta_k(t) = k\theta(kt),$$

and we set

$$\forall k \in \mathbb{N} : R_k : [0,T[\to \mathbb{R}, \qquad R_k(t) = 1 - \int_{[0,t[} \theta_k(s)\,ds.$$

For $\Phi \in C_c^\infty(\Omega)$ and $k \in \mathbb{N}$ take $\eta_k = \Phi(\cdot)R_k(\cdot)$ in (5). We obtain for all $k \in \mathbb{N}$, and for all $t_0 < t < T$,

$$\int_{\Omega_t} R_k \nabla_x u \cdot \nabla_x \Phi\,d(\tau,x) = \int_\Omega u_0\,\Phi\,g\,dx - \int_{]0,t[} \theta_k(s)\left(\int_\Omega u\,\Phi\,g\,dx\right)ds. \tag{13}$$

Obviously, since for all $k \in \mathbb{N}$, $\operatorname{supp}(R_k) \subset [0,\frac{t_0}{k}]$, $0 \leqslant R_k \leqslant 1$, for all $k \in \mathbb{N}$

$$\int_{\Omega_t} R_k \nabla_x u \cdot \nabla_x \Phi\,d(\tau,x) \leqslant \int_{]0,\frac{t_0}{k}[\,\times\,\Omega} |\nabla_x u \cdot \nabla_x \Phi|\,d(\tau,x),$$

where $\lim_{k\uparrow\infty} \lambda(]0,\frac{t_0}{k}[\times\Omega) = 0$, such that due to $\nabla_x u \cdot \nabla_x \Phi \in L^1(Q_T)$,

$$\lim_{k\uparrow\infty} \int_{\Omega_t} R_k(s) \nabla_x u \cdot \nabla_x \Phi\,d(\tau,x) = 0. \tag{14}$$

Since for all $k \in \mathbb{N}$, $\int_{]0,t[} \theta_k\,ds = 1$, we can rewrite the right-hand side of (13) as

$$\begin{aligned}
&\int_\Omega u_0 \Phi g\,dx - \int_{]0,t[} \theta_k(s)\left(\int_\Omega u\,\Phi g\,dx\right)ds \\
&\quad = \int_{]0,t[} \theta_k(s)\left(\int_\Omega (u_0 - u(s))\Phi g\,dx\right)ds \\
&\quad = \int_{]0,t[} \theta_k(s)\left(\int_\Omega (u_0 - w)\Phi g\,dx\right)ds + \int_{]0,t[} \theta_k(s)\left(\int_\Omega (w - u(s))\Phi g\,dx\right)ds \\
&\quad = \int_\Omega (u_0 - w)\,\Phi\,g\,dx + \int_{]0,t[} \theta_k(s)\left(\int_\Omega (w - u(s))\Phi g\,dx\right)ds.
\end{aligned}$$

Since $\lim_{t\downarrow 0} \|w - u(t)\|_{L^2(g\,dx)} = 0$, we have

$$\begin{aligned}
&\lim_{k\uparrow\infty} \int_{]0,t[} \theta_k(s)\left(\int_\Omega (w - u(s))\Phi g\,dx\right) \\
&\quad = \lim_{k\uparrow\infty} \int_{]0,\frac{t_0}{k}[} \theta_k(s)\left(\int_\Omega (w - u(s))\Phi g\,dx\right) \\
&\quad \leqslant \lim_{k\uparrow\infty} \left(\int_{[0,\frac{t_0}{k}]} \theta_k(s)\|w - u(s)\|_{L^2(gdx)}\|\phi\|_{L^2(gdx)}\right) = 0.
\end{aligned}$$

From the above and (13) due to (14), by carrying out the limit $k \uparrow \infty$,

$$\int_\Omega (u_0 - w)\Phi g\, dx = 0. \tag{15}$$

Equation (15) holds for each $\Phi \in C_c^\infty(\Omega)$. Thus $w = u_0$ almost everywhere. □

Corollary 5. *If* $u \in C(0, T; L^2(g\, dx))$ *is a weak solution of* (2), *then for all* $\eta \in C_c^\infty(Q_{\overline{T}})$, *the function*

$$B(\eta) : [0, T[\to \mathbb{R}, \qquad B(\eta)(t) = \int_\Omega u(t)\eta(t) g\, dx,$$

is continuous and for all $t \in [0, T[$,

$$\int_\Omega u(t)\eta(t) g\, dx + \int_{Q_t} (\nabla_x u \cdot \nabla_x \eta)\, d(\tau, x) = \int_\Omega u_0 \eta(0) g\, dx + \int_{Q_t} u\dot{\eta} g\, d(\tau, x). \tag{16}$$

2.2. The energy estimate. In this section we shall prove an energy estimate for weak solutions of (2). In a preparational step we establish the following auxiliary result.

Lemma 6. *Let* $u \in C(0, T; L^2(g\, dx))$ *be a weak solution of* (2). *Then for all* $t \in]0, T[$, $h \in]0, T - t[$, *and* $\eta \in L^2(0, T; H_0^1(\Omega))$,

$$\sum_{j=1}^d \int_{Q_t} I_h^+(\partial_j u)(\partial_j \eta)\, d(\tau, x) = -\int_{Q_t} D_h^+(u)\eta g\, d(\tau, x). \tag{17}$$

Proof. Let $\eta \in C_c^\infty(Q)$. Testing (16) with $I_h^-(\eta) \downharpoonright Q_{\overline{T}}$, $0 < h < t < T$, gives

$$\begin{aligned} \int_\Omega u(t) I_h^-(\eta)(t) g\, dx &+ \sum_{j=1}^d \int_{Q_t} (\partial_j u) \cdot (I_h^-(\partial_j \eta))\, d(\tau, x) \\ &= \int_\Omega u_0 I_h^-(\eta)(0) g\, dx + \int_{Q_t} u D_h^-(\eta) g\, d(\tau, x), \end{aligned}$$

from which we deduce by interchanging orders of integration and transforming integration variables,

$$\begin{aligned} \int_\Omega u(t) I_h^-(\eta)(t) g\, dx &+ \sum_{j=1}^d \int_{]-h,0[\times\Omega} (\partial_j \eta)(\tau, x) \left[\frac{1}{h}\int_0^{\tau+h} (\partial_j u)(\sigma, x)\, d\sigma\right] d(\tau, x) \\ &+ \sum_{j=1}^d \int_{Q_{t-h}} I_h^+(\partial_j u)(\partial_j \eta)\, d(\tau, x) \\ &+ \sum_{j=1}^d \int_{]t-h,t[\times\Omega} (\partial_j \eta)(\tau, x) \left[\frac{1}{h}\int_\tau^t (\partial_j u)(\sigma, x)\, d\sigma\right] d(\tau, x) \\ = \int_\Omega u_0 I_h^-(\eta)(0) g\, dx \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{h}\int_{]t-h,t[\,\times\,\Omega} u\eta g\,d(\tau,x)-\int_{Q_{t-h}} D_h^+(u)\eta g\,d(\tau,x)\\
&-\frac{1}{h}\int_{]-h,0[\,\times\,\Omega} u(\tau+h)\eta(\tau)g\,d(\tau,x).
\end{aligned}
\tag{18}
$$

