

Multiple Positive Solutions for Classes of p-Laplacian Equations

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Abstract

We study positive $C^1(\bar{\Omega})$ solutions to classes of boundary value problems of the form

$$\begin{aligned} -\Delta_p u &= \lambda f(u) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

where Δ_p denotes the p-Laplacian operator defined by $\Delta_p z := \operatorname{div}(|\nabla z|^{p-2}\nabla z)$; $p > 1$, $\lambda > 0$ is a parameter and Ω is a bounded domain in R^N ; $N \geq 2$ with $\partial\Omega$ of class C^2 and connected. (If $N = 1$, we assume that Ω is a bounded open interval.) In particular, we establish existence of three positive solutions for classes of nondecreasing, p-sublinear functions f belonging to $C^1([0, \infty))$. Our proofs are based on sub-super solution techniques.

1 Introduction

We consider weak solutions to classes of boundary value problems of the form

$$\begin{aligned} -\Delta_p u &= \lambda f(u) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{1.1}$$

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where Δ_p denotes the p -Laplacian operator defined by $\Delta_p z := \operatorname{div}(|\nabla z|^{p-2} \nabla z)$; $p > 1$, λ is a positive parameter and Ω is a bounded domain in R^N ; $N \geq 2$ with $\partial\Omega$ in class C^2 and connected. (If $N = 1$, we assume that Ω is a bounded open interval.) By a weak solution of (1.1), we mean, a function $u \in W_0^{1,p}(\Omega)$ that satisfies

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \omega = \int_{\Omega} \lambda f(u) \omega, \quad \forall \omega \in C_0^\infty(\Omega).$$

However, in this paper, we in fact study the existence and multiplicity of $C^1(\bar{\Omega})$ solutions, that are strictly positive in Ω . Throughout this paper our classes of functions f satisfy:

(A1) $f \in C^1([0, \infty))$ is a nondecreasing function such that $f(0) > 0$ and $\lim_{v \rightarrow \infty} \frac{f(v)}{v^{p-1}} = 0$ (p -sublinear).

For such positive p -sublinear nonlinearities, it is easy to establish that there is a positive solution for every $\lambda > 0$. Further when $\frac{v^{p-1}}{f(v)}$ is nondecreasing, uniqueness of the positive solution for every λ follows from [DS]. In this paper we will consider the case when $\frac{v^{p-1}}{f(v)}$ is not monotonic. In particular, we consider f for which there exists a and b such that $0 < a < b$ and

$$Q(a, b) := \left(\frac{a^{p-1}}{f(a)} \right) / \left(\frac{b^{p-1}}{f(b)} \right)$$

is sufficiently large. For such classes of nonlinearities we discuss the existence of three positive solutions for a certain range of λ . Our work extends the multiplicity result of [BIS], where the authors study S-shaped bifurcation curves for the Laplacian case ($p = 2$). In [BIS] the Green's function played a crucial role in the proof. However, here in the p -Laplacian case new ideas are required to overcome the non availability of the Green's function.

We now state our main result.

Theorem 1.1 *There exists a positive constant $C = C(p, N, \Omega)$ such that if*

$$Q(a, b) > C$$

for some points a and b , $a < b$, then the equation 1.1 has at least three positive solutions for a certain range of λ .

Remark : Recently in [COS], the authors study a multiplicity result for a class of positive p -sublinear problems via the antimaximum principle. However this requires rather restrictive assumptions on f for small u and also do not extend the work in [BIS] in a natural way.

