POSITIVE SOLUTIONS FOR A CLASS OF INFINITE SEMIPOSITONE PROBLEMS

MYTHILY RAMASWAMY
TATA Institute of Fundamental Research, IISC Campus
Bangalore - 560012, India

R. SHIVAJI AND JINGLONG YE
Department of Mathematics and Statistics, Mississippi State University
Mississippi State, MS 39762

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Abstract. We analyze the positive solutions to the singular boundary value problem

\[-\Delta u = \lambda [f(u) - 1/u^\alpha]; x \in \Omega,\]
\[u = 0; x \in \partial \Omega,\]

where \(f\) is a \(C^2\) function in \((0, \infty), f(0) \geq 0, f' > 0, \lim_{s \to \infty} \frac{f(s)}{s} = 0, \lambda\)

is a positive parameter, \(\alpha \in (0, 1)\) and \(\Omega\) is a bounded region in \(\mathbb{R}^n, n \geq 1\) with \(C^{2+\gamma}\) boundary for some \(\gamma \in (0, 1)\). In the case \(n = 1\) we use the quadrature method and for \(n > 1\) we use the method of sub-super solution to establish our results.

1. INTRODUCTION

We consider

\[-\Delta u = \lambda [f(u) - 1/u^\alpha]; x \in \Omega,\]
\[u = 0; x \in \partial \Omega,\]

(1.1)

where \(\alpha \in (0, 1), \lambda\) is a positive parameter, and \(\Omega\) is a bounded region in \(\mathbb{R}^n, n \geq 1\) with \(C^{2+\gamma}\) boundary for some \(\gamma \in (0, 1)\). Throughout this paper we assume:

(H) \(f \in C^2(0, \infty), f'(s) > 0; s > 0\) and \(\lim_{s \to \infty} \frac{f(s)}{s} = 0.\)

Let \(g(u) := f(u) - 1/u^\alpha.\) Then \(\lim_{u \to 0} g(u) = -\infty,\) and hence (1.1) is a singular boundary value problem. In recent years, there is a rich history of research for the case when \(g(0) < 0\) but finite (see [1-12, 21]). Such non-singular problems are referred to as semipositone problems. It is well
known in the literature that the study of positive solutions to such semipositone problems are mathematically very challenging (see [5, 19]). In this paper, we consider the even more challenging semipositone problem when $\lim_{u \to 0} g(u) = -\infty$, which we refer to as an infinite semipositone problem. We will seek positive solutions in $C^2(\Omega) \cap C(\overline{\Omega})$. Our aim in this paper is to establish non existence of positive solution for $\lambda$ near zero and existence of positive solution for $\lambda$ large. In the case when $n = 1$ we will also discuss multiplicity and uniqueness results. In particular, in Section 2 we study

$$-u'' = \lambda[f(u) - 1/u^\alpha]; x \in (0,1)$$

$$u(0) = 0 = u(1),$$

(1.2)

with $f(0) > 0$ and establish the following result:

**Theorem 1.1.** Assume hypothesis (H) holds, $f(0) > 0, f''(s) < 0$, and $f(\theta) \leq \theta f'(\theta) + \frac{1+\alpha}{\theta^\beta}$, where $\theta$ is the positive real zero of $F(s) := \int_0^s [f(z) - 1/z^\alpha]dz$. Then there exists positive constants $\mu_i, i = 1,2,3$ such that $\mu_1 < \mu_2 < \mu_3$ and the BVP (1.2) has no positive solution for $\lambda < \mu_1$, has at least one positive solution for $\lambda \geq \mu_1$, has at least two positive solutions for $\mu_1 < \lambda \leq \mu_2$ and a unique positive solution for $\lambda > \mu_3$.

Next, in Section 3, we study (1.1) in the case when $n \geq 1$ and prove our main Theorems. Namely, we establish the following results:

**Theorem 1.2.** Assume hypothesis (H) holds and $f(0) > 0$. Then there exists positive constants $\mu_1, \mu_2$ such that $\mu_1 < \mu_2$ and the BVP (1.1) has no positive solution for $\lambda < \mu_1$ and has at least one positive solution for $\lambda > \mu_2$.

