Multiple Positive Solutions for Classes of p-Laplacian Equations

Mythily Ramaswamy * and Ratnasingham Shivaji †

Abstract

We study positive $C^1(\Omega)$ solutions to classes of boundary value problems of the form

$$\begin{align*}
-\Delta_p u &= \lambda f(u) \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega
\end{align*}$$

where $\Delta_p$ denotes the p-Laplacian operator defined by $\Delta_p z := \text{div}(|\nabla z|^{p-2}\nabla z)$; $p > 1$, $\lambda > 0$ is a parameter and $\Omega$ is a bounded domain in $\mathbb{R}^N$; $N \geq 2$ with $\partial \Omega$ of class $C^2$ and connected. (If $N = 1$, we assume that $\Omega$ is a bounded open interval.) In particular, we establish existence of three positive solutions for classes of nondecreasing, p-sublinear functions $f$ belonging to $C^1([0, \infty))$. Our proofs are based on sub-super solution techniques.

1 Introduction

We consider weak solutions to classes of boundary value problems of the form

$$\begin{align*}
-\Delta_p u &= \lambda f(u) \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega
\end{align*}$$  \hspace{1cm} (1.1)$$

---

1AMS subject classification 35J70, 35J55

* TATA Institute for Fundamental Research Centre, IISc Campus, Bangalore - 560012, India, e-mail: mythily@math.tifrbng.res.in

† Department of Mathematics and Statistics, Mississippi State University, Mississippi State, MS 39762, USA, e-mail: shivaji@ra.msstate.edu
where $\Delta_p$ denotes the p-Laplacian operator defined by $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$; $p > 1$, $\lambda$ is a positive parameter and $\Omega$ is a bounded domain in $\mathbb{R}^N$; $N \geq 2$ with $\partial \Omega$ in class $C^2$ and connected. (If $N = 1$, we assume that $\Omega$ is a bounded open interval.) By a weak solution of (1.1), we mean, a function $u \in W^{1,p}_0(\Omega)$ that satisfies

$$\int_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla \omega = \int_{\Omega} \lambda f(u) \omega, \quad \forall \omega \in C^\infty_0(\Omega).$$

However, in this paper, we in fact study the existence and multiplicity of $C^1(\bar{\Omega})$ solutions, that are strictly positive in $\Omega$. Throughout this paper our classes of functions $f$ satisfy:

(A1) $f \in C^1([0, \infty))$ is a nondecreasing function such that $f(0) > 0$ and $\lim_{v \to \infty} \frac{f(v)}{v^{p-1}} = 0$ ($p$-sublinear).

For such positone $p$-sublinear nonlinearities, it is easy to establish that there is a positive solution for every $\lambda > 0$. Further when $\frac{v^{p-1}}{f(v)}$ is nondecreasing, uniqueness of the positive solution for every $\lambda$ follows from [DS]. In this paper we will consider the case when $\frac{v^{p-1}}{f(v)}$ is not monotonic. In particular, we consider $f$ for which there exists $a$ and $b$ such that $0 < a < b$ and

$$Q(a, b) := \frac{(a^{p-1})}{f(a)} / \frac{(b^{p-1})}{f(b)}$$

is sufficiently large. For such classes of nonlinearities we discuss the existence of three positive solutions for a certain range of $\lambda$. Our work extends the multiplicity result of [BIS], where the authors study S-shaped bifurcation curves for the Laplacian case ($p = 2$). In [BIS] the Green’s function played a crucial role in the proof. However, here in the $p$-Laplacian case new ideas are required to overcome the non availability of the Green’s function.

We now state our main result.

**Theorem 1.1** There exists a positive constant $C = C(p, N, \Omega)$ such that if

$$Q(a, b) > C$$

for some points $a$ and $b$, $a < b$, then the equation 1.1 has at least three positive solutions for a certain range of $\lambda$.

