Local stabilization of the compressible Navier–Stokes system, around null velocity, in one dimension

Shirshendu Chowdhury a,1, Debayan Maity b,1, Mythily Ramaswamy b,1, Jean-Pierre Raymond a,*,1

a Institut de Mathématiques de Toulouse, Université Paul Sabatier & CNRS, 31062 Toulouse Cedex, France
b TIFR Centre for Applicable Mathematics, Post Bag No. 6503, GKV Post Office, Bangalore 560065, India

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Abstract

In this paper we study the exponential stabilization of the one dimensional compressible Navier–Stokes system, in a bounded interval \((0, \pi)\), locally around a constant steady state \((\bar{\rho}, 0)\), \(\bar{\rho} > 0\), by a localized distributed control acting only in the velocity equation. We determine a linear feedback law able to stabilize a nonlinear transformed system. Coming back to the original nonlinear system, we obtain a nonlinear feedback law able to stabilize locally this nonlinear system. To the best of our knowledge, the results of the paper are the first ones providing feedback control laws stabilizing compressible fluid flows.

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* Corresponding author.
E-mail addresses: shirshendu.tifr@gmail.com (S. Chowdhury), debayan@math.tifrbng.res.in (D. Maity), mythily@math.tifrbng.res.in (M. Ramaswamy), jean-pierre.raymond@math.univ-toulouse.fr (J.-P. Raymond).

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1. Introduction

The goal of this paper is to study the local stabilization of the one dimensional compressible Navier–Stokes equations in a bounded interval \((0, \pi)\), around a steady state solution \((\bar{\rho}, 0)\) by a distributed control acting only in the velocity equation. More precisely we consider the system

\[
\begin{align*}
\rho_t + (\rho v)_y &= 0, \quad \text{in } (0, \pi) \times (0, \infty), \\
\rho (v_t + vv_y) + (p(\rho))_y - vv_{yy} &= \rho \chi(\ell_1, \ell_2) g, \quad \text{in } (0, \pi) \times (0, \infty), \\
\rho(0) &= \rho_0, \quad v(0) = v_0, \quad \text{in } (0, \pi), \\
v(0, t) &= 0, \quad v(\pi, t) = 0, \quad \forall t > 0,
\end{align*}
\]

(1.1)

where \(\rho(y, t)\) is the density, \(v(y, t)\) is the velocity of the fluid, \(v > 0\) is the viscosity constant, the pressure \(p\) satisfies the following constitutive law

\[p(\rho) = a \rho^\gamma \text{ for } a > 0, \quad \gamma \geq 1,\]

\(g\) is a distributed control acting in the subset \((\ell_1, \ell_2) \subset (0, \pi)\), and \(\chi(\ell_1, \ell_2)\) is the characteristic function of the interval \((\ell_1, \ell_2)\). Eq. (1.1) is a model for compressible isentropic fluid flows \((\gamma > 1)\) or isothermal fluid flows \((\gamma = 1)\) (see [12]). Let us first notice that for any constant \(\bar{\rho} > 0\), the pair \((\bar{\rho}, 0)\) is a steady state solution to system (1.1) when \(g = 0\).

Integrating the first equation in (1.1) over \((0, \pi)\) and using the boundary conditions, we obtain

\[
\int_0^\pi \rho(y, t) \, dy = \int_0^\pi \rho_0(y) \, dy.
\]

Therefore, the control \(g\) has no action on the mean value of the density. As our goal is to study the stabilization of the system locally around a constant steady state solution \((\bar{\rho}, 0)\), we choose the initial density \(\rho_0\) such that

\[
\min_{y \in [0, \pi]} \rho_0(y) > 0.
\]

(1.2)

The stabilization of the compressible Navier–Stokes system linearized around a constant \((\bar{\rho}, 0)\) has been studied in [2] and [6]. In particular, from [6] we know that the linearized system around a constant steady state solution \((\bar{\rho}, 0)\) is exponentially stabilizable with any decay rate \(-\omega\) with \(0 < \omega < \omega_0\), and not stabilizable with boundary controls with a decay rate \(-\omega\) if \(\omega > \omega_0\), where \(\omega_0\) is an accumulation point of the spectrum of the linearized operator, precisely defined by

\[
\omega_0 := \frac{a\gamma \bar{\rho}^\gamma}{\nu}.
\]

(1.3)

Since the stabilizability results will be similar for localized controls, in the present paper, we are looking for local stabilization results of system (1.1), with an exponential decay rate \(-\omega\) such that \(0 < \omega < \omega_0\).
In order to study the local stabilization of the nonlinear system (1.1), locally around the constant steady state solution \((\bar{\rho}, 0)\), we first write the system satisfied by \((\sigma, v) = (\rho - \bar{\rho}, v)\):

\[
\sigma_t + \bar{\rho}v_y + (\sigma v)_y = 0, \quad \text{in } (0, \pi) \times (0, \infty),
\]

\[
v_t + vv_y + a\gamma(\bar{\rho} + \sigma)^{\gamma-2}v_y - \frac{v}{\bar{\rho} + \sigma} v_{yy} = \chi(\ell_1, \ell_2) \sigma_y, \quad \text{in } (0, \pi) \times (0, \infty),
\]

\[
\sigma(0) := \sigma_0 = \rho_0 - \bar{\rho}, \quad v(0) = v_0, \quad \text{in } (0, \pi),
\]

\[
v(0, t) = 0, \quad v(\pi, t) = 0, \quad \forall t > 0, \quad \int_0^\pi \sigma_0(y)dy = 0. \tag{1.4}
\]

We now have a zero average condition for \(\sigma\):

\[
\frac{1}{\pi} \int_0^\pi \sigma(y, t)dy = 0, \quad \forall t > 0. \tag{1.5}
\]

We set \(\Omega_y = (0, \pi)\) and \(Q_\infty := \Omega_y \times (0, \infty)\). Since we are looking for a control \(g\) stabilizing the system (1.4) with the exponential decay rate \(e^{-\omega t}\), it is convenient to introduce

\[
\hat{\sigma}(y, t) = e^{\omega t} \sigma(y, t), \quad \hat{v}(y, t) = e^{\omega t} v(y, t), \quad \hat{g}(y, t) = e^{\omega t} g(y, t).
\]