Now let $t\in\,]0,T[$ be fixed. Then for all $\eta\in C_c^\infty(Q)$ with $\operatorname{supp}(\eta)\subseteq[0,t]$, if we evaluate (18) at $t+h$, $0<h<T-t$,

$$
\sum_{j=1}^{d}\int_{Q_t} I_h^+(\partial_j u)(\partial_j\eta)\,d(\tau,x)=-\int_{Q_t} D_h^+(u)\eta g\,d(\tau,x). \tag{19}
$$

If $\omega\in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}(\omega)\subseteq[0,t]$, and if $\phi\in C_c^\infty(\Omega_{\overline{T}})$, then we can use $\omega\phi$ as test function in (19) and we deduce for all $h\in\,]0,T-t[$,

$$
\sum_{j=1}^{d}\int_{Q_t}\omega I_h^+(\partial_j u)(\partial_j\phi)\,d(\tau,x)=-\int_{Q_t}\omega D_h^+(u)\phi g\,d(\tau,x).
$$

As a consequence, for all $\phi\in C_c^\infty(Q_{\overline{T}})$ and for all $h\in\,]0,T-t[$,

$$
\sum_{j=1}^{d}\int_{Q_t} I_h^+(\partial_j u)(\partial_j\phi)\,d(\tau,x)=-\int_{Q_t} D_h^+(u)\phi g\,d(\tau,x). \tag{20}
$$

Now let $\eta\in L^2(0,T;H_0^1(\Omega))$. Then there is a sequence $(\phi(n))_{n\in\mathbb{N}}$ in $C_c^\infty(Q_{\overline{T}})$ with $\lim_{n\uparrow\infty}\|\eta-\phi(n)\|_{L^2(0,T;H_0^1(\Omega))}=0$, in particular, by using (1) for the fourth time, $\lim_{n\uparrow\infty}\|\eta-\phi(n)\|_{L^2(0,T;L^2(g\,dx))}=0$. Hence, if we use for each $n\in\mathbb{N}$ the function $\phi(n)$ as test function in (20), we can pass to the limit $n\uparrow\infty$ in (20). □

Lemma 7. *If $u\in C(0,T;L^2(g\,dx))$ is a weak solution of* (2), *then*

$$
\forall t\in[0,T[:\ E(0)=E(t)+\int_{Q_t}|\nabla_x u|^2\,d(\tau,x), \tag{21}
$$

and

$$
\forall t_1,t_2\in[0,T[:\ \int_{t_1}^{t_2}\|\nabla_x u(\tau)\|^2_{L^2(\Omega:\mathbb{R}^d)}\,d\tau=E(t_1)-E(t_2). \tag{22}
$$

Proof. The case $t=0$ is trivial. If $t\in\,]0,T[$ and if $0<h<T-t$, then we can use $I_h^+(u)$ as test function in (17). We obtain

$$
\sum_{j=1}^{d}\int_{Q_t} I_h^+(\partial_j u)I_h^+(\partial_j u)\,d(\tau,x)=-\int_{Q_t} D_h^+(u)I_h^+(u)g\,d(\tau,x). \tag{23}
$$

One easily verifies using (I4) that

$$
A:\,]0,t[\,\to\mathbb{R},\qquad A(\tau)=\|I_h^+(u)(\tau)\|^2_{L^2(g\,dx)}
$$

is also differentiable even in the classical sense since $u \in C(0,T;L^2(gdx))$ with continuous

$$\dot{A}:]0,t[\to \mathbb{R}, \qquad \dot{A}(\tau) = 2\int_\Omega D_h^+(u) I_h^+(u) g\,dx.$$

Consequently, if $0 < t_1 < t_2 < t$, then

$$2\int_{t_1}^{t_2} \left[\int_\Omega D_h^+(u) I_h^+(u) g\,dx \right] ds = \|I_h^+(u)(t_2)\|^2_{L^2(g\,dx)} - \|I_h^+(u)(t_1)\|^2_{L^2(g\,dx)}. \tag{24}$$

Actually, function $A(\tau)$ can be continuously extended to $[0,t]$. Thus, we can consider the limit $t_1 \downarrow 0$ and $t_2 \uparrow t$ in (24) to deduce

$$2\int_{Q_t} D_h^+(u) I_h^+(u) g\,d(\tau,x) = \|I_h^+(u)(t)\|^2_{L^2(g\,dx)} - \|I_h^+(u)(0)\|^2_{L^2(g\,dx)},$$

such that via (23),

$$\|I_h^+(u)(t)\|^2_{L^2(g\,dx)} + 2\sum_{j=1}^d \int_{Q_t} I_h^+(\partial_j u) I_h^+(\partial_j u)\,d(\tau,x) = \|I_h^+(u)(0)\|^2_{L^2(g\,dx)}. \tag{25}$$

Due to $u \in C(0,T;L^2(g\,dx))$ and since $\lim_{t\downarrow 0} \|u(t) - u_0\|_{L^2(g\,dx)} = 0$ we have

$$\forall \tau \in [0,t[: \lim_{h\downarrow 0} \|u(\tau) - I_h^+(u)(\tau)\|_{L^2(g\,dx)} = 0,$$

and since

$$\lim_{h\downarrow 0} \|(\partial_j u) - I_h^+(\partial_j u)\|_{L^2(Q_t)} = 0,$$

as well, we can pass to the limit $h \downarrow 0$ in (25).

Finally, (22) follows from evaluation (21) at t_1, t_2, respectively. □

2.3. The maximum/minimum principle. Due to the classical maximum principle, the maximum of the solution of a parabolic equation never exceeds the maximum of the initial data and of the boundary data. A similar result holds for the minimum.