**Theorem 1.3.** Assume hypothesis (H) holds, $f(0) = 0$ and $\lim_{s \to 0} \frac{f(s)}{s^\beta} = k > 0 (0 < \beta \leq 1)$. Then there exists positive constants $\mu_1, \mu_2$ such that $\mu_1 < \mu_2$ and the BVP (1.1) has no positive solution for $\lambda < \mu_1$ and has at least one positive solution for $\lambda > \mu_2$.

Finally, in Section 4, we discuss some examples satisfying our hypothesis.

In [22], the author considers the boundary value problem

$$-\Delta u = \lambda u^p - 1/u^\alpha; x \in \Omega$$

$$u = 0; x \in \partial \Omega.$$

Note that in their equation the parameter $\lambda$ multiplies the term $u^p$ and not the singular term $1/u^\alpha$. See also [16] for some extensions of the results in [22]. Here we extend this study to the more challenging problem (1.1) where we deal with a class of non-linear $f(u)$ rather than the specific $u^p$ non-linear,
and the parameter $\lambda$ also multiplying the singular term $1/u^\alpha$. Note that this more difficult case was not treated in [22] and [16].

We prove Theorem 1.1 via the quadrature method developed in [7, 12, 17]. See also [13] where exact multiplicity results for such problems in the case $n = 1$ was recently discussed under more restrictive conditions. We prove our main results (Theorem 1.2 and Theorem 1.3) by the method of sub-super solution discussed in [14]. Here a subsolution is a function $u : \Omega \to \mathbb{R}$ such that

$$-\Delta u \leq \lambda [f(u) - 1/u^\alpha]; \quad x \in \Omega$$

$$u > 0; \quad x \in \Omega$$

$$u = 0; \quad x \in \partial \Omega,$$

and a supersolution is a function $\bar{u} : \bar{\Omega} \to \mathbb{R}$ such that $\bar{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ and

$$-\Delta \bar{u} \geq \lambda [f(\bar{u}) - 1/\bar{u}^\alpha]; \quad x \in \Omega$$

$$\bar{u} > 0; \quad x \in \Omega$$

$$\bar{u} = 0; \quad x \in \partial \Omega.$$

Then the following results hold:

**Lemma 1.1.** (See Lemma 3 in [14]) For $\lambda > 0$, if there exist a subsolution $u_\lambda$ and a supersolution $\bar{u}_\lambda$ of (1.1) such that $u_\lambda \leq \bar{u}_\lambda$ on $\overline{\Omega}$, then (1.1) has at least one solution $u_\lambda \in C^{2+\gamma}(\Omega) \cap C(\overline{\Omega})$ satisfying $u_\lambda \leq \bar{u}_\lambda \leq \lambda^{\frac{1}{2+\alpha}}$.

**Lemma 1.2.** (See [18]) Let $\phi$ denote an eigenfunction corresponding to the first eigenvalue of the operator $-\Delta$ with Dirichlet boundary conditions, then $\int_\Omega (1/\phi)^s dx < +\infty$ if and only if $s < 1$.

**Lemma 1.3.** (See [15]) Let $g \in L^1(\Omega)$, $g \geq 0$ and $f \in L^\infty(\Omega)$. Let $u \geq 0$ be a solution of

$$-\Delta u + g = f; \quad x \in \Omega$$

$$u = 0; \quad x \in \partial \Omega,$$

then $u \in H^1_0(\Omega)$.

We prove Theorems 1.2-1.3 by establishing a crucial subsolution of the form $\lambda^{\frac{1}{2+\alpha}} \phi^\frac{2}{2+\alpha}$ and applying Lemma 1.1. Hence, our solution $u$ will be in the space $C^{2+\gamma}(\Omega) \cap C(\overline{\Omega})$ and satisfies $u \geq \lambda^{\frac{1}{2+\alpha}} \phi^\frac{2}{2+\alpha}$. Here $\phi > 0$; $\Omega$ is an eigenfunction corresponding to the first eigenvalue of the operator $-\Delta$ with Dirichlet boundary conditions. By Lemma 1.2, $1/\phi^\frac{2}{2+\alpha} \in L^1(\Omega)$ since $\frac{2\alpha}{1+\alpha} < $
1 when $\alpha < 1$. Thus, $\frac{\lambda}{u^\alpha} \in L^1(\Omega)$ and by Lemma 1.3 we also have that $u \in H_0^1(\Omega)$. Hence, our classical solution is indeed in a much better space, namely $C^{2+\gamma}(\Omega) \cap C(\Omega) \cap H_0^1(\Omega)$. The non existence result when $\lambda$ small follows easily by using a linear upper bound for $g(u)$.