The system satisfied by \((\hat{\sigma}, \hat{v})\) is

\[
\hat{\sigma}_t - \omega \hat{\sigma} + \hat{\rho} \hat{v}_y + e^{-\omega t} \hat{\sigma}_y \hat{v} + e^{-\omega t} \hat{\sigma} \hat{v}_y = 0, \quad \text{in } Q_\infty,
\]

\[
\hat{v}_t - \omega \hat{v} + e^{-\omega t} \hat{\sigma} \hat{v}_y + a\gamma(\hat{\rho} + e^{-\omega t} \hat{\sigma})^{\gamma-2} \hat{\sigma}_y - \frac{v}{\hat{\rho} + e^{-\omega t} \hat{\sigma}} \hat{v}_{yy} = \chi(\ell_1, \ell_2) \hat{g}, \quad \text{in } Q_\infty,
\]

\[
\hat{\sigma}(0) = \sigma_0, \quad \hat{v}(0) = v_0, \quad \text{in } \Omega_y,
\]

\[
\hat{v}(0, t) = 0, \quad \hat{v}(\pi, t) = 0, \quad \forall t > 0, \quad \int_{\Omega_y} \sigma_0(y)dy = 0. \tag{1.6}
\]

And we have

\[
\frac{1}{\pi} \int_0^\pi \hat{\sigma}(y, t)dy = 0 \quad \forall t > 0. \tag{1.7}
\]

For the incompressible Navier–Stokes system, local stabilization results are usually proved by using a linear feedback control law stabilizing the linearized system and applied to the nonlinear system. The local stabilization is proved by a fixed point argument \([4,3,9,13,14]\). Here we would like to follow the same approach.

This approach consisting in linearizing system (1.6) around \((0, 0)\), stabilizing the linearized system, and considering the nonlinear terms as source terms of a nonhomogeneous linearized system, cannot work here. Indeed, the nonlinear term \(e^{-\omega t} \hat{\sigma}_y \hat{v}\) cannot be treated, in a fixed point...
method, as a source term of the linearized transport equation, because there is no regularizing effect in transport equations, and the regularity of $e^{-\omega t}\tilde{\sigma}_y\tilde{v}$ will be necessarily worse than the regularity of $\tilde{\sigma}$.

To overcome this difficulty, we introduce a change of unknowns in order to eliminate this nonlinear term. We are going to obtain a new nonlinear system. The classical method above described can now be applied by determining a feedback control law stabilizing the new linearized system, and next using it to stabilize the new nonlinear system.

In order to clearly define the change of variables used to transform the initial nonlinear system into a new one, we introduce

$$\Omega_x := (0, \pi) \quad \text{and} \quad Q_x^\infty := \Omega_x \times (0, \infty),$$

and, for any $\tilde{v}$ smooth enough and bounded in some norm, we consider the mapping $Y_\tilde{v}(\cdot, t)$ from $\Omega_x$ to $\Omega_y$ satisfying the following ordinary differential equation, parametrized by $x \in \Omega_x$

$$\frac{\partial Y_\tilde{v}}{\partial t}(x, t) = e^{-\omega t}\tilde{v}(Y_\tilde{v}(x, t), t), \quad Y_\tilde{v}(x, 0) = x, \quad \text{for } t > 0. \quad (1.8)$$

Let us notice that the change of variables induced by the mapping $Y_\tilde{v}$ is very similar to that used to write the Navier–Stokes equations in Lagrangian variables, but it is slightly different due to the factor $e^{-\omega t}$.

Of course we have to show that Eq. (1.8) is well posed. This is why some conditions are required on $\tilde{v}$. Under these conditions, we shall prove that, for each $t > 0$, $Y_\tilde{v}(\cdot, t)$ is a $C^1$-diffeomorphism from $\Omega_x$ onto $\Omega_y$. For $t > 0$, we denote by $X_\tilde{v}(\cdot, t)$ the inverse mapping of $Y_\tilde{v}(\cdot, t)$. Next, by setting

$$\tilde{\sigma}(x, t) = \tilde{\sigma}(Y_\tilde{v}(x, t), t), \quad \tilde{v}(x, t) = \tilde{v}(Y_\tilde{v}(x, t), t), \quad \tilde{g}(x, t) = \tilde{g}(Y_\tilde{v}(x, t), t),$$

$$\tilde{\ell}_{1, \tilde{v}}(t) = X_\tilde{v}(\ell_1, t) \quad \text{and} \quad \tilde{\ell}_{2, \tilde{v}}(t) = X_\tilde{v}(\ell_2, t), \quad (1.9)$$

we can easily check that $(\tilde{\sigma}, \tilde{v}, \tilde{g})$, together with $(X, Y) = (X_\tilde{v}, Y_\tilde{v})$, satisfy the following new nonlinear system

$$\tilde{\sigma}_x + \rho \tilde{v}_x - \omega \tilde{\sigma} = \mathcal{F}_1(\tilde{\sigma}, \tilde{v}, t), \quad \text{in } Q_x^\infty,$$

$$\tilde{v}_t + b\tilde{\sigma}_x - v_0\tilde{v}_{xx} - \omega \tilde{v} = \mathcal{F}_2(\tilde{\sigma}, \tilde{v}, t) + \chi(\tilde{\ell}_{1, \tilde{v}}(t), \tilde{\ell}_{2, \tilde{v}}(t)) \tilde{g}, \quad \text{in } Q_x^\infty,$$

$$\tilde{\sigma}(0) = \sigma_0, \quad \tilde{v}(0) = v_0 \quad \text{in } \Omega_x, \quad \int_{\Omega_x} \sigma_0(x) dx = 0,$$

$$\tilde{v}(0, t) = 0, \quad \tilde{v}(\pi, t) = 0, \quad \forall t > 0,$$

$$Y(x, t) = x + \int_0^t e^{-\omega s} \tilde{v}(x, s) ds, \quad t > 0, \quad x \in \Omega_x,$$

$$X(Y(x, t), t) = x, \quad x \in \Omega_x, \quad Y(X(y, t), t) = y, \quad y \in \Omega_y, \quad t > 0,$$