In this section we extend these maximum/minimum principles to the degenerate situation (2). Since the boundary data vanish, the contribution of the boundary data is in our setting simply 0, see (26). The strategy of the proof is classical: We use an apropriate "cut-off" function as test-function and prove that this cut-off function must vanish almost everywhere. The hard part of the proof is already done: We have already proved that we can use the "standard" cut-off function as test function, see (17).

Lemma 8. *If* $u \in C(0,T;L^2(g\,dx))$ *is a weak solution of* (2), *then for all* $t \in [0,T[$,

$$\begin{aligned} -\infty \leqslant \min\left\{0, \operatorname*{ess\,inf}_\Omega u_0\right\} &\leqslant \operatorname*{ess\,inf}_\Omega u(t) \\ &\leqslant \operatorname*{ess\,sup}_\Omega u(t) \leqslant \max\left\{0, \operatorname*{ess\,sup}_\Omega u_0\right\} \leqslant +\infty. \end{aligned} \tag{26}$$

Proof. We prove $\operatorname{ess\,sup}_\Omega u(t) \leqslant \max\{0, \operatorname{ess\,sup}_\Omega u_0\}$. The other inequality follows in analogy. Naturally, only the case $K = \operatorname{ess\,sup}_\Omega u_0 < +\infty$ is of interest. The case $t_0 = 0$ is trivial. We introduce for $\max\{0,K\} < \kappa$ the function

$$H_\kappa : \mathbb{R} \to \mathbb{R}, \qquad H_\kappa(x) = \max\{x-\kappa, 0\}.$$

Let $0 < h < T-t$. We introduce

$$I_h^+(u) : Q_{\bar{t}} \to \mathbb{R}, \qquad I_h^+(u)(\tau,x) = \frac{1}{h}\int_\tau^{\tau+h} u(\sigma,x)\,d\sigma.$$

We observe: $I_h^+(u)$ is in $C^1(0,t;L^2(g\,dx)) \cap L^2(0,t;H_0^1(\Omega))$. Since H_κ is uniformly Lipschitz continuous (1 is a global Lipschitz constant) we have in the sense of $(C_c^\infty(Q_t))'$,

$$\forall j \in \{1,\ldots,d\} : \partial_j(H_\kappa(I_h^+(u))) = H_\kappa'(I_h^+(u)) \cdot (\partial_j I_h^+(u)),$$
$$\partial_t[H_\kappa(I_h^+(u))]^2 = 2H_\kappa'(I_h^+(u)) \cdot H_\kappa(I_h^+(u)) \cdot (\partial_t I_h^+(u)).$$

Since $0 < \kappa$, we have $H_\kappa(0) = 0$. Thus, $h_\kappa(w) \in H_0^1(\Omega)$ for all $w \in H_0^1(\Omega)$. Consequently, $H_\kappa(I_h^+(u)) \in C(0,t;H_0^1(\Omega))$, and we can use $H_\kappa(I_h^+(u))$ as test function in (17). We obtain

$$\sum_{j=1}^d \int_{Q_t} \partial_j I_h^+(u)\, \partial_j(H_\kappa(I_h^+(u)))\,d(\tau,x) = -\int_{Q_t} D_h^+(u) H_\kappa(I_h^+(u)) g\,d(\tau,x). \tag{27}$$

Since $H_\kappa'(x) = 1$ for $x > \kappa$, we have

$$2\sum_{j=1}^d \int_{Q_t} \partial_j I_h^+(u) \partial_j(H_\kappa(I_h^+(u)))\,d(\tau,x) = 2\int_{Q_t} |\nabla_x H_\kappa(I_h^+(u))|^2\,d(\tau,x). \tag{28}$$

Furthermore, we have in the sense of $(C_c^\infty(]0,t[))'$,

$$\partial_t \|H_\kappa(I_h^+(u))\|^2_{L^2(g\,dx)} = 2\int_\Omega D_h^+(u) H_\kappa(I_h^+(u)) g\,dx, \tag{29}$$

where we made use of $H_\kappa'(\sigma) = 1$ for $\sigma > \kappa$, $H_\kappa'(\sigma) = 0$ for $\sigma < \kappa$ and $H_\kappa(\kappa) = 0$. We note that due to $u \in C(0,T;L^2(g\,dx))$ the left-hand side of (29) is actually continuously differentiable. Consequently, if $0 < t_1 < t_2 < t$, then

$$\begin{aligned} 2\int_{t_1}^{t_2} \int_\Omega & D_h^+(u) H_\kappa(I_h^+(u)) g\,dx\,d\tau \\ &= \|H_\kappa(I_h^+(u))(t_2)\|^2_{L^2(g\,dx)} - \|H_\kappa(I_h^+(u))(t_1)\|^2_{L^2(g\,dx)}. \end{aligned} \tag{30}$$

As before, we can consider the limit $t_1 \downarrow 0$ and $t_2 \uparrow t$ in (30) to deduce

$$\begin{aligned} 2\int_{Q_t} & D_h^+(u) H_\kappa(I_h^+(u)) g\,d(\tau,x) \\ &= \|H_\kappa(I_h^+(u))(t)\|^2_{L^2(g\,dx)} - \|H_\kappa(I_h^+(u))(0)\|^2_{L^2(g\,dx)}, \end{aligned}$$

such that via (27), (28)

$$\|H_\kappa(I_h^+(u))(t)\|^2_{L^2(g\,dx)} + 2\sum_{j=1}^{d}\int_{Q_t} |\nabla_x H_\kappa(I_h^+(u))|^2\, d(\tau,x)$$
$$= \|H_\kappa(I_h^+(u))(0)\|^2_{L^2(g\,dx)}. \tag{31}$$

Due to continuity of H_κ, due to $u \in C(0,T;L^2(g\,dx))$ and since $\lim_{t\downarrow 0}\|u(t) - u_0\|_{L^2(g\,dx)} = 0$, we have

$$\forall \tau \in [0,t[: \lim_{h\downarrow 0} \|H_\kappa(u)(\tau) - H_\kappa(I_h^+(u))(\tau)\|_{L^2(g\,dx)} = 0.$$

We deduce from (31) by carrying out the limit $h \downarrow 0$,

$$\|H_\kappa(u)(\tau)\|^2_{L^2(g\,dx)} \leqslant \|H_\kappa(u_0)\|^2_{L^2(g\,dx)} = 0,$$

for all $\tau \in [0,t[$, because $\operatorname{ess\,sup}_\Omega u_0 < \kappa$. □

3. The Existence- and Uniqueness Result and Exponential Decay of the Entropy

In this section we prove our main result, Theorem 8. A difficulty for the proof is the lack of an existence- and uniqueness theory for (2). Such a theory is developed during the proof. For the sake of being self-contained we include the assumptions on Ω, g and on u_0 explicitly in the theorem.