Remark : Recently in [COS], the authors study a multiplicity result for a class of positone $p$-sublinear problems via the antimaximum principle. However this requires rather restrictive assumptions on $f$ for small $u$ and also do not extend the work in [BIS] in a natural way.
We establish Theorem 1.1 by the method of sub-super solutions. By a super solution \( \phi \) we mean a function \( \phi \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \) such that \( \phi = 0 \) on \( \partial \Omega \) and
\[
\int_{\Omega} |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla \omega \geq \int_{\Omega} \lambda f(\phi) \omega, \ \forall \omega \in W
\] (1.2)
and by a sub solution \( \psi \) we mean a function \( \psi \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \) such that \( \psi = 0 \) on \( \partial \Omega \) and
\[
\int_{\Omega} |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla \omega \leq \int_{\Omega} \lambda f(\psi) \omega, \ \forall \omega \in W,
\] (1.3)
where \( W = \{ v \in C^\infty_0(\Omega) \mid v \geq 0 \text{ in } \Omega \} \). Then by the weak comparison principle (see [FT] or [DKT]), if there exist sub and super solutions \( \psi \) and \( \phi \) respectively such that \( \psi \leq \phi \) in \( \Omega \) then (1.1) has a \( C^1(\Omega) \) solution \( u \) such that \( \psi \leq u \leq \phi \).

We prove multiplicity by a sub-super solution result for the \( p \)-Laplacian case discussed in [COS]. This result extends the corresponding result for the Laplacian (\( p=2 \)) case proved in [A] and [S]. The result is as follows:

**Lemma 1.1** Let \( f \) be nonnegative and nondecreasing and suppose there exist a sub solution \( \psi_1 \), a strict super solution \( \phi_1 \), a strict sub solution \( \psi_2 \) and a super solution \( \phi_2 \) for (1.1) such that \( \psi_1 < \phi_1 < \phi_2, \ \psi_1 < \psi_2 < \phi_2 \text{ and } \psi_2 \not\leq \phi_1 \). Then (1.1) has at least three distinct solutions \( u_i (i = 1, 2, 3) \) such that \( \psi_1 \leq u_1 < u_2 < u_3 \leq \phi_2 \).

We will prove Theorem 1.1, for the case when \( \Omega \) is a ball in Section 2. Here the proof depends heavily on the construction of a crucial positive subsolution. In Section 3, we extend the theorem for general domains, by using a simple variant of this subsolution. Finally in Section 4 we will discuss a popular example arising in combustion theory.

## 2 Case when \( \Omega \) is a ball

In this section we shall prove the theorem in the case when \( \Omega \) is a ball \( B_R \) of radius \( R \).

**Lemma 2.1** There exists a positive constant \( C_1 = C_1((p, N, R)) \) such that for any positive number \( b \) if
\[ \lambda > C_1 \frac{b^{p-1}}{f(b)} \]
then there exists a subsolution \( \psi \) of the equation (1.1) on \( B_R \), with \( \|\psi\|_\infty \geq b \).
Proof of Lemma 2.1: Let us define, for some $\alpha, \beta > 1$ and $\varepsilon > 0$,

$$v(r) := \begin{cases} 1 & r \leq \varepsilon \\ 1 - (1 - (\frac{R-r}{R-\varepsilon})^\alpha) & \varepsilon < r \leq R \end{cases}$$

and let $\tilde{v}(r) = b v(r)$. Denoting

$$\mu_1(r) = \frac{R-r}{R-\varepsilon}, \quad \mu_2(r) = 1 - (\frac{R-r}{R-\varepsilon})^\beta$$

we have that for $\varepsilon < r < R$,

$$-\tilde{v}'(r) = b \frac{\alpha \beta}{R-\varepsilon} (\mu_2(r))^{\alpha-1}(\mu_1(r))^{\beta-1},$$

and hence

$$|\tilde{v}'(r)| \leq b \frac{\alpha \beta}{R-\varepsilon}.$$ 

Then define $\psi$ as the radially symmetric solution of

$$-\Delta_p \psi(x) = \lambda f(\bar{\psi}(|x|)) \text{ in } B_R \\
\psi = 0 \text{ on } \partial B_R. \quad (2.1)$$

Then $\psi$ satisfies

$$-(r^{N-1}G(\psi'(r)))' = \lambda r^{N-1} f(\bar{\psi}(r)) \\
\psi'(0) = 0; \psi(R) = 0,$$

where for any real $t$,

$$G(t) = |t|^{p-2} t.$$ 

Integrating once, we get for $0 < r < R$,

$$-G(\psi'(r)) = \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} f(\bar{\psi}(s)) \, ds.$$ 

Observe that $G$ being monotone $G^{-1}$ also is continuous and monotone. Hence,

$$-\psi'(r) = G^{-1}\{ \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} f(\bar{\psi}(s)) \, ds \}. \quad (2.2)$$