$$\tilde{\ell}_{j, \tilde{v}}(t) = X(\ell_j, t), \quad j = 1, 2, \quad (1.10)$$

where the nonlinear terms $\mathcal{F}_1$ and $\mathcal{F}_2$ are precisely defined in (2.8).
We notice that the control interval \((\ell_1, \ell_2)\), which is fixed in system (1.6), is now transformed into a time dependent interval \((\tilde{\ell}_1(t), \tilde{\ell}_2(t))\). This introduces an additional source of difficulty. We overcome it by looking for controls localized in a fixed interval \(O\), independent of time, strictly contained in \((\tilde{\ell}_1(t), \tilde{\ell}_2(t))\). For such controls, the linearized system corresponding to (1.10) is

\[
\begin{align*}
\tilde{\sigma}_t + \tilde{\rho}_t \tilde{v}_x - \omega \tilde{\sigma} &= 0, \quad \text{in } Q^\infty_x, \\
\tilde{v}_t + b \tilde{\sigma}_x - v_0 \tilde{v}_{xx} - \omega \tilde{v} &= \chi_0 \tilde{g}, \quad \text{in } Q^\infty_x, \\
\tilde{\sigma}(0) &= \sigma_0, \quad \tilde{v}(0) = v_0, \quad \text{in } \Omega_x, \quad \int_{\Omega_x} \sigma_0(x) dx = 0, \\
\tilde{v}(0, t) &= 0, \quad \tilde{v}(\pi, t) = 0, \quad \forall t > 0.
\end{align*}
\]

(1.11)

Another difficulty comes from the fact that, even if we look for a solution to the nonlinear system for which the first component is with mean value zero, the component \(\tilde{\sigma}\) in the deformed configuration \(\Omega_x\) is not necessarily with mean value zero. So we have to split the first component \(\tilde{\sigma}\) of the solution of the nonhomogeneous closed loop linear system into a solution with mean value zero and a solution which only depends on \(t\).

We introduce the space

\[
L^2_m(\Omega_x) = \left\{ f \in L^2(\Omega_x), \quad \int_0^\pi f(x) dx = 0 \right\}
\]

and we set

\[
H^1_m(\Omega_x) = H^1(\Omega_x) \cap L^2_m(\Omega_x).
\]

We can now state the main results of the paper.

**Theorem 1.1.** Let \(\omega\) belong to \((0, \omega_0)\). There exists a bounded linear operator \(K\) from \(L^2(\Omega_x) \times L^2(\Omega_x)\) into \(L^2(\Omega_x)\) of the form

\[
K(\sigma, v)(x) = \int_0^\pi k_\sigma(x, \xi) \sigma(\xi) d\xi + \int_0^\pi k_v(x, \xi) v(\xi) d\xi,
\]

(1.12)

with \(k_\sigma \in L^2(\Omega_x \times \Omega_x)\) and \(k_v \in L^2(\Omega_x \times \Omega_x)\), and there exist constants \(\mu_0 > 0\) and \(\tilde{C}_1 > 0\), depending on \(K\), such that, for all \(0 < \bar{\mu} < \mu_0\) and all initial conditions \((\sigma_0, v_0)\) satisfying

\[
\| (\sigma_0, v_0) \|_{H^1_m(\Omega_x) \times H^1_m(\Omega_x)} \leq \tilde{C}_1 \bar{\mu},
\]

the closed loop nonlinear system obtained by setting

\[
\tilde{g}(t) = K(\tilde{\sigma}(t), \tilde{v}(t))
\]
in system (1.10), admits a unique solution \((\tilde{\sigma}, \tilde{v}, X, Y)\) such that \((\tilde{\sigma}, \tilde{v})\) belongs to the ball \(D_{\bar{\mu}}\), \(X\) belongs to \(C^1_b([0, \infty); H^1(\Omega_y)) \cap C_b([0, \infty); H^2(\Omega_y))\), \(Y\) belongs to \(C^1_b([0, \infty); H^1(\Omega_x)) \cap C_b([0, \infty); H^2(\Omega_x))\), and \(D_{\bar{\mu}}\) is precisely defined in (4.4).

From Theorem 1.1, we can find a feedback control for system (1.6) by making a reverse change of variables. For that, we set

\[
\tilde{\sigma}(\zeta, t) = \tilde{\sigma}(X(\zeta, t), t) \quad \text{and} \quad \tilde{v}(\zeta, t) = \tilde{v}(X(\zeta, t), t), \quad \forall \zeta \in \Omega_y, \forall t \in (0, \infty).
\]

The feedback control law for system (1.6), obtained through the change of variables associated with \(Y\) in (1.12), is

\[
\tilde{K}(\tilde{\sigma}(t), \tilde{v}(t), Y(t), X(t))(y) = \int_0^\pi \left| \frac{\partial X}{\partial y}(\zeta, t) \right| \tilde{k}_\sigma(y, \zeta, t) \tilde{\sigma}(\zeta, t) d\zeta + \int_0^\pi \left| \frac{\partial X}{\partial y}(\zeta, t) \right| \tilde{k}_v(y, \zeta, t) \tilde{v}(\zeta, t) d\zeta, \quad (1.13)
\]

where

\[
\tilde{k}_\sigma(y, \zeta, t) = k_\sigma(X(y, t), X(\zeta, t)) \quad \text{and} \quad \tilde{k}_v(y, \zeta, t) = k_v(X(y, t), X(\zeta, t)).
\]

The feedback control law \(\tilde{K}\) is linear with respect to \(\tilde{\sigma}(t)\) and \(\tilde{v}(t)\), but it is nonlinear with respect to \(X(t)\) and \(Y(t)\).