Theorem 9. *Assume $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is a smooth, bounded, nonvoid domain and*

1. *$g \in L^1_{\mathrm{loc}}(\Omega)$ is non-negative.*
2. *There is a generalized Hardy-Sobolev inequality with respect to $g\,dx$, i.e. there is $C \in \mathbb{R}^+$ such that*

$$\forall v \in H^1_0(\Omega) : \int_\Omega v^2 g\,dx \leqslant C\int_\Omega |\nabla_x v|^2 dx. \tag{32}$$

3. *$u_0 \in L^2(g\,dx)$.*

Then the (possibly degenerated) parabolic PDE

$$g(x)\dot{u} = \Delta u, \qquad u(t=0) = u_0, \quad u(t,.) \in H^1_0(\Omega), \tag{33}$$

has a unique global weak solution. Furthermore,

1. *$u \in C(0,\infty;L^2(g\,dx)) \cap L^2(0,\infty;H^1_0(\Omega))$ and $\lim_{t\downarrow 0}\|u(t) - u_0\|_{L^2(g\,dx)} = 0$.*
2. *The weak maximum/minimum principle*

$$\forall t \in \mathbb{R}^+_0 : -\infty \leqslant \min\left\{0, \operatorname*{ess\,inf}_\Omega u_0\right\} \leqslant \operatorname*{ess\,inf}_\Omega u(t)$$
$$\leqslant \operatorname*{ess\,sup}_\Omega u(t) \leqslant \max\left\{0, \operatorname*{ess\,sup}_\Omega u_0\right\} \leqslant +\infty$$

holds.

3. *The entropy*

$$E(t) = \frac{1}{2}\int_\Omega u^2(t) g\, dx, \quad t \in \mathbb{R}_0^+, \qquad E(0) = \frac{1}{2}\int_\Omega u_0^2 g\, dx,$$

is continuous on $\mathbb{R}_0^+$ *and for all* $t \in \mathbb{R}_0^+$, *the energy estimate*

$$E(0) = E(t) + \int_{Q_t} |\nabla_x u|^2\, d(\tau, x),$$

the exponential decay law

$$E(t) \leqslant E(0)\, e^{-2t/C},$$

and the exponential decay of the (*squared*) *gradient norm*

$$\int_t^\infty \|\nabla_x u(\tau)\|^2_{L^2(\Omega:\mathbb{R}^d)}\, d\tau \leqslant E(0)\, e^{-2t/C}$$

hold, *where* C *is the constant of the Hardy-Sobolev inequality* (32).

Proof. The proof is divided into several steps.

Step 1. We construct a sequence of smooth functions $\gamma(n)$, $n \in \mathbb{N}$, such that $\gamma(n)$ converges in $L^1_{\mathrm{loc}}(\Omega)$ to g as $n \to \infty$. We take the truncated function γ_n^* and convolve it with a smoothing kernel to obtain $\gamma(n)$. Let $r \in]2, (2d/d-2)[$ such that $H_0^1(\Omega)$ is continuously embedded in $L^r(\Omega)$. If $d = 1$ or 2, then any r in $]2, \infty[$ does.

We set $s = \frac{r}{r-2}$. Then $s \in]1, +\infty[$ and there is $C_0 \in \mathbb{R}^+$ such that

$$\forall (v, w) \in H_0^1(\Omega) \times L^s(\Omega) : \int_\Omega |v^2 w|\, dx \leqslant C_0 \|\nabla_x v\|^2_{L^2(\Omega:\mathbb{R}^d)} \|w\|_{L^s(\Omega)}. \tag{34}$$

We introduce for $n \in \mathbb{N}$,

$$\gamma_n^* : \mathbb{R}^d \to \mathbb{R}, \qquad \gamma_n^*(x) = \begin{cases} \max\left\{\frac{1}{n}, \min\{g(x), n\}\right\}, & x \in \Omega \\ 1, & x \notin \Omega \end{cases},$$

such that

$$\forall (x, n) \in \mathbb{R}^d \times \mathbb{N} : \frac{1}{n} \leqslant \gamma_n^*(x) \leqslant n. \tag{35}$$

Let us take a function $\phi \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(\phi) \subseteq \{z \in \mathbb{R}^d : |z| \leqslant 1\}$, $0 \leqslant \phi \leqslant 1$, and $\int_{\mathbb{R}^d} \phi\, dx = 1$. We set

$$\forall \varepsilon \in \mathbb{R}^+ : \phi_\varepsilon : \mathbb{R}^d \to \mathbb{R}_0^+, \quad \phi_\varepsilon(x) = \varepsilon^d \phi(\varepsilon x).$$

We set for $(n, \varepsilon) \in \mathbb{N} \times \mathbb{R}^+$, $\gamma^*_{n,\varepsilon} = \gamma_n^* \star \phi_\varepsilon$ (convolution in $\mathbb{R}^d$). Then, due to (35), we have

$$\forall (n, \varepsilon) \in \mathbb{N} \times \mathbb{R}^+ : \gamma^*_{n,\varepsilon} \restriction \Omega \in C^\infty_{\mathrm{unif}}(\Omega),$$

and

$$\forall n \in \mathbb{N} : \lim_{\varepsilon \downarrow 0} \int_\Omega |\gamma^*_{n,\varepsilon}(x) - \gamma_n^*(x)|^s = 0.$$

Now, for $n \in \mathbb{N}$ we choose $\varepsilon(n) \in \mathbb{R}^+$ with

$$\int_\Omega |\gamma^*_{n,\varepsilon(n)}(x) - \gamma^*_n(x)|^s \leqslant \frac{1}{n}.$$

We set

$$\forall n \in \mathbb{N} : \gamma(n) = \gamma^*_{n,\varepsilon(n)} \lfloor \Omega.$$