We claim that

$$\psi(r) \geq \bar{\psi}(r) \ \forall \ 0 \leq r \leq R. \quad (2.3)$$
Then from 2.1 it follows that that $\psi(r)$ is a subsolution since $f$ is monotone. Both $\psi$ and $\tilde{v}$ vanish at $R$. Thus, in order to show 2.3, it is enough to show that

$$
\psi'(r) \leq \tilde{v}'(r) \quad \forall \ 0 \leq r \leq R.
$$

(2.4)

Since $f > 0$, from equation (2.2) $\psi'(r) \leq 0$ while $\tilde{v}'(r) = 0$ for $0 \leq r \leq \varepsilon$, it is enough to verify equation (2.4) in the range $\varepsilon \leq r \leq R$. For $r \geq \varepsilon$ we have

$$
\int_0^r s^{N-1} f(\tilde{v}(s)) ds \geq \int_0^\varepsilon s^{N-1} f(\tilde{v}(s)) \geq f(b) \varepsilon^N
$$

and hence from (2.2), using the monotonicity of $G^{-1}$

$$
-\psi'(r) \geq G^{-1}\{ \frac{\lambda}{R^{N-1}} f(b) \frac{\varepsilon^N}{N} \}.
$$

Thus (2.4) will hold also for all $\varepsilon \leq r \leq R$, if

$$
G^{-1}\left( \frac{\lambda}{R^{N-1}} f(b) \frac{\varepsilon^N}{N} \right) \geq \frac{\alpha \beta b}{R - \varepsilon}
$$

which is the same as

$$
\frac{\lambda}{R^{N-1}} f(b) \frac{\varepsilon^N}{N} \geq G\left( \frac{\alpha \beta b}{R - \varepsilon} \right) = \left( \frac{\alpha \beta b}{R - \varepsilon} \right)^{p-1}.
$$

Thus if

$$
\lambda \geq \frac{b^{p-1} R^{N-1} N}{\varepsilon^N} \left( \frac{\alpha \beta}{R - \varepsilon} \right)^{p-1}
$$

then 2.4 holds. To get the best possible lower bound for $\lambda$, we can take $\alpha = \beta = 1$, since $v$ is $C^1$ for any $\alpha = \beta = 1 + \delta$. Define

$$
C_1 := \inf_{\varepsilon} \varepsilon^N \left( \frac{1}{R - \varepsilon} \right)^{p-1} R^{N-1}.
$$

Then

$$
C_1 = \frac{N}{R^p} \inf_{t} \frac{1}{(1-t)^{p-1} t^N}.
$$

This infimum is attained for the value

$$
t = \frac{N}{N + p - 1},
$$

which gives $C_1(p, N, R)$. This proves the lemma.
Theorem 2.1 There exists a positive constant \( C = C(p, N, R) \) such that if

\[
Q(a, b) > C
\]

for some points \( a \) and \( b, a < b \), then the equation (1.1) on \( B_R \) has at least three positive solutions for a certain range of \( \lambda \).

Proof: We shall construct supersolutions \( \phi_1 \) and \( \phi_2 \) and subsolutions \( \psi_1 \) and \( \psi_2 \) as in Lemma 1.1. Clearly \( \psi_1 = 0 \) is a subsolution for every \( \lambda > 0 \) since \( f(0) > 0 \). Let \( \phi_1 = a \frac{e}{\|e\|_\infty} \) where \( e \in C^1(\Omega) \) is the solution of \(-\Delta_p e = 1 \) in \( \Omega \), \( e = 0 \) on \( \partial\Omega \). Then \(-\Delta_p \phi_1 = \left( \frac{e}{\|e\|_\infty} \right)^{p-1} \geq \lambda f(a) \geq \lambda f(\phi_1) \), and hence a supersolution if \( \lambda \leq \left( \frac{a^{p-1}}{f(a)} \right) \left( \frac{1}{\|e\|_\infty} \right)^{p-1} = B \) (say). Note that \( \|\phi_1\|_\infty = a \). Next let \( \psi_2 = \psi \). Then by Lemma 2.1, \( \psi_2 \) is a subsolution such that \( \|\psi_2\|_\infty \geq b \) if \( \lambda \geq C_1 \frac{b^{p-1}}{f(b)} = A \) (say). Note that if \( Q(a, b) > C \) where \( C = C(p, N, R) = C_1(\|e\|_\infty)^{p-1} \), then \( A < B \). Finally, let \( \phi_2 = M(\lambda) \frac{e}{\|e\|_\infty} \). Then \(-\Delta_p \phi_2 = \left( \frac{M(\lambda)}{\|e\|_\infty} \right)^{p-1} \geq \lambda f(M(\lambda)) \geq \lambda f(\phi_2) \), and hence a supersolution for any given \( \lambda \), if \( M(\lambda) \) is chosen sufficiently large so that \( \frac{(M(\lambda))^{p-1}}{f(M(\lambda))} \geq \lambda \left( \|e\|_\infty \right)^{p-1} \). This is possible since the function \( f \) is \( p \)-sublinear. Here since \( \frac{\partial e}{\partial n} < 0 \) on \( \partial\Omega \), we can also choose \( M(\lambda) \) large enough so that \( \phi_2 > \psi_2 \) and \( \phi_2 > \phi_1 \). Hence there exist three positive solutions for \( \lambda \) in \([A, B]\).