The nonlinear closed loop system for \((\tilde{\sigma}, \tilde{v}, Y, X)\) is therefore

\[
\begin{align*}
\hat{\sigma}_t - \omega \hat{\sigma} + \hat{\rho} \hat{v}_y + e^{-\omega t} \hat{\sigma}_y \hat{v} + e^{-\omega t} \hat{\sigma}_y \hat{v}_y &= 0, \quad \text{in} \ Q^\infty_y, \\
\hat{v}_t - \omega \hat{v} + e^{-\omega t} \hat{v}_y \hat{v} + a\gamma (\hat{\rho} + e^{-\omega t} \hat{\sigma}) \hat{\sigma}_y \hat{v} - \frac{v}{\hat{\rho} + e^{-\omega t} \hat{\sigma}} \hat{v}_{yy} &= \chi(\ell_1, \ell_2) \tilde{K}(\tilde{\sigma}(t), \tilde{v}(t), Y(t), X(t)) \quad \text{in} \ Q^\infty_y, \\
\hat{\sigma}(0) = \sigma_0, \quad \hat{v}(0) = v_0, \quad \text{in} \ \Omega_y, \quad \hat{v}(0, t) = 0, \quad \hat{v}(\pi, t) = 0, \quad \forall t > 0,
\end{align*}
\]

\[
Y(x, t) = x + \int_0^t e^{-\omega s} \hat{v}(Y(x, s), s) ds, \quad t > 0, \ x \in \Omega_x,
\]

\[
X(Y(x, t), t) = x, \ x \in \Omega_x, \quad Y(X(y, t), t) = y, \ y \in \Omega_y, \ t > 0. \quad (1.14)
\]

We obtain the following stabilization result.

**Theorem 1.2.** Let \(\omega\) belong to \((0, \omega_0)\). There exist constants \(\hat{\mu}_0 > 0\) and \(\hat{C}_1 > 0\) such that, for \(0 < \hat{\mu} < \hat{\mu}_0\) and any initial condition \((\sigma_0, v_0) \in H^1_m(\Omega_y) \times H^1_0(\Omega_y)\) satisfying

\[
\| (\sigma_0, v_0) \|_{H^1_m(\Omega_y) \times H^1_0(\Omega_y)} \leq \hat{C}_1 \hat{\mu},
\]

...
the nonlinear closed loop system (1.14) admits a unique solution $(\hat{\sigma}, \hat{v}, X, Y)$ in the class of functions satisfying

$$\| (\hat{\sigma}, \hat{v}) \| _{H^1(0,\infty;H^1_b(\Omega_y)) \times H^1(0,\infty;L^2(\Omega_y))} \leq \hat{\mu}.$$  

Moreover $X \in C^1_b([0,\infty); H^1(\Omega_y)) \cap C_b([0,\infty); H^2(\Omega_y)), \quad Y \in C^1_b([0,\infty); H^1(\Omega_x)) \cap C_b([0,\infty); H^2(\Omega_x)).$

The closed loop nonlinear system for Eq. (1.1) reads as follows

$$\rho_t + (\rho v)_y = 0, \quad \text{in } (0,\pi) \times (0,\infty),$$

$$\rho(v_t + vv_y) + \left( p(\rho) \right)_y - v v_{yy} = \rho \chi_{(\epsilon_1,\epsilon_2)} \hat{K}(e^{ot}(\rho(t) - \hat{\rho}), e^{ot}v(t), Y(t), X(t))$$  

in $(0,\pi) \times (0,\infty),$$

$$\rho(0) = \rho_0, \quad v(0) = v_0, \quad \text{in } (0,\pi),$$

$$v(0, t) = 0, \quad v(\pi, t) = 0, \quad \forall t > 0,$$

$$Y(x, t) = x + \int_0^t v(Y(x, s), s) \, ds, \quad t > 0, \quad x \in \Omega_x,$$

$$X(Y(x, t), t) = x, \quad x \in \Omega_x, \quad Y(X(y, t), t) = y, \quad y \in \Omega_y, \quad t > 0. \quad (1.15)$$

This leads to the stabilization result for the original system (1.1).

**Theorem 1.3.** Let $\omega$ belong to $(0,\omega_0)$. There exists a continuous nonlinear mapping $\hat{K}$ from $H^1_m(\Omega_y) \times H^1_b(\Omega_y) \times H^1(\Omega_x) \times H^1(\Omega_y)$ into $L^2(\Omega_x)$, and there exist $\hat{\mu}_0 > 0$ and $\hat{C}_1 > 0$, such that, for all $0 < \hat{\mu} < \hat{\mu}_0$, for all initial condition $(\rho_0, v_0) \in H^1(\Omega_y) \times H^1_b(\Omega_y)$ satisfying

$$\| (\rho_0 - \hat{\rho}, v_0) \| _{H^1_m(\Omega_y) \times H^1_b(\Omega_y)} \leq \hat{C}_1\hat{\mu},$$

the nonlinear closed loop system (1.15) admits a unique solution $(\rho, v, X, Y)$ in the class of functions satisfying

$$\| (e^{ot}(\rho - \hat{\rho}), e^{ot}v) \| _{H^1(0,\infty;H^1_m(\Omega_y)) \times H^1(0,\infty;L^2(\Omega_y))} \leq \hat{\mu}.$$  

Moreover, we have

$$\| (\rho(\cdot, t) - \hat{\rho}, v(\cdot, t)) \| _{H^1_m(\Omega_y) \times H^1_b(\Omega_y)} \leq C \hat{\mu} e^{-ot},$$

and $\rho(y, t) \geq \frac{\hat{\rho}}{2}$ for all $(y, t) \in Q^\infty_y$.

Before ending this introduction let us mention some related works from the literature. The local exact controllability of systems similar to (1.1) has been recently studied by Ervedoza et al. [8] using two boundary controls both for the density and the velocity. The stabilizability of the
system linearized about a steady state solution of the form \((\rho_s, v_s) = (\bar{\rho}, 0)\) has been proved in [2] and [6]. In [1], Amosova considers a compressible viscous fluid in one dimension in Lagrangian coordinates with zero boundary condition for the velocity on the boundaries of the interval \((0, 1)\) and an interior control on the velocity equation. She proves local exact controllability to trajectories for velocity, provided that the initial density is already on the “targeted trajectory”. Some stabilization results for hyperbolic systems may also be found in [7, Chapter 13].

To the best of our knowledge, the results of the paper are the first ones providing feedback control laws stabilizing compressible fluid flows. Moreover, since the class of feedback laws that we determine are quite explicit, they could be used numerically to determine these feedback controls.

The plan of the paper is as follows. In Section 2 we establish the equivalence between systems \((1.6)\) and \((1.10)\). In Section 3 we recall some properties of the linearized operator. The stabilizability of the linearized controlled system is studied in Section 3.3. The closed loop non-homogeneous linearized system is studied in Section 3.4. The proof of the main results are given in Section 4. We have collected in Appendix A the proofs of some estimates and Lipschitz properties of the nonlinear terms of system \((1.10)\).