Then

$$\forall n \in \mathbb{N} : \gamma(n) \in C^\infty_{\text{unif}}(\Omega) \qquad \text{and} \qquad \frac{1}{n} \leqslant \gamma(n) \leqslant n.$$

Furthermore,

$$\forall n \in \mathbb{N} : \|\gamma(n) - \gamma^*(n)\|_{L^s(\Omega)} \leqslant \frac{1}{\sqrt[s]{n}}, \tag{36}$$

where

$$\forall n \in \mathbb{N} : \gamma^*(n) = \gamma^*_n \lfloor \Omega = \max\left\{\frac{1}{n}, \min\{g, n\}\right\}.$$

If A is a compact subset of Ω, and if $m \in \mathbb{N}$, then we set

$$r(A,m) = \int_A [g - m]^+ \, dx.$$

Since $g \in L^1_{\text{loc}}(\Omega)$, we have for each compact $A \subset \Omega$,

$$\lim_{m \uparrow \infty} r(A,m) = 0. \tag{37}$$

We estimate for each compact $A \subset \Omega$, $\forall n \in \mathbb{N}$:

$$\begin{aligned}
\int_A |g - \gamma(n)|\, dx &\leqslant \int_A |g - \gamma^*(n)|\, dx + \int_A |\gamma^*(n) - \gamma(n)|\, dx \\
&= \int_{\{x \in A : g(x) < 1/n\}} |g - \gamma^*(n)|\, dx + \int_{\{x \in A : g(x) > n\}} |g - \gamma^*(n)|\, dx \\
&\quad + \int_A |\gamma^*(n) - \gamma(n)|\, dx \\
&= \int_{\{x \in A : g(x) < 1/n\}} \left(\frac{1}{n} - g(x)\right) dx + \int_{\{x \in A : g(x) > n\}} [g - n]^+ \, dx \\
&\quad + \int_A |\gamma^*(n) - \gamma(n)| dx \\
&\leqslant \int_A \frac{1}{n}\, dx + \int_A [g - n]^+ \, dx + \int_\Omega |\gamma^*(n) - \gamma(n)| dx \\
&\leqslant \frac{|\Omega|}{n} + r(A,n) + \|\gamma^*(n) - \gamma(n)\|_{L^s(\Omega)} \cdot |\Omega|^{1-\frac{1}{s}},
\end{aligned}$$

such that for each compact $A \subset \Omega$ via (36),

$$\forall n \in \mathbb{N} : \int_A |g - \gamma(n)|\, dx \leqslant \frac{|\Omega|}{n} + r(A,n) + \frac{|\Omega|^{1-\frac{1}{s}}}{\sqrt[s]{n}}. \tag{38}$$

Step 2. We define a sequence of smooth functions approximating u_0. Since $u_0\sqrt{g}\in L^2(\Omega)$, there is a sequence $(w_n^*)_{n\in\mathbb{N}}$ in $C_c^\infty(\Omega)$ such that w_n^* converges in $L^2(\Omega)$ to $u_0\sqrt{g}$ as $n\uparrow\infty$. We set for $n\in\mathbb{N}$, $w_0(n)=w_n^*/\sqrt{\gamma(n)}$. Then

$$w_0(n)\sqrt{\gamma(n)}\to u_0\sqrt{g},\quad \text{strongly in } L^2(\Omega) \text{ as } n\uparrow\infty,$$

where for each $n\in\mathbb{N}$, $w_0(n)\in C_c^\infty(\Omega)$. In particular, we have

$$\lim_{n\uparrow\infty}\int_\Omega (w_0(n))^2\gamma(n)\,dx=\int_\Omega u_0^2 g\,dx. \tag{39}$$

Step 3. We consider for $n\in\mathbb{N}$ the (non-degenerated) initial value problem

$$\gamma(n)(x)\frac{\partial w(n)}{\partial t}=\Delta w(n),\qquad w(n)(t=0)=w_0(n),\quad w(n)(t,.)\in H_0^1(\Omega), \tag{40}$$

where due to construction, $\gamma(n)$, $w_0(n)$ satisfy for each $n\in\mathbb{N}$,

A.1 $\gamma(n)\in C^\infty_{\text{unif}}(\Omega)$ and $\frac{1}{n}\leqslant\gamma(n)\leqslant n$.
A.2 $w_0(n)\in C_c^\infty(\Omega)$.

We introduce for $n\in\mathbb{N}$

$$C(n)=\sup\left\{\int_\Omega v^2\gamma(n)\,dx : v\in H_0^1(\Omega)\quad\text{and}\quad\int_\Omega|\nabla_x v|^2\,dx=1\right\}. \tag{41}$$

Due to A.1, A.2, by means of Poincaré's inequality, we have $C(n)\in\mathbb{R}^+$. Naturally, for each $n\in\mathbb{N}$, $C(n)$ is the optimal constant in the corresponding Hardy-Sobolev inequality

$$\int_\Omega v^2\gamma(n)\,dx\leqslant C(n)\int_\Omega|\nabla_x v|^2\,dx,\quad v\in H_0^1(\Omega). \tag{42}$$

Following [5], for each $n\in\mathbb{N}$, the initial value problem (40) has a unique global weak solution $w(n)$ which is smooth and bounded. The weak maximum principle holds for $w(n)$, $n\in\mathbb{N}$. By integration by parts one easily verifies for each $n\in\mathbb{N}$ the energy estimate

$$\begin{aligned}\forall t\in\mathbb{R}_0^+ : \int_\Omega (w_0(n))^2\gamma(n)\,dx &= \int_\Omega (w(n)(t))^2\gamma(n)\,dx\\ &\quad + 2\int_{Q_t}|\nabla_x w(n)|^2\,d(\tau,x),\end{aligned} \tag{43}$$

and in a straight-forward manner – as indicated in the introduction – we obtain by making use of the smoothness of $w(n)$,

$$\forall t\in\mathbb{R}_0^+ : \int_\Omega (w(n)(t))^2\gamma(n)\,dx\leqslant e^{-2t/C(n)}\int_\Omega (w_0(n))^2\gamma(n)\,dx. \tag{44}$$

Step 4. Now we shall establish the existence of a weak solution of

$$g(x)\frac{\partial u}{\partial t}=\Delta u,\qquad u(t=0)=u_0,\quad u(t,.)\in H_0^1(\Omega), \tag{45}$$

where we recall $u_0\in L^2(g\,dx)$. The strategy is to establish several a priori estimates on the sequence $(w(n))_{n\in\mathbb{N}}$ and to pass to the limit $n\uparrow\infty$ then.