3 Proof of Theorem 1.1

In this section we will prove Theorem 1.1 in general domains.

First we construct a positive subsolution \( z(x) \) in \( \Omega \) with \( \|z\|_\infty \geq b \). Let \( B_R \) be the largest inscribed ball in \( \Omega \) and \( C_1(p, N, R) \) be as in Lemma 2.1. Assume \( Q(a, b) > C_1, \lambda \geq C_1 \frac{b^{p-1}}{f(b)} = A \) and let \( \psi(r) \) be the subsolution constructed in \( B_R \) in Lemma 2.1. Now define \( z(x) = \psi(|x|) \) if \( x \in B_R \) and \( z(x) = 0 \) if \( x \in \Omega - B_R \). Then \( z \in W^{1,p}(\Omega) \cap C(\bar{\Omega}) \) and \( z = 0 \) on \( \partial\Omega \). Further, on \( B_R \) we have \(-\Delta_p z(x) = -\Delta_p \psi(|x|) \leq \lambda f(\psi(|x|)) = \lambda f(z(x)) \), while outside \( B_R \) we have \(-\Delta_p z(x) = 0 < \lambda f(0) = \lambda f(z(x)) \) (since \( f(0) > 0 \)). Hence \( z(x) \) is a subsolution in \( \Omega \) for \( \lambda \geq C_1 \frac{b^{p-1}}{f(b)} = A \) with \( \|z\|_\infty \geq b \).

The rest of the proof of Theorem 1.1 is identical to that of the proof of Theorem 2.1 except that here we define \( \psi_2 = z \).
4 Application in Combustion Theory

Here we consider the example

\[-\Delta_p u = \lambda \exp[\frac{\alpha u}{\alpha + u}] \text{ in } \Omega \]
\[u = 0 \text{ on } \partial \Omega\]

The nonlinearity \( f(u) = \exp[\frac{\alpha u}{\alpha + u}] \) arises in the theory of combustion and it was discussed in [BIS] and many references cited within for the case when \( p = 2 \) (Laplacian case). In [BIS] the authors prove that a necessary condition for multiple positive solutions is \( \alpha > 4 \). Further they prove that if \( \alpha \) is large enough then there are at least three positive solutions for a certain range of \( \lambda \). Here we will establish similar results for the \( p \)-Laplacian case.

Clearly \( f \) satisfies hypothesis (A1). Also a simple calculation shows that \( \frac{u^{p-1}}{f(u)} \) is nondecreasing if \( \alpha \leq 4(p-1) \). Hence a necessary condition for multiplicity is \( \alpha > 4(p-1) \). Further choosing \( a = 1 \) and \( b = \alpha \) we have

\[Q(a, b) := \left( \frac{a^{p-1}}{f(a)} \right) \bigg/ \left( \frac{b^{p-1}}{f(b)} \right) = (\alpha)^{1-p} \exp[\frac{\alpha}{2} \frac{\alpha}{\alpha + 1}],\]

Hence given any positive constant \( C = C(p, N, \Omega) \), for \( \alpha \) large we have \( Q(1, \alpha) > C \), and thus there exists at least three positive solutions for a certain range of \( \lambda \) by Theorem 1.1.

References