2. Transformed system

We introduce the positive constants

\[
b = a \gamma \bar{\rho}^{\gamma - 2}, \quad v_0 = \frac{\nu}{\bar{\rho}},
\]

We endow the space \(Z = L^2(\Omega_x) \times L^2(\Omega_x)\) with the inner product

\[
\left\langle \left( \begin{array}{c} \rho \\ u \\ \sigma \\ v \end{array} \right) , \left( \begin{array}{c} \rho \\ u \\ \sigma \\ v \end{array} \right) \right\rangle_Z = b \int_0^\pi \rho(x)\sigma(x) \, dx + \bar{\rho} \int_0^\pi u(x)v(x) \, dx.
\]

We also introduce the space \(Z_m = L^2_m(\Omega_x) \times L^2(\Omega_x)\), equipped with the \(Z\)-inner product defined above.

As explained in the introduction, in order to deal with the nonlinear term \(e^{-\omega t} \hat{\sigma}_y \hat{v}\) in system \((1.6)\), we are going to make a change of variable associated with the flow corresponding to \(\hat{v}\). For any \(\hat{v}\), we consider the mapping \(Y_{\hat{v}}(\cdot, t)\) from \(\Omega_x\) to \(\Omega_y\) satisfying the ordinary differential equation

\[
\frac{\partial Y_{\hat{v}}}{\partial t}(x, t) = e^{-\omega t} \hat{v}(Y_{\hat{v}}(x, t), t), \quad Y_{\hat{v}}(x, 0) = x, \quad \text{for } t > 0.
\]

In order to study Eq. \((2.2)\), we introduce the function space

\[
\hat{V} = C_b([0, \infty); H^1_0(\Omega_y)) \cap L^2(0, \infty; H^2(\Omega_y)),
\]

and the subset

\[
\hat{V}_\omega = \left\{ \hat{v} \in \hat{V} \mid \|\hat{v}\|_{L^2(0, \infty; H^2(\Omega_y))} \leq \frac{\sqrt{\omega}}{2 \sqrt{2} S_0} \right\}.
\]
where \( s_0 \) is the continuity constant of the Sobolev embedding \( H^1(0, \pi) \hookrightarrow L^\infty(0, \pi) \). We are now in a position to state an existence and uniqueness result for Eq. (2.2).

**Lemma 2.1.** If \( \hat{v} \) belongs to \( \hat{V}_\omega \), then Eq. (2.2) admits a unique solution in \( C_b([0, \infty); L^2(\Omega_x)) \), and this solution obeys

\[
Y_{\hat{v}}(x, t) = x + \int_0^t e^{-\alpha s} \hat{v}(Y_{\hat{v}}(x, s), s) \, ds, \quad \text{for all } (x, t) \in Q_x^\infty. \tag{2.3}
\]

Moreover, for all \( (x, t) \in Q_x^\infty \), we have \( Y_{\hat{v}}(x, t) \in \Omega_y \) and \( Y_{\hat{v}} \in C_b([0, \infty); H^1(\Omega_y)) \cap C_b([0, \infty); H^2(\Omega_x)) \) with

\[
Y_{\hat{v}}(0, t) = 0, \quad Y_{\hat{v}}(\pi, t) = \pi, \quad \text{for all } t > 0.
\]

**Proof.** We will use an iterative process to show the existence and uniqueness of solution of (2.2). We start with \( Y^0_{\hat{v}}(x, t) = x \) and define the next iterates by

\[
Y^{j+1}_{\hat{v}}(x, t) = x + \int_0^t e^{-\alpha s} \hat{v}(Y^j_{\hat{v}}(x, s), s) \, ds, \quad j \geq 0. \tag{2.4}
\]

Then

\[
\left\| Y^1_{\hat{v}}(\cdot, t) - Y^0_{\hat{v}}(\cdot, t) \right\|_{L^2(\Omega_x)} = \left\| \int_0^t e^{-\alpha s} \hat{v}(Y^0_{\hat{v}}(\cdot, s), s) \, ds \right\|_{L^2(\Omega_x)} \\
\leq \sqrt{\pi} \int_0^t e^{-\alpha s} \| \hat{v}(\cdot, s) \|_{L^\infty(\Omega_y)} \, ds \\
\leq \frac{s_0 \sqrt{\pi}}{\sqrt{2\alpha}} \| \hat{v} \|_{L^2(0, \infty; H^2(\Omega_y))} \leq \frac{\sqrt{\pi}}{4}.
\]

Next we have

\[
\left\| Y^2_{\hat{v}}(\cdot, t) - Y^1_{\hat{v}}(\cdot, t) \right\|_{L^2(\Omega_x)} = \left\| \int_0^t e^{-\alpha s} (\hat{v}(Y^1_{\hat{v}}(\cdot, s), s) - \hat{v}(Y^0_{\hat{v}}(\cdot, s), s)) \, ds \right\|_{L^2(\Omega_x)} \\
\leq \int_0^t e^{-\alpha s} \| \hat{v} \|_{L^\infty(\Omega_y)} \| Y^1_{\hat{v}}(x, s) - Y^0_{\hat{v}}(x, s) \|_{L^2(\Omega_x)} \, ds \\
\leq \frac{\sqrt{\pi} s_0}{4 \sqrt{2\alpha}} \| \hat{v} \|_{L^2(0, \infty; H^2(\Omega_y))} \leq \frac{\sqrt{\pi}}{16}.
\]
Continuing this procedure we find
\[ \left\| Y_{j+1}^f (\cdot, t) - Y_j^f (\cdot, t) \right\|_{L^2 (\Omega_x)} \leq \frac{\sqrt{\pi}}{4j+1}. \]

Now, let \( Y_1, \hat{v} \) and \( Y_2, \hat{v} \) be two solutions of (2.2). Proceeding as above we obtain
\[ \left\| Y_1 (\cdot, t) - Y_2 (\cdot, t) \right\|_{L^2 (\Omega_x)}^2 \leq C \int_0^t \left\| Y_1 (\cdot, s) - Y_2 (\cdot, s) \right\|_{L^2 (\Omega_x)}^2 ds, \quad \text{for all } t > 0. \]

Thus by using Gronwall’s inequality, we deduce that (2.2) admits a unique solution in \( C_b ([0, \infty); L^2 (\Omega_x) \cap C_b ([0, \infty); H^2 (\Omega_x)) \) can be obtained with similar calculations. \( \square \)

We now discuss some properties of this transformation.