We note that our results for these infinite semipositone problems (with nonlinearities $f$ that are sublinear at $\infty$) resemble results for the corresponding non-singular semipositone problems (See [7, 9]).

2. Proof Of Theorem 1.1

We first recall some of the results of the Quadrature method introduced by Laetsch in [17] and extended to semipositone problems in [7]. See also [20] where this quadrature method was used to study classes of singular positone problems.

If (1.2) has a positive solution $u$, then $u$ increases in $[0,1/2)$ and is symmetric about $x = 1/2$. Multiplying (1.2) by $u'(x)$ and integrating we obtain

$$\frac{[u'(x)]^2}{2} = \lambda[F(\rho) - F(u(x))], \quad (2.1)$$

where $F(s) = \int_0^s [f(z) - 1/z^\alpha]dz$ and $\rho = u(1/2) = \|u\|_\infty$. Further, integrating (2.1), $u(x)$ is determined by the equation

$$\int_0^{u(x)} \frac{1}{\sqrt{F(\rho) - F(s)}} ds = \sqrt{2\lambda} x; \quad 0 \leq x \leq 1/2. \quad (2.2)$$

Setting $x = 1/2$, $\lambda$ and $\rho$ must satisfy

$$\sqrt{\lambda} = \sqrt{2} \int_0^\rho \frac{1}{\sqrt{F(\rho) - F(s)}} ds := G(\rho). \quad (2.3)$$

Note that $G(\rho)$ is well defined for $\rho \in [\theta, \infty)$ where $\theta$ is the positive zero of $F$. This follows from the fact that $F'(\rho) > 0$ for $\rho \in [\theta, \infty)$. Also clearly $G(\rho) > 0$. Further, since $F(\rho) < 0$ for $0 < \rho < \theta$, by (2.1) there are no positive solutions with $\|u\|_\infty = \rho < \theta$. On the other hand if $\sqrt{\lambda} = G(\rho)$ for some $\rho \in [\theta, \infty)$, then (2.2) defines the positive solution of (1.2) with $u(1/2) = \rho$. Therefore, (2.3) describes the bifurcation diagram for positive solutions of (1.2). Further, rewriting $G(\rho)$ as

$$G(\rho) = \sqrt{2} \rho \int_0^1 \frac{1}{\sqrt{F(\rho) - F(\rho v)}} dv,$$
it follows that
\[ G'(\rho) = \sqrt{2} \int_0^1 \frac{H(\rho) - H(\rho v)}{[F(\rho) - F(\rho v)]^{3/2}} dv, \]
where \( H(s) = F(s) - \frac{1}{2}s[f(s) - \frac{1}{4v}] \) (see [7]). Thus, \( G'(\rho) > 0 \) if \( H(\rho) > H(s); 0 \leq s < \rho \) and \( G'(\rho) < 0 \) if \( H(\rho) < H(s); 0 \leq s < \rho \).

In order to prove Theorem 1.1, we first analyze the properties of the function \( H \). In fact, \( H'(s) = \frac{1}{2}[f(s) - s f'(s) - \frac{1+\alpha}{s^\alpha}] \) and \( H''(s) = -\frac{1}{2} f''(s) + \frac{1}{2} \frac{\alpha(1+\alpha)}{s^\alpha + 1} \). Hence, \( H''(s) > 0; s > 0, \lim_{s \to 0^+} H'(s) = -\infty \) and
\[ \lim_{s \to \infty} H'(s) = \lim_{s \to \infty} \frac{s}{2} \left[ \frac{f(s) - f(0)}{s} - f'(s) \right] + \frac{1}{2} f(0) \geq \frac{1}{2} f(0) > 0 \text{ (since } f'' < 0). \]

Also \( H'(\theta) = \frac{1}{2} [f(\theta) - \theta f'(\theta) - \frac{1+\alpha}{\theta^\alpha}] \leq 0 \). Thus, \( H \) has the described shape in Figure 1. Here \( \gamma \) is the positive zero of \( H \). Now Theorem 1.1 follows easily by proving the following (see Figure 2):

(a) \( G'(\theta) < 0 \), (b) \( \lim_{\rho \to \infty} G(\rho) = \infty \) and (c) \( G'(\rho) > 0 \) if \( \rho \) is large.