We establish Theorem 1.1 by the method of sub-super solutions. By a super solution ϕ we mean a function $\in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ such that $\phi = 0$ on $\partial\Omega$ and

$$\int_{\Omega} |\nabla\phi|^{p-2} \nabla\phi \cdot \nabla\omega \geq \int_{\Omega} \lambda f(\phi) \omega, \quad \forall \omega \in W \quad (1.2)$$

and by a sub solution ψ we mean a function $\in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ such that $\psi = 0$ on $\partial\Omega$ and

$$\int_{\Omega} |\nabla\psi|^{p-2} \nabla\psi \cdot \nabla\omega \leq \int_{\Omega} \lambda f(\psi) \omega, \quad \forall \omega \in W, \quad (1.3)$$

where $W = \{v \in C_0^\infty(\Omega) \mid v \geq 0 \text{ in } \Omega\}$. Then by the weak comparison principle (see [FT] or [DKT]), if there exist sub and super solutions ψ and ϕ respectively such that $\psi \leq \phi$ in Ω then (1.1) has a $C^1(\bar{\Omega})$ solution u such that $\psi \leq u \leq \phi$.

We prove multiplicity by a sub-super solution result for the p -Laplacian case discussed in [COS]. This result extends the corresponding result for the Laplacian ($p=2$) case proved in [A] and [S]. The result is as follows:

Lemma 1.1 *Let f be nonnegative and nondecreasing and suppose there exist a sub solution ψ_1 , a strict super solution ϕ_1 , a strict sub solution ψ_2 and a super solution ϕ_2 for (1.1) such that $\psi_1 < \phi_1 < \phi_2$, $\psi_1 < \psi_2 < \phi_2$ and $\psi_2 \not\leq \phi_1$. Then (1.1) has at least three distinct solutions u_i ($i = 1, 2, 3$) such that $\psi_1 \leq u_1 < u_2 < u_3 \leq \phi_2$.*

We will prove Theorem 1.1, for the case when Ω is a ball in Section 2. Here the proof depends heavily on the construction of a crucial positive subsolution. In Section 3, we extend the theorem for general domains, by using a simple variant of this subsolution. Finally in Section 4 we will discuss a popular example arising in combustion theory .

2 Case when Ω is a ball

In this section we shall prove the theorem in the case when Ω is a ball B_R of radius R .

Lemma 2.1 *There exists a positive constant $C_1 = C_1((p, N, R))$ such that for any positive number b if*

$$\lambda > C_1 \frac{b^{p-1}}{f(b)}$$

then there exists a subsolution ψ of the equation (1.1) on B_R , with $\|\psi\|_\infty \geq b$.

Proof of Lemma 2.1 : Let us define, for some $\alpha, \beta > 1$ and $\varepsilon > 0$,

$$v(r) := \begin{cases} 1 & r \leq \varepsilon \\ 1 - (1 - (\frac{R-r}{R-\varepsilon})^\beta)^\alpha & \varepsilon < r \leq R \end{cases}$$

and let $\tilde{v}(r) = b v(r)$. Denoting

$$\mu_1(r) = \frac{R-r}{R-\varepsilon}, \quad \mu_2(r) = 1 - (\frac{R-r}{R-\varepsilon})^\beta$$

we have that for $\varepsilon < r < R$,

$$-\tilde{v}'(r) = b \frac{\alpha\beta}{R-\varepsilon} (\mu_2(r))^{\alpha-1} (\mu_1(r))^{\beta-1},$$

and hence

$$|\tilde{v}'(r)| \leq b \frac{\alpha\beta}{R-\varepsilon}.$$

Then define ψ as the radially symmetric solution of

$$\begin{aligned} -\Delta_p \psi(x) &= \lambda f(\tilde{v}(|x|)) \text{ in } B_R \\ \psi &= 0 \text{ on } \partial B_R. \end{aligned} \tag{2.1}$$

Then ψ satisfies

$$\begin{aligned} -(r^{N-1} G(\psi'(r)))' &= \lambda r^{N-1} f(\tilde{v}(r)) \\ \psi'(0) &= 0; \psi(R) = 0, \end{aligned}$$

where for any real t ,

$$G(t) = |t|^{p-2} t.$$

Integrating once, we get for $0 < r < R$,

$$-G(\psi'(r)) = \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} f(\tilde{v}(s)) ds.$$

Observe that G being monotone G^{-1} also is continuous and monotone. Hence,

$$-\psi'(r) = G^{-1} \left\{ \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} f(\tilde{v}(s)) ds \right\}. \tag{2.2}$$