**Lemma 2.2.** Let \( \hat{v} \) belong to \( \hat{V}_\omega \). Then the solution \( Y_\hat{v} \) to Eq. (2.2) obeys
\[ \left\| \frac{\partial Y_\hat{v}}{\partial x} - 1 \right\|_{L^\infty (\Omega_x^\infty)} \leq \frac{1}{2}. \] (2.5)

**Proof.** Differentiating the ordinary differential equation (2.2) with respect to \( x \), we first obtain
\[ \frac{\partial}{\partial t} \frac{\partial Y_\hat{v}}{\partial x} (x, t) = e^{-\omega t} \frac{\partial \hat{v}}{\partial y} (Y_\hat{v}(x, t), t) \frac{\partial Y_\hat{v}}{\partial x} (x, t), \quad \frac{\partial Y_\hat{v}}{\partial x} (x, 0) = 1. \]

Hence we have
\[ \frac{\partial Y_\hat{v}}{\partial x} (x, t) = \exp \left( \int_0^t e^{-\omega s} \frac{\partial \hat{v}}{\partial y} (Y_\hat{v}(x, s), s) \, ds \right). \]

Notice that
\[ \left| \int_0^t e^{-\omega s} \frac{\partial \hat{v}}{\partial y} (Y_\hat{v}(x, s), s) \, ds \right| \leq \frac{1}{\sqrt{2\omega}} \left\| \hat{v} \right\|_{L^2 (0, \infty; L^\infty (\Omega_y))} \leq \frac{s_0}{\sqrt{2\omega}} \left\| \hat{v} \right\|_{L^2 (0, \infty; H^2 (\Omega_y))} \leq \frac{1}{4}. \]

Therefore (2.5) is proved. \( \square \)

**Corollary 2.3.** Let \( \hat{v} \) belong to \( \hat{V}_\omega \) and let \( Y_\hat{v} \) be the solution to (2.2). Then, for each \( t > 0 \), the mapping \( x \mapsto Y_\hat{v}(x, t) \) is a \( C^1 \) diffeomorphism from \( (0, \pi) \) onto \( (0, \pi) \). For \( t > 0 \), the inverse mapping of \( Y_\hat{v}(\cdot, t) \) is denoted by \( X_\hat{v}(\cdot, t) \). Therefore we have
\[ X_\hat{v}(Y_\hat{v}(x, t), t) = x \text{ for all } x \in \Omega_x, \quad Y_\hat{v}(X_\hat{v}(y, t), t) = y \text{ for all } y \in \Omega_y. \] (2.6)
If \((\tilde{\sigma}, \tilde{v})\) is a solution to system (1.6) corresponding to the control \(\hat{g}\) and if \(\tilde{v}\) belongs to \(\tilde{V}_\omega\), we set for \((x, t) \in Q_x^\infty\)

\[
\tilde{\sigma}(x, t) = \tilde{\sigma}(Y_{\tilde{T}}(x, t), t), \quad \tilde{v}(x, t) = \tilde{v}(Y_{\tilde{T}}(x, t), t) \quad \text{and} \quad \tilde{g}(x, t) = \tilde{g}(Y_{\tilde{T}}(x, t), t),
\]

(2.7)

where \(Y_{\tilde{T}}\) is the solution to (2.2).

When we use the change of variables introduced in (2.7), the nonlinear terms \(F_1\) and \(F_2\) appearing in system (1.10) are defined by

\[
F_1(\tilde{\sigma}, \tilde{v}, t) = \tilde{\rho} \tilde{v}_x \left( 1 - \left( \frac{\partial Y}{\partial x} \right)^{-1} \right) - e^{-\omega t} \tilde{\sigma} \tilde{v}_x \left( \frac{\partial Y}{\partial x} \right)^{-1},
\]

\[
F_2(\tilde{\sigma}, \tilde{v}, t) = \tilde{\sigma}_x \left( b - a \gamma (\tilde{\rho} + e^{-\omega t} \tilde{\sigma}) \gamma - 2 \left( \frac{\partial Y}{\partial x} \right)^{-1} - \frac{\gamma}{\tilde{\rho} + e^{-\omega t} \tilde{\sigma}} \frac{\partial^2 Y}{\partial x^2} \left( \frac{\partial Y}{\partial x} \right)^{-3} \tilde{v}_x \right)
\]

\[- \nu \tilde{v}_{xxx} \left( \frac{1}{\tilde{\rho}} - \frac{1}{\tilde{\rho} + e^{-\omega t} \tilde{\sigma}} \left( \frac{\partial Y}{\partial x} \right)^{-2} \right).\]

(2.8)

**Theorem 2.4.** Let \((\tilde{\sigma}, \tilde{v}) \in H^1(0, \infty; H^1_0(\Omega_y)) \times H^1(0, \infty; L^2(\Omega_y)) \cap L^2(0, \infty; H^2(\Omega_y)) \cap H^1_0(\Omega_y))\) be a solution to system (1.6) corresponding to the control \(\hat{g} \in L^2(0, \infty; L^2(\Omega_y))\). If \(\tilde{v} \in \tilde{V}_\omega\), then the triple \((\tilde{\sigma}, \tilde{v}, \tilde{g})\) defined in (2.7) belongs to \(H^1(0, \infty; H^1(\Omega_y)) \times (H^1(0, \infty; L^2(\Omega_y)) \cap L^2(0, \infty; H^2(\Omega_y)) \cap H^1_0(\Omega_y))) \times L^2(0, \infty; L^2(\Omega_y))\), and \((\tilde{\sigma}, \tilde{v}, \tilde{g})\) together with \((Y, X) = (Y_{\tilde{T}}, X_{\tilde{T}})\), as defined in Corollary 2.3, satisfy the system (1.10).

**Proof.** The proof follows from Lemma 2.2 and the chain rule differentiation formula. \(\Box\)

**Remark 2.5.** Notice that \(\tilde{\sigma}\) does not satisfy a zero average condition like (1.5) since the nonlinear term \(F_1\) does not satisfy a similar zero average condition.