In fact, if (a), (b) and (c) are proven then \( \mu_1 \) in Theorem 1.1 turn out to be \( \mu_1 = (\min\{G(\rho) : \rho \in [\theta, \gamma]\})^2 \). Now from the properties of \( H \) and the expression for \( G'(\rho) \) it is easy to see that \( G'(\theta) < 0 \) and \( G'(\rho) > 0; \rho \geq \gamma \). Hence (a) and (c) are proven. Next note that
\[ F(\rho v) = \int_0^{\rho v} [f(s) - \frac{1}{s^\alpha}] ds \geq [f(0) - \frac{1}{1-\alpha} (\rho v)^\alpha] \rho v. \]
≥ \frac{1}{2} f(0)(\rho v) \text{ if } v > [\frac{2}{(1-\alpha)f(0)}]^{1/\alpha} \frac{1}{\rho} = \beta \text{ (say).}

Thus, for $\rho$ large

\[ G(\rho) \geq \sqrt{2} \frac{\rho}{\sqrt{F(\rho)}} \int_{1}^{\infty} \frac{dw}{1 - \frac{f(0)}{2F(\rho)} \rho w} \geq 2\sqrt{2} \frac{\sqrt{F(\rho)}}{f(0)} \int_{\frac{f(0)\rho}{2F(\rho)}}^{\infty} \frac{dw}{\sqrt{1 - w}} \]

\[ = 4\sqrt{2} \left[ \frac{\sqrt{F(\rho)}}{f(0)} \right] \left[ \sqrt{1 - \frac{f(0)\rho\beta}{2F(\rho)}} - \sqrt{1 - \frac{f(0)\rho}{2F(\rho)}} \right] \]

\[ = 2\sqrt{2} \frac{\rho}{\sqrt{F(\rho)}} (1 - \beta) \frac{1}{\sqrt{1 - \frac{f(0)\rho\beta}{2F(\rho)}} + \sqrt{1 - \frac{f(0)\rho}{2F(\rho)}}} \]

But $\lim_{\rho \to \infty} \frac{\rho^2}{F(\rho)} = \lim_{\rho \to \infty} \frac{2\rho}{f(\rho) - \frac{1}{x^2}} = \infty$. Hence, $\lim_{\rho \to \infty} G(\rho) = \infty$ and (b) and Theorem 1.1 is proven.

3. Proof Of Theorem 1.2 and Theorem 1.3

3.1. Proof of Theorem 1.2. Since $\lim_{s \to \infty} \frac{f(s)}{s} = 0$, there exists constants $a > 0, b > 0$ such that $f(s) - \frac{1}{s^2} < as - b$. Let $\lambda_1 > 0$ be the principle eigenvalue and $\phi > 0$; $\Omega$ be a corresponding eigenfunction of $-\Delta$ with Dirichlet boundary conditions. Suppose $u > 0$; $\Omega$ is a positive solution of (1.1). Then

\[ \int_{\Omega} (-\Delta u) \phi dx \leq \lambda \int_{\Omega} (au - b) \phi dx. \]
But
\[ \int_\Omega (-\Delta u)\phi \, dx = \int_\Omega u(-\Delta \phi) \, dx = \int_\Omega \lambda_1 u \phi \, dx. \]
Thus
\[ \int_\Omega (\lambda_1 - \lambda a) u \phi \, dx \leq \int_\Omega (-\lambda b) \phi \, dx. \]
This is impossible if \( \lambda < \lambda_1/a \) and the first part of Theorem 1.2 is proven.