We claim that

$$\psi(r) \geq \tilde{v}(r) \quad \forall 0 \leq r \leq R. \tag{2.3}$$

Then from 2.1 it follows that that $\psi(r)$ is a subsolution since f is monotone. Both ψ and \tilde{v} vanish at R . Thus, in order to show 2.3, it is enough to show that

$$\psi'(r) \leq \tilde{v}'(r) \quad \forall 0 \leq r \leq R. \quad (2.4)$$

Since $f > 0$, from equation (2.2) $\psi'(r) \leq 0$ while $\tilde{v}'(r) = 0$ for $0 \leq r \leq \varepsilon$, it is enough to verify equation(2.4) in the range $\varepsilon \leq r \leq R$. For $r \geq \varepsilon$ we have

$$\int_0^r s^{N-1} f(\tilde{v}(s)) ds \geq \int_0^\varepsilon s^{N-1} f(\tilde{v}(s)) \geq f(b) \frac{\varepsilon^N}{N}$$

and hence from (2.2), using the monotonicity of G^{-1}

$$-\psi'(r) \geq G^{-1} \left\{ \frac{\lambda}{R^{N-1}} f(b) \frac{\varepsilon^N}{N} \right\}.$$

Thus (2.4) will hold also for all $\varepsilon \leq r \leq R$, if

$$G^{-1} \left(\frac{\lambda}{R^{N-1}} f(b) \frac{\varepsilon^N}{N} \right) \geq \frac{\alpha \beta b}{R - \varepsilon}$$

which is the same as

$$\begin{aligned} \frac{\lambda}{R^{N-1}} f(b) \frac{\varepsilon^N}{N} &\geq G \left(\frac{\alpha \beta b}{R - \varepsilon} \right) \\ &= \left(\frac{\alpha \beta b}{R - \varepsilon} \right)^{p-1}. \end{aligned}$$

Thus if

$$\lambda \geq \frac{b^{p-1}}{f(b)} R^{N-1} \frac{N}{\varepsilon^N} \left(\frac{\alpha \beta}{R - \varepsilon} \right)^{p-1}$$

then 2.4 holds. To get the best possible lower bound for λ , we can take $\alpha = \beta = 1$, since v is C^1 for any $\alpha = \beta = 1 + \delta$. Define

$$C_1 := \inf_{\varepsilon} \frac{N}{\varepsilon^N} \left(\frac{1}{R - \varepsilon} \right)^{p-1} R^{N-1}.$$

Then

$$C_1 = \frac{N}{R^p} \inf_t \frac{1}{(1-t)^{p-1} t^N}.$$

This infimum is attained for the value

$$t = \frac{N}{N + p - 1},$$

which gives $C_1(p, N, R)$. This proves the lemma.

Theorem 2.1 *There exists a positive constant $C = C(p, N, R)$ such that if*

$$Q(a, b) > C$$

for some points a and b , $a < b$, then the equation (1.1) on B_R has at least three positive solutions for a certain range of λ .