Now we want to prove the converse of Theorem 2.4. In order to do that, we introduce

\[
\tilde{V} = L^2(0, \infty; H^2(\Omega_x) \cap H^1_0(\Omega_x)) \cap C_b([0, \infty); H^1_0(\Omega_x)),
\]

and

\[
\tilde{V}_\omega = \{ \tilde{v} \in \tilde{V} | \| \tilde{v} \|_{L^2(0, \infty; H^2(\Omega_x))} \leq \frac{\sqrt{\omega}}{2\delta_0} \}\).
\]

For all \(\tilde{v} \in \tilde{V}\), we set

\[
Y_{\tilde{T}}(x, t) := x + \int_0^t e^{-\omega s} \tilde{v}(x, s) \, ds.
\]

(2.9)

We have the following lemma.
Lemma 2.6. Let \( \tilde{v} \) belong to \( \tilde{V} \), and let \( Y_{\tilde{v}} \) be defined by (2.9). Then
\[
\left\| \frac{\partial Y_{\tilde{v}}}{\partial x} - 1 \right\|_{L^\infty(0,\infty;L^2(\Omega_\epsilon))} \leq \frac{1}{\sqrt{2\omega}} \| \tilde{v} \|_{L^2(0,\infty;H^2(\Omega_\epsilon))},
\]
(2.10)
\[
\left\| \frac{\partial^2 Y_{\tilde{v}}}{\partial x^2} \right\|_{L^\infty(0,\infty;L^2(\Omega_\epsilon))} \leq \frac{1}{\sqrt{2\omega}} \| \tilde{v} \|_{L^2(0,\infty;H^2(\Omega_\epsilon))},
\]
(2.11)
\[
\left\| \frac{\partial Y_{\tilde{v}}}{\partial x} - 1 \right\|_{L^\infty(0,\infty;H^1(\Omega_\epsilon))} \leq \frac{1}{\sqrt{2\omega}} \| \tilde{v} \|_{L^2(0,\infty;H^2(\Omega_\epsilon))}.
\]
(2.12)

Further if \( \tilde{v} \in \tilde{V}_\omega \), then we have
\[
\left\| \frac{\partial Y_{\tilde{v}}}{\partial x} - 1 \right\|_{L^\infty(Q_\infty)} \leq \frac{1}{2}.
\]
(2.13)

Proof. By differentiating (2.9), we have
\[
\frac{\partial Y_{\tilde{v}}}{\partial x}(x,t) = 1 + \int_0^t e^{-\omega s} \tilde{v}_x(x,s) \, ds.
\]
(2.14)

Now
\[
\left\| \frac{\partial Y_{\tilde{v}}}{\partial x}(\cdot,t) - 1 \right\|_{L^2(\Omega_\epsilon)}^2 = \int_{\Omega_\epsilon} \left( \int_0^t e^{-\omega s} \tilde{v}_x(x,s) \, ds \right)^2 \, dx
\]
\[
\leq \int_{\Omega_\epsilon} \left[ \left( \int_0^t e^{-2\omega s} \, ds \right) \left( \int_0^t |\tilde{v}_x(x,s)|^2 \, ds \right) \right] \, dx
\]
\[
\leq \frac{1 - e^{-2\omega t}}{2\omega} \| \tilde{v}_x \|_{L^2(0,\infty;L^2(\Omega_\epsilon))}^2.
\]

This implies (2.10). Similarly we can show (2.11). Thus (2.12) is proved. We also have
\[
\left\| \frac{\partial Y_{\tilde{v}}}{\partial x} - 1 \right\|_{L^\infty(Q_\infty)} \leq s_\theta \left\| \frac{\partial Y_{\tilde{v}}}{\partial x} - 1 \right\|_{L^\infty(0,\infty;H^1(\Omega_\epsilon))} \leq \frac{s_\theta}{\sqrt{\omega}} \| \tilde{v} \|_{L^2(0,\infty;H^2(\Omega_\epsilon))} \leq \frac{1}{2}. \quad \Box
\]

Corollary 2.7. Let \( \tilde{v} \) belong to \( \tilde{V}_\omega \) and let \( Y_{\tilde{v}} \) be defined by (2.9). Then, for each \( t > 0 \), the mapping \( x \mapsto Y_{\tilde{v}}(x,t) \) is a \( C^1 \) diffeomorphism from \( (0,\pi) \) onto \( (0,\pi) \). For \( t > 0 \), the inverse mapping of \( Y_{\tilde{v}}(\cdot,t) \) is denoted by \( X_{\tilde{v}}(\cdot,t) \). Therefore we have
\[
X_{\tilde{v}}(Y_{\tilde{v}}(x,t),t) = x \text{ for all } x \in \Omega_x, \quad Y_{\tilde{v}}(X_{\tilde{v}}(y,t),t) = y \text{ for all } y \in \Omega_y.
\]
(2.15)
Theorem 2.8. Let \( \tilde{g} \in L^2(0, \infty; L^2(\Omega_x)) \), \( (\tilde{\sigma}, \tilde{v}) \in H^1(0, \infty; H^1(\Omega_x)) \times H^1(0, \infty; L^2(\Omega_x)) \cap L^2(0, \infty; H^2(\Omega_x) \cap H^1(\Omega_x)) \), \( Y \in C^b_b([0, \infty); H^1(\Omega_x)) \cap C^b([0, \infty); H^2(\Omega_x)) \) and \( X \in C^1([0, \infty); H^1(\Omega_y)) \cap C^b_b([0, \infty); H^2(\Omega_y)) \) be a solution to the transformed system (1.10). Let us assume in addition that \( \tilde{v} \in \tilde{V}_\omega \), and let us set for \((y, t) \in Q^\infty \)

\[
\tilde{\sigma}(y, t) = \tilde{\sigma}(X(y, t), t), \\
\tilde{v}(y, t) = \tilde{v}(X(y, t), t), \\
\tilde{g}(y, t) = \tilde{g}(X(y, t), t).
\] (2.16)