Next choose the eigenfunction \( \phi > 0 \) such that \( \| \phi \|_\infty = 1 \) and let \( \psi := \lambda^r \phi^{2/(1+\alpha)} \) where the parameter \( r \in (1/(1+\alpha), 1) \). Then
\[ \nabla \psi = \lambda^r \left( \frac{2}{1+\alpha} \right) \phi^{1/(1+\alpha)} \nabla \phi \]
and
\[ \Delta \psi = \lambda^r \left( \frac{2}{1+\alpha} \right) \left\{ \phi^{2/(1+\alpha)} \Delta \phi + \frac{1-\alpha}{1+\alpha} \phi^{1/(1+\alpha)} \ | \nabla \phi|^2 \right\} \]
\[ = \lambda^r \left( \frac{2}{1+\alpha} \right) \left\{ -\lambda_1 \phi^{2/(1+\alpha)} + \frac{1-\alpha}{1+\alpha} \frac{|\nabla \phi|^2}{\phi^{2/(1+\alpha)}} \right\}. \]
Thus,
\[ -\Delta \psi = \lambda^r \left( \frac{2}{1+\alpha} \right) \left\{ \lambda_1 \phi^{2/(1+\alpha)} - \frac{1-\alpha}{1+\alpha} \frac{|\nabla \phi|^2}{\phi^{2/(1+\alpha)}} \right\}. \]

Let \( \delta > 0, \mu > 0, m > 0 \) be such that \( |\nabla \phi|^2 \geq m \), in \( \overline{\Omega_\delta} \), and \( \phi^{2/(1+\alpha)} \in [\mu, 1] \) in \( \Omega - \overline{\Omega_\delta} \) where \( \Omega_\delta := \{ x \in \Omega \mid d(x, \partial \Omega) \leq \delta \} \). This is possible since \( |\nabla \phi| \neq 0 \); \( \partial \Omega \). Then in \( \overline{\Omega_\delta} \) if \( \lambda \gg 1 \)
\[ -\lambda^r \left( \frac{2}{1+\alpha} \right) \frac{1-\alpha}{1+\alpha} \frac{|\nabla \phi|^2}{\phi^{2/(1+\alpha)}} \leq \lambda \left[ -\frac{1}{\left( \lambda^r \phi^{2/(1+\alpha)} \right)^\alpha} \right] \]
since \( 1 - r - r\alpha < 0 \). Also in \( \overline{\Omega_\delta} \) (in fact in \( \Omega \)),
\[ \lambda^r \left( \frac{2}{1+\alpha} \right) \lambda_1 \phi^{2/(1+\alpha)} \leq \lambda f(0) \leq \lambda f(\lambda^r \phi^{2/(1+\alpha)}) \text{ if } \lambda \gg 1. \]

Hence, in \( \overline{\Omega_\delta} \),
\[ -\Delta \psi \leq \lambda f(\lambda^r \phi^{2/(1+\alpha)}) - \frac{1}{\left( \lambda^r \phi^{2/(1+\alpha)} \right)^\alpha} = \lambda \left[ f(\psi) - \frac{1}{\psi^\alpha} \right]. \quad (3.1) \]
Next in \( \Omega - \overline{\Omega_\delta} \), since \( \phi^{2/(1+\alpha)} \geq \mu, \lambda \left[ f(\psi) - \frac{1}{\psi^\alpha} \right] \geq \lambda \left[ f(\lambda^r \mu) - \frac{1}{(\lambda^r \mu)^\alpha} \right] \). But if \( \lambda \gg 1, -\Delta \psi \leq \lambda^r \left( \frac{2}{1+\alpha} \right) \lambda_1 \leq \lambda \left[ f(\lambda^r \mu) - \frac{1}{(\lambda^r \mu)^\alpha} \right] \), since \( r < 1 \). Hence, if \( \lambda \gg 1 \),
in \( \Omega - \Omega_\delta \), we have

\[
-\Delta \psi \leq \lambda [f(\psi) - \frac{1}{\psi^{\alpha}}].
\]  (3.2)

Combining (3.1) and (3.2) we see that \( \psi = \lambda^* \phi^{\frac{2}{1+\alpha}} \) is a positive subsolution of (1.1).

Now we construct a supersolution \( Z \geq \psi \).

Since \( \lim_{s \to \infty} \frac{f(s)}{s} = 0, \forall \lambda > 0, \exists m(\lambda) > 0 \) such that \( m(\lambda) \geq \lambda f(m(\lambda)) \parallel e \parallel_\infty \) where \( e \) is the unique positive solution of \( -\Delta e = 1; x \in \Omega, \ e = 0; x \in \partial \Omega \).