Proof: We shall construct supersolutions ϕ_1 and ϕ_2 and subsolutions ψ_1 and ψ_2 as in Lemma 1.1. Clearly $\psi_1 = 0$ is a subsolution for every $\lambda > 0$ since $f(0) > 0$. Let $\phi_1 = a \frac{e}{\|e\|_\infty}$ where $e \in C^1(\bar{\Omega})$ is the solution of $-\Delta_p e = 1$ in Ω , $e = 0$ on $\partial\Omega$. Then $-\Delta_p \phi_1 = \left(\frac{a}{\|e\|_\infty}\right)^{p-1} \geq \lambda f(a) \geq \lambda f(\phi_1)$, and hence a supersolution if $\lambda \leq \left(\frac{a^{p-1}}{f(a)}\right) \left(\frac{1}{(\|e\|_\infty)^{p-1}}\right) = B$ (say). Note that $\|\phi_1\|_\infty = a$. Next let $\psi_2 = \psi$. Then by Lemma 2.1, ψ_2 is a subsolution such that $\|\psi_2\|_\infty \geq b$ if $\lambda \geq C_1 \frac{b^{p-1}}{f(b)} = A$ (say). Note that if $Q(a, b) > C$ where $C = C(p, N, R) = C_1(\|e\|_\infty)^{p-1}$ then $A < B$. Finally, let $\phi_2 = M(\lambda) \frac{e}{\|e\|_\infty}$. Then $-\Delta_p \phi_2 = \left(\frac{M(\lambda)}{\|e\|_\infty}\right)^{p-1} \geq \lambda f(M(\lambda)) \geq \lambda f(\phi_2)$, and hence a supersolution for any given λ , if $M(\lambda)$ is chosen sufficiently large so that $\frac{(M(\lambda))^{p-1}}{f(M(\lambda))} \geq \lambda(\|e\|_\infty)^{p-1}$. This is possible since the function f is p -sublinear. Here since $\frac{\partial e}{\partial n} < 0$ on $\partial\Omega$, we can also choose $M(\lambda)$ large enough so that $\phi_2 > \psi_2$ and $\phi_2 > \phi_1$. Hence there exist three positive solutions for λ in $[A, B]$.

3 Proof of Theorem 1.1

In this section we will prove Theorem 1.1 in general domains.

First we construct a positive subsolution $z(x)$ in Ω with $\|z\|_\infty \geq b$. Let B_R be the largest inscribed ball in Ω and $C_1(p, N, R)$ be as in Lemma 2.1. Assume $Q(a, b) > C_1$, $\lambda \geq C_1 \frac{b^{p-1}}{f(b)} = A$ and let $\psi(r)$ be the subsolution constructed in B_R in Lemma 2.1. Now define $z(x) = \psi(|x|)$ if $x \in B_R$ and $z(x) = 0$ if $x \in \Omega - B_R$. Then $z \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ and $z = 0$ on $\partial\Omega$. Further, on B_R we have $-\Delta_p z(x) = -\Delta_p \psi(|x|) \leq \lambda f(\psi(|x|)) = \lambda f(z(x))$, while outside B_R we have $-\Delta_p z(x) = 0 < \lambda f(0) = \lambda f(z(x))$ (since $f(0) > 0$). Hence $z(x)$ is a subsolution in Ω for $\lambda \geq C_1 \frac{b^{p-1}}{f(b)} = A$ with $\|z\|_\infty \geq b$.

The rest of the proof of Theorem 1.1 is identical to that of the proof of Theorem 2.1 except that here we define $\psi_2 = z$.

4 Application in Combustion Theory

Here we consider the example

$$\begin{aligned} -\Delta_p u &= \lambda \exp\left[\frac{\alpha u}{\alpha + u}\right] \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

The nonlinearity $f(u) = \exp\left[\frac{\alpha u}{\alpha + u}\right]$ arises in the theory of combustion and it was discussed in [BIS] and many references cited within for the case when $p = 2$ (Laplacian case). In [BIS] the authors prove that a necessary condition for multiple positive solutions is $\alpha > 4$. Further they prove that if α is large enough then there are at least three positive solutions for a certain range of λ . Here we will establish similar results for the p -Laplacian case.

Clearly f satisfies hypothesis (A1). Also a simple calculation shows that $\frac{u^{p-1}}{f(u)}$ is nondecreasing if $\alpha \leq 4(p-1)$. Hence a necessary condition for multiplicity is $\alpha > 4(p-1)$. Further choosing $a = 1$ and $b = \alpha$ we have

$$Q(a, b) := \left(\frac{a^{p-1}}{f(a)}\right) / \left(\frac{b^{p-1}}{f(b)}\right) = (\alpha)^{1-p} \exp\left[\frac{\alpha}{2} - \frac{\alpha}{\alpha + 1}\right].$$

Hence given any positive constant $C = C(p, N, \Omega)$, for α large we have $Q(1, \alpha) > C$, and thus there exists at least three positive solutions for a certain range of λ by Theorem 1.1.

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