According to (39) and due the energy estimate (43), we have for all $n \in \mathbb{N}$ and $\forall t \in \mathbb{R}_0^+$, the estimate

$$\int_\Omega (w(n)(t))^2 \gamma(n)\, dx + 2 \int_{Q_t} |\nabla_x w(n)|^2\, d(s,x) = \int_\Omega (w_0(n))^2 \gamma(n)\, dx \leqslant K_1 \in \mathbb{R}^+,$$

where

$$K_1 = 1 + \max_{n \in \mathbb{N}} \|w_0(n) \sqrt{\gamma(n)}\|^2_{L^2(\Omega)}.$$

We deduce

$$\forall n \in \mathbb{N} : \|w(n)\sqrt{\gamma(n)}\|_{L^\infty(\mathbb{R}_0^+; L^2(\Omega))} \leqslant K_1, \quad \|\nabla_x w(n)\|_{L^2(\tilde{Q})} \leqslant K_1,$$

where here and in the sequel

$$\tilde{Q} = \mathbb{R}^+ \times \Omega.$$

Furthermore, due to Poincaré's inequality there is $C_P \in \mathbb{R}^+$ such that

$$\forall v \in H_0^1(\Omega) : \int_\Omega v^2\, dx \leqslant C_P \int_\Omega |\nabla_x v|^2\, dx, \tag{46}$$

hence for each $n \in \mathbb{N}$, and for each $t \in \mathbb{R}_0^+$,

$$\begin{aligned}\int_{Q_t} (w(n))^2\, d(\tau,x) &= \int_{[0,t[} \left[\int_\Omega (w(n)(\tau))^2\, dx \right] d\tau \\ &\leqslant C_P \int_{[0,t[} \left[\int_\Omega |\nabla_x w(n)(\tau)|^2\, dx \right] d\tau \leqslant C_P \cdot K_1^2,\end{aligned}$$

and therefore

$$\forall n \in \mathbb{N} : \|w(n)\|^2_{L^2(\tilde{Q})} \leqslant C_P \cdot K_1^2.$$

We observe,

1. $L^\infty(0,\infty; L^2(\Omega))$ is isometrically isomorphic to the dual space of the separable space $L^1(0,\infty; L^2(\Omega))$, thus each sequence in $L^\infty(0,\infty; L^2(\Omega))$ which bounded in $L^\infty(0,\infty; L^2(\Omega))$ has a subsequence which converges weak* in $L^\infty(0,\infty; L^2(\Omega))$,
2. since $L^2(\tilde{Q} : \mathbb{R}^d)$ is a Hilbert space, each sequence in $L^2(\tilde{Q} : \mathbb{R}^d)$ which is bounded in $L^2(\tilde{Q} : \mathbb{R}^d)$ has a subsequence which converges weakly in $L^2(\tilde{Q} : \mathbb{R}^d)$,
3. and $L^2(\tilde{Q})$ is also a Hilbert space such that each sequence in $L^2(\tilde{Q})$ which is bounded in $L^2(\tilde{Q})$ has a subsequence which converges weakly in $L^2(\tilde{Q})$.

As a consequence, we can extract, if necessary, a subsequence of $(w(n))_{n \in \mathbb{N}}$ – but we do not change notations here – and there are v, V, u, such that

(C1) $v \in L^\infty(0,\infty; L^2(\Omega))$ and

$$w(n)\sqrt{\gamma(n)} \rightharpoonup v, \quad \text{weak}^* \text{ in } L^\infty(0,\infty; L^2(\Omega)),$$

(C2) $V \in L^2(\tilde{Q} : \mathbb{R}^d)$ and

$$\nabla_x w(n) \rightharpoonup V, \quad \text{weakly in } L^2(\tilde{Q} : \mathbb{R}^d),$$

(C3) $u \in L^2(\tilde{Q})$ and

$$w(n) \rightharpoonup u, \quad \text{weakly in } L^2(\tilde{Q}).$$

Certainly, via (C2) and (C3), $V = \nabla_x u$, and since for each $n \in \mathbb{N}$, $w(n) \in L^2(0, T; H_0^1(\Omega))$, we have

$$u \in L^2(0, \infty; H_0^1(\Omega)), \qquad w(n) \rightharpoonup u, \quad \text{weakly in } L^2(0, \infty; H_0^1(\Omega)).$$

Now let $\phi \in C_c^\infty(\mathbb{R}_0^+ \times \Omega)$. Since $\gamma(n) \to g$ in $L^1_{\mathrm{loc}}(\Omega)$ and since ϕ is compactly supported, we have

$$\phi\sqrt{\gamma(n)} \to \phi\sqrt{g}, \quad \text{strongly in } L^2(\tilde{Q}),$$

such that via (C1) and (C3),

$$\lim_{n\uparrow\infty} \int_{\tilde{Q}} w(n)\phi\sqrt{\gamma(n)}\, d(\tau, x) = \int_{\tilde{Q}} v\phi\, d(\tau, x) = \int_{\tilde{Q}} u\phi\sqrt{g}\, d(\tau, x),$$

and since this identity holds for all $\phi \in C_c^\infty(\mathbb{R}_0^+ \times \Omega)$, we deduce

$$v = u\sqrt{g}.$$

We summarize for later reference:

(D0) $u \in L^\infty(0, \infty, L^2(g\, dx)) \cap L^2(0, \infty, H_0^1(\Omega))$,
(D1) $w(n)\sqrt{\gamma(n)} \rightharpoonup u\sqrt{g}$, weak* in $L^\infty(0, \infty; L^2(\Omega))$,
(D2) $w(n) \rightharpoonup u$, weakly in $L^2(0, \infty; H_0^1(\Omega))$.