Let \( Z := m(\lambda)e \). Then

\[
-\Delta Z = m(\lambda) \geq \lambda f(m(\lambda) \parallel e \parallel_\infty) \geq \lambda f(m(\lambda)e) = \lambda f(Z)
\]

Thus \( Z \) is a supersolution. Further, \( m(\lambda) \) can be chosen large enough so that \( Z = m(\lambda)e \geq \psi \) in \( \Omega \). Hence for \( \lambda \gg 1 \), (1.1) has a positive solution \( u \in [\psi, Z] \) and the second part of Theorem 1.2 is proven.

3.2. Proof of Theorem 1.3. Let \( \psi = \lambda^* \phi^{\frac{2}{1+\alpha}} \) as earlier. In the proof of Theorem 1.2, when proving \( \psi \) is a subsolution of (1.1) for \( \lambda \gg 1 \), we used the fact that \( f(0) > 0 \) to show \( \lambda^* (\frac{2}{1+\alpha}) \lambda_1 \phi^{\frac{2}{1+\alpha}} \leq \lambda f(\lambda^* \phi^{\frac{2}{1+\alpha}}) \) in \( \Omega_\delta \). Here with \( f(0) = 0 \) we establish the above inequality by using \( \lim_{s \to 0} \frac{f(s)}{s} = k > 0 \).

(The rest of the proof of Theorem 1.3 is exactly as the proof of Theorem 1.2.) Let \( A > 0 \) be such that \( f(x) \geq \frac{k}{2} x^\beta \) for \( x \in [0, A] \). Choose \( \lambda^* \gg 1 \) such that

\[
\lambda^* (\frac{2}{1+\alpha}) \lambda_1 \phi^{\frac{2}{1+\alpha}} \leq \lambda \{ \frac{k}{2} \lambda^* \phi^{\frac{2}{1+\alpha}} \} \quad \forall \lambda \geq \lambda^*.
\]

Hence, if \( (\lambda^*)^* \phi^{\frac{2}{1+\alpha}} \leq A \), then

\[
f((\lambda^*)^* \phi^{\frac{2}{1+\alpha}}) \geq (\lambda^*)^{r-1} (\frac{2}{1+\alpha}) \lambda_1 \phi^{\frac{2}{1+\alpha}}.
\]

Further, for \( \lambda \geq \lambda^* \) we have

\[
f(\lambda^* \phi^{\frac{2}{1+\alpha}}) \geq f((\lambda^*)^* \phi^{\frac{2}{1+\alpha}}) \geq (\lambda^*)^{r-1} (\frac{2}{1+\alpha}) \lambda_1 \phi^{\frac{2}{1+\alpha}}
\]

\[
\geq \lambda^{r-1} (\frac{2}{1+\alpha}) \lambda_1 \phi^{\frac{2}{1+\alpha}}.
\]

Multiplying by \( \lambda \) we get

\[\lambda f(\psi) \geq \lambda^r (\frac{2}{1+\alpha}) \lambda_1 \phi^{\frac{2}{1+\alpha}}.\]
Next if \((\lambda^*)\phi^{\frac{2}{1+\alpha}} > A\), then \(\forall \lambda(\geq \lambda^*)\) and sufficiently large we have
\[
\lambda f(\psi) \geq \lambda f(A) \geq \lambda^*(\frac{2}{1 + \alpha})\lambda_1^{\phi^{\frac{2}{1+\alpha}}}.
\]
Hence, here for \(\lambda \gg 1\)
\[
\lambda f(\psi) \geq \lambda^*(\frac{2}{1 + \alpha})\lambda_1^{\phi^{\frac{2}{1+\alpha}}}
\]
must hold in \(\Omega_0\) (in fact throughout \(\Omega\)). Thus Theorem 1.3 is proven.

4. SOME EXAMPLES

In this section we discuss the following examples:

(A) \(f(u) = (u + 1)^{\frac{1}{2}}\),
(B) \(f(u) = e^{\frac{u}{u+1}}\),
(C) \(f(u) = u^{\frac{1}{2}}\).

4.1. Example (A): \(f(u) = (u + 1)^{\frac{1}{2}}\). Here \(f(0) > 0, f'(u) = \frac{1}{2\sqrt{u+1}} > 0\)
and \(\lim_{u \to \infty} \frac{f(u)}{u} = 0\). Hence, Theorem 1.2 holds \(\forall \alpha \in (0, 1)\). Next for \(\alpha = \frac{1}{8}\)
we estimate the root \(\theta\) for \(F(\theta) = 0\). Here,
\[
F(s) = \int_0^s (\sqrt{u+1} - \frac{1}{u^{0.125}})du = \frac{2}{3}(s+1)^{\frac{3}{2}} - \frac{2}{3} - \frac{8\sqrt{7}}{7}s^{\frac{7}{8}},
\]
and \(F(1) = \frac{4}{3}\sqrt{2} - \frac{2}{3} - \frac{s}{7} > 0\). But \(F'' = f'' + \frac{\alpha}{s^{\alpha + 1}} > 0\). Hence, \(\theta < 1\). Let
\(T(s) = f(s) - sf'(s) - \frac{1}{u^{0.125}}\). Then \(T'(s) = -sf''(s) + \frac{1.140625}{s^{1.125}} > 0\), and
\(T(1) = \sqrt{2} - \frac{1}{2\sqrt{2}} - 1.125 < 0\). So \(T(\theta) < 0\). Thus, \(f(\theta) < \theta f'(\theta) + \frac{1+\alpha}{\theta^{\alpha}}\). Also
\(f''(u) = -\frac{1}{4(u+1)} < 0\). Therefore, this example satisfies all the hypotheses
in Theorem 1.1 when \(\alpha = \frac{1}{8}\).

4.2. Example (B): \(f(u) = e^{\frac{u}{u+1}}\). Here, \(f(0) > 0, f'(u) = e^{\frac{u}{u+1}}\frac{1}{(u+1)^2} > 0\)
and \(\lim_{u \to \infty} \frac{f(u)}{u} = 0\). Hence, Theorem 1.2 holds \(\forall \alpha \in (0, 1)\). Next for \(\alpha = 0.3\)
we estimate the root \(\theta\) for \(F(\theta) = 0\). Here,
\[
F(s) = \int_0^{s} (e^{\frac{u}{u+1}} - \frac{1}{u^{0.3}})du
\geq \int_0^{s} [1 + \frac{u}{u+1} + \frac{1}{2} \frac{u^2}{(u+1)^2} + \frac{1}{6} \frac{u^3}{(u+1)^3}]du - \frac{10}{7}s^{0.7}
\geq \int_0^{s} [2 - \frac{1}{u+1} + \frac{1}{2} \frac{u^2}{u^2} + \frac{1}{2} \frac{u^2}{(u+1)^2} + \frac{1}{6} \frac{u^3}{(u+1)^3}]du - \frac{10}{7}s^{0.7}
\]
\[ F(1.09) > 2.725 - 1.4744 + 0.5 - 0.2393 + \frac{1}{6} \int_0^s \frac{u^3}{(u+1)^3} \, du - \frac{10}{7} s^{0.7}. \]

Hence,
\[ F(1.09) > 1.5113 + 0.0064 - 1.5175 = 0.0002 > 0. \]

So \( \theta < 1.09 \). Let \( T(s) = f(s) - s f'(s) - \frac{1+0.3}{s^{0.7}} \), then \( T'(s) = -s f''(s) + \frac{0.3}{s^{1.7}} > 0 \), and \( T(1.09) = e^{1.09} - 1.09 e^{1.09} (\frac{1}{1.09})^2 - \frac{1.3}{1.09^{1.7}} < 1.6864 - 0.4204 < 1.2669 = -0.0027 < 0 \). So \( T(\theta) < 0 \). Thus \( f(\theta) < \theta f'(\theta) + \frac{1+0.3}{\theta^{0.7}} \). Also \( f''(u) = e^{u^{1.7}} \left( \frac{1}{(u+1)^3} - \frac{2}{(u+1)^2} \right) = e^{u^{1.7}} \left( \frac{1}{(u+1)^3} - 2u - 1 \right) < 0 \), for \( u > 0 \). Therefore this example satisfies all hypotheses in Theorem 1.1 when \( \alpha = 0.3 \).

4.3. **Example (C):** \( f(u) = u^{1.7} \). Here, \( f(0) = 0, f'(u) = \frac{1}{2u^{0.3}} > 0 \) and \( \lim_{s \to \infty} \frac{f(s)}{s} = 0 \). Also since \( f(s) = s^{1.7} = 1 \), choose \( \beta = 1/2 \) all the hypotheses of Theorem 1.3 are satisfied for all \( \alpha \in (0, 1) \).

**References**