Next, we shall pass to the limit $n \uparrow \infty$ in the weak formulation of (40). Let $\eta \in C_c^\infty(\mathbb{R}_0^+ \times \Omega)$. Then we have for each $n \in \mathbb{N}$, and for almost all $t \in \mathbb{R}_0^+$,

$$\begin{aligned} 0 &= T_1(n,\eta)(t) + T_2(n,\eta)(t) - T_3(n,\eta) - T_4(n,\eta)(t) \\ &= \int_\Omega w(n)(t)\eta(t)\gamma(n)\, dx + \int_{Q_t} (\nabla_x w(n) \cdot \nabla_x \eta)\, d(\tau, x) \\ &\quad - \int_\Omega w_0(n)\eta(0)\gamma(n)\, dx - \int_{Q_t} w(n)\dot{\eta}\gamma(n)\, d(\tau, x). \end{aligned} \tag{47}$$

$T_1(n,\eta)(t)$: η is compactly supported in $\mathbb{R}_0^+ \times \Omega$ and $\gamma(n)$ converges in $L^1_{\mathrm{loc}}(\Omega)$ with limit g. Hence $\eta\sqrt{\gamma(n)}$ converges strongly in $L^1(0, \infty; L^2(\Omega))$ with limit $\eta\sqrt{g}$. Thus via (D1),

$$\lim_{n\uparrow\infty} \int_{\tilde{Q}} (w(n)\sqrt{\gamma(n)})(\eta\sqrt{\gamma(n)})\, d(\tau, x) = \int_{\tilde{Q}} u\sqrt{g}\eta\sqrt{g}\, d(\tau, x) = \int_{\tilde{Q}} u\eta g\, d(\tau, x),$$

and consequently for almost all $t \in \mathbb{R}_0^+$,

$$\lim_{n\uparrow\infty} T_1(n,\eta)(t) = \lim_{n\uparrow\infty} \int_\Omega w(n)(t)\eta(t)\gamma(n)\, dx = \int_\Omega u(t)\eta(t)g\, dx.$$

$T_2(n,\eta)(t)$: $\nabla_x w(n) \rightharpoonup \nabla_x u$ weakly in $L^2(\tilde{Q}:\mathbb{R}^d)$ (see (D2)). We deduce for almost all $t\in\mathbb{R}_0^+$,

$$\lim_{n\uparrow\infty} T_2(n,\eta)(t) = \lim_{n\uparrow\infty}\int_{Q_t}(\nabla_x w(n)\cdot\nabla_x\eta)\,d(\tau,x) = \int_{Q_t}(\nabla_x u\cdot\nabla_x\eta)\,d(\tau,x).$$

$T_3(n,\eta)$: We recall $w_0(n)\sqrt{\gamma(n)} \to u_0\sqrt{g}$ strongly in $L^2(\Omega)$ and $\gamma(n)\to g$ in $L^1_{\mathrm{loc}}(\Omega)$. Since $\eta(0)$ is compactly supported we deduce $\eta(0)\sqrt{\gamma(n)}\to\eta(0)\sqrt{g}$ strongly in $L^2(\Omega)$. Consequently,

$$\lim_{n\uparrow\infty} T_3(n,\eta)(t) = \lim_{n\uparrow\infty}\int_{\Omega} w_0(n)\eta(0)\gamma(n)\,dx = \int_{\Omega} u_0\eta(0)g\,dx.$$

$T_4(n,\eta)(t)$: $\dot{\eta}$ is compactly supported in $\mathbb{R}_0^+\times\Omega$ and $\gamma(n)$ converges in $L^1_{\mathrm{loc}}(\Omega)$ with limit g. Hence $\dot{\eta}\sqrt{\gamma(n)}$ converges strongly in $L^1(0,\infty;L^2(\Omega))$ with limit $\dot{\eta}\sqrt{g}$. Thus via (D1),

$$\lim_{n\uparrow\infty}\int_{\tilde{Q}}(w(n)\sqrt{\gamma(n)})(\dot{\eta}\sqrt{\gamma(n)})\,d(\tau,x) = \int_{\tilde{Q}} u\sqrt{g}\dot{\eta}\sqrt{g}\,d(\tau,x) = \int_{\tilde{Q}} u\dot{\eta}g\,d(\tau,x),$$

and consequently for almost all $t\in\mathbb{R}_0^+$,

$$\lim_{n\uparrow\infty} T_4(n,\eta)(t) = \int_{Q_t} u\dot{\eta}g\,d(\tau,x).$$

Putting the limits together, we deduce: u is a weak solution of (45).

Step 5. Conclusions (1), (2) and (3) of the theorem follow from the previous section's results.

Step 6. It remains to prove exponential decay of the entropy and the corresponding formula for the integrated (squared) gradient norm of u.

Concerning the decay of the entropy, we already know that the function $\|u(\cdot)\|_{L^2(g\,dx)}$ is continuous in time. Thus, E is a continuous function. The idea is to pass to the limit $n\uparrow\infty$ in (44). However, we can not directly pass to this limit in (44), because there is no result which concerns the pointwise (in time) behaviour of $\frac{1}{2}\int_\Omega (w(n)(t))^2\gamma(n)\,dx$ as $n\uparrow\infty$.

We define

$$C^* = \sup\left\{\int_\Omega v^2 g\,dx : v\in H_0^1(\Omega) \text{ and } \int_\Omega |\nabla_x v|^2\,dx = 1\right\}.$$

Then

$$\forall v\in H_0^1(\Omega):\int_\Omega v^2 g\,dx \leqslant C^*\int_\Omega |\nabla_x v|^2\,dx. \tag{48}$$

Step 6a. We shall prove $C^* = \lim_{n\uparrow\infty} C(n)$, where for $n\in\mathbb{N}$, $C(n)$ is as in (41). If $\Phi\in C_c^\infty(\Omega)$ with $\int_\Omega |\nabla_x\Phi|^2\,dx = 1$, then, since $\mathrm{supp}(\Phi)\subset\Omega$ is compact, via (38) for all $n\in\mathbb{N}$,

$$\begin{aligned}\int_\Omega \Phi^2 g\,dx &= \int_{\mathrm{supp}(\Phi)}\Phi^2\cdot(g-\gamma(n))\,dx + \int_{\mathrm{supp}(\Phi)}\Phi^2\gamma(n)\,dx\\ &\leqslant \|\Phi^2\|_{L^\infty(\Omega)}\cdot\left(\frac{|\Omega|}{n} + r(\mathrm{supp}(\Phi),n) + \frac{|\Omega|^{1-\frac{1}{s}}}{\sqrt[s]{n}}\right) + C(n),\end{aligned}$$

hence

$$\begin{aligned}\int_\Omega \Phi^2 g\,dx &\leqslant \liminf_{n\uparrow\infty}\left(\|\Phi^2\|_{L^\infty(\Omega)}\cdot\left(\frac{|\Omega|}{n}+r(\operatorname{supp}(\Phi),n)+\frac{|\Omega|^{1-\frac{1}{s}}}{\sqrt[s]{n}}\right)+C(n)\right)\\ &=\liminf_{n\uparrow\infty} C(n),\end{aligned}$$

such that due to the definition of C^*,

$$C^* \leqslant \liminf_{n\uparrow\infty} C(n). \tag{49}$$

On the other hand, if $\Phi\in C_c^\infty(\Omega)$ with $\int_\Omega |\nabla_x\Phi|^2\,dx=1$, then for all $n\in\mathbb{N}$, via (34), (36), via Poincaré's inequality (46) and via the optimized Hardy-Sobolev inequality (48) (thus, we implicitly apply (1) for the fifth time),

$$\begin{aligned}\int_\Omega \Phi^2\gamma(n)\,dx &= \int_{\operatorname{supp}(\Phi)}\Phi^2\gamma(n)\,dx\\ &= \int_{\operatorname{supp}(\Phi)}\Phi^2\cdot(\gamma(n)-\gamma^*(n))\,dx+\int_{\operatorname{supp}(\Phi)}\Phi^2\cdot(\gamma^*(n)-g)\,dx\\ &\quad+\int_{\operatorname{supp}(\Phi)}\Phi^2 g\,dx\\ &\leqslant C_0\|\nabla_x\Phi\|_{L^2(\Omega:\mathbb{R}^d)}\|\gamma(n)-\gamma^*(n)\|_{L^s(\Omega)}\\ &\quad+\int_{\operatorname{supp}(\Phi)}\Phi^2\cdot[\gamma^*(n)-g]^+dx+C^*\\ &\leqslant \frac{C_0}{\sqrt[s]{n}}+\frac{1}{n}\int_\Omega\Phi^2\,dx+C^*\leqslant\frac{C_0}{\sqrt[s]{n}}+\frac{C_P}{n}+C^*,\end{aligned}$$

such that due to the definition of $C(n)$,

$$\forall n\in\mathbb{N}: C(n)\leqslant\frac{C_0}{\sqrt[s]{n}}+\frac{C_P}{n}+C^*,$$

and therefore

$$\limsup_{n\uparrow\infty} C(n)\leqslant C^*,$$

and we deduce $\lim_{n\uparrow\infty} C(n)=C^*$ via (49).

Step 6b. As shown in Step 4 we have $w(n)\sqrt{\gamma(n)}\rightharpoonup u\sqrt{g}$ weak* in $L^\infty(0,\infty;L^2(\Omega))$. Consequently, $w(n)\sqrt{\gamma(n)}\rightharpoonup u\sqrt{g}$ weakly in $L^2(0,\infty;L^2(\Omega))$ and we deduce

for all $t_1,t_2\in\mathbb{R}^+$: if $t_1\leqslant t_2$, then

$$\int_{t_1}^{t_2}E(\tau)\,d\tau=\frac{1}{2}\int_{[t_1,t_2]\times\Omega}u^2 g\,dx\leqslant\liminf_{n\uparrow\infty}\frac{1}{2}\int_{[t_1,t_2]\times\Omega}(w(n)(t))^2\gamma(n)\,dx. \tag{50}$$

Step 6c. Since $E\in C(\mathbb{R}_0^+)$, each $t\in\mathbb{R}_0^+$ is a Lebesgue point of E. In particular,

$$\forall t\in\mathbb{R}^+: E(t)=\lim_{h\downarrow 0}\frac{1}{2h}\int_{t-h}^{t+h}E(\tau)\,d\tau. \tag{51}$$

We deduce from (50), (51) via (44),

$$\forall (t,h) \in \mathbb{R}^+ \times \mathbb{R}^+ : \text{If } 0 < t-h, \text{ then}$$

$$\begin{aligned} \frac{1}{2h}\int_{t-h}^{t+h} E(\tau)\,d\tau &\leqslant \liminf_{n\uparrow\infty} \frac{1}{4h}\int_{[t-h,t+h]\times\Omega} (w(n)(t))^2\gamma(n)\,dx \\ &\leqslant \liminf_{n\uparrow\infty} \frac{1}{2h}\int_{t-h}^{t+h} E(n)(0)e^{-2\tau/C(n)}\,d\tau \\ &= \liminf_{n\uparrow\infty}\left(e^{-2t/C(n)}E(n)(0)\cdot\frac{\sinh(2h/C(n))}{2h/C(n)}\right), \end{aligned} \tag{52}$$

where we put for the sake of brevity

$$\forall n \in \mathbb{N} : E(n)(0) = \frac{1}{2}\int_\Omega (w_0(n))^2\gamma(n)\,dx.$$

According to (39) we have

$$\lim_{n\uparrow\infty} E(n)(0) = E(0),$$

and according to Step 6a we have $\lim_{n\uparrow\infty} C(n) = C^*$. Therefore we can easily carry out the limit $n \uparrow \infty$ in (52) to obtain

$$\forall (t,h) \in \mathbb{R}^+ \times \mathbb{R}^+ : \text{If } 0 < t-h, \text{ then}$$

$$\begin{aligned} \frac{1}{2h}\int_{t-h}^{t+h} E(\tau)\,d\tau &\leqslant \liminf_{n\uparrow\infty} \frac{1}{4h}\int_{[t-h,t+h]\times\Omega} (w(n)(t))^2\gamma(n)\,dx \\ &\leqslant E(0)e^{-2t/C^*}\cdot\frac{\sinh(2h/C^*)}{2h/C^*}. \end{aligned} \tag{53}$$

Passing to the limit $h \downarrow 0$ in (53) we obtain via (51),

$$\forall t \in \mathbb{R}^+ : E(t) \leqslant E(0)e^{-2t/C^*},$$

which is due to $C^* \leqslant C$ an estimate not worse than stated in the theorem.

Step 7. For all t, $T \in \mathbb{R}^+$ with $t \leqslant T$, due to (22),

$$\int_t^T \|\nabla_x u(\tau)\|^2_{L^2(\Omega:\mathbb{R}^d)}\,d\tau = E(t) - E(T),$$

such that via $\lim_{T\uparrow\infty} E(T) = 0$,

$$\int_t^\infty \|\nabla_x u(\tau)\|^2_{L^2(\Omega:\mathbb{R}^d)}\,d\tau = E(t) \leqslant E(0)e^{-2C/t}.$$

□

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