Then, \( (\tilde{\sigma}, \tilde{v}) \in H^1(0, \infty; H^1_m(\Omega_y)) \times H^1(0, \infty; L^2(\Omega_y)) \cap L^2(0, \infty; H^2(\Omega_y) \cap H^1_0(\Omega_y)) \), \( \tilde{g} \in L^2(0, \infty; L^2(\Omega_y)) \), \( (\tilde{\sigma}, \tilde{v}, \tilde{g}) \) satisfy the original system (1.6),

\[
Y_{\tilde{v}} = Y \quad \text{and} \quad X_{\tilde{v}} = X.
\]

Proof. The proof follows from Lemma 2.6 and the chain rule differentiation formula. \( \square \)

Remark 2.9. Notice that \( \tilde{\ell}_j, \tilde{v}(t) \) satisfies the following identity

\[
\ell_j = \tilde{\ell}_j, \tilde{v}(t) + \int_0^t e^{-\omega s} \tilde{v}(\tilde{\ell}_j, \tilde{v}(t), s) \, ds \quad \text{for} \ j = 1, 2.
\] (2.17)

We now state a lemma which will be useful to determine a fixed control zone.

Lemma 2.10. Let \( \tilde{v} \in C_b([0, \infty); H^1_0(\Omega_x)) \cap L^2(0, T; H^2(\Omega_x)) \) and satisfy

\[
\| \tilde{v} \|_{L^2(0, \infty; H^2(\Omega_x))} \leq \min \left\{ \frac{\sqrt{2 \omega (\ell_2 - \ell_1)}}{8 s_0}, \frac{\sqrt{\omega}}{2 s_0} \right\}.
\] (2.18)

Then, we have

\[
| \tilde{\ell}_j, \tilde{v}(t) - \ell_j | \leq \frac{\ell_2 - \ell_1}{8}, \quad \text{for} \ j = 1, 2, \quad \text{and all} \ t > 0.
\]

If we set \( \mathcal{O} = \left( \frac{7 \ell_1 + \ell_2}{8}, \frac{7 \ell_2 + \ell_1}{8} \right) \), we have

\[
\mathcal{O} \subset \left( \tilde{\ell}_1, \tilde{v}(t), \tilde{\ell}_2, \tilde{v}(t) \right) \quad \text{for all} \ t > 0.
\]

Proof. Using (2.17), for \( j = 1, 2 \), we obtain

\[
| \tilde{\ell}_j, \tilde{v}(t) - \ell_j | = \left| \int_0^t e^{-\omega s} \tilde{v}(\tilde{\ell}_j, \tilde{v}(t), s) \, ds \right|.
\]
The proof is complete. □

We notice that if \( \tilde{v} \) belongs to \( C_b([0, \infty); H^1_0(\Omega_x)) \cap L^2(0, T; H^2(\Omega_x)) \) and satisfies (2.18), and if

\[
O = \left( \frac{7\ell_1 + \ell_2}{8}, \frac{7\ell_2 + \ell_1}{8} \right),
\]

(2.20)

then

\[
\chi(\ell_1, \ell_2) \tilde{g} = \chi O \tilde{g},
\]

(2.21)

for all \( \tilde{g} \in L^2(0, \infty; L^2(\Omega_x)) \) such that \( \text{supp} \tilde{g}(\cdot, t) \subset \chi O \). In that case, we can replace \( \chi(\ell_1, \ell_2, \ell_3, \ell_4) \tilde{g} \) by \( \chi O \tilde{g} \) in system (1.10).

From now on, we assume that \( O \) is defined by (2.20).

**Remark 2.11.** In Section 4, we prove Theorem 1.1 by the Banach fixed point theorem. Thus we obtain a unique solution \((\hat{\sigma}, \hat{v}, X, Y)\) to system (1.10) such that \( \hat{v} \) belongs to \( \hat{V}_0 \). Now applying Theorem 2.8, we obtain a solution \((\hat{\sigma}, \hat{v}, X, Y)\) to system (1.14), which is not necessarily unique. In order to obtain uniqueness of solution to system (1.14), we have to ensure that after the change of variable (2.16), \( \hat{v} \) belongs to \( \hat{V}_0 \). In the following lemma we will obtain a precise bound for \( \| \hat{v} \|_{L^2(0, \infty; H^2(\Omega_x))} \), denoted by \( C_\omega \), so that after the change of variable (2.16), \( \hat{v} \) belongs to \( \hat{V}_0 \). We will introduce the constant \( C_\omega \) in the definition of the ball \( D_\mu \), in which we seek the unique solution to system (1.10). This fact is taken into account in (4.3).

**Lemma 2.12.** Let \( \tilde{v} \) belong to \( \tilde{V}_0 \) and let us set \( \hat{v}(y, t) = \tilde{v}(X(y, t), t) \). If in addition

\[
\| \hat{v} \|_{L^2(0, \infty; H^2(\Omega_x))} \leq C_\omega = \frac{\sqrt{\omega}}{8(\sqrt{2} + 1)s_0},
\]

then \( \hat{v} \) belongs to \( \hat{V}_0 \).

**Proof.** We have \( \hat{v}(y, t) = \tilde{v}(X(y, t), t) \) and \( Y(X(y, t), t) = y \) for all \( y \in \Omega_y \). Differentiating with respect to \( y \), we obtain

\[
\frac{\partial \hat{v}}{\partial y} = \frac{\partial \tilde{v}}{\partial x} \frac{\partial x}{\partial y}, \quad \frac{\partial Y}{\partial x} = 1.
\]

(2.22)
Using Lemma 2.6, in particular $\frac{\partial Y}{\partial x} \geq \frac{1}{2}$, we have
\[\|\frac{\partial \hat{v}}{\partial y}\|_{L^2(\Omega^\infty_y)} \leq \|\frac{\partial \tilde{v}}{\partial y}\|_{L^2(\Omega^\infty_x)} \|\frac{\partial X}{\partial y}\|_{L^\infty(\Omega^\infty_y)} \leq 2 \|\frac{\partial \tilde{v}}{\partial x}\|_{L^2(\Omega^\infty_x)}. \tag{2.23}\]

Differentiating (2.22) with respect to $y$, we obtain
\[\frac{\partial^2 \hat{v}}{\partial y^2} = \frac{\partial^2 \tilde{v}}{\partial x^2} \left(\frac{\partial X}{\partial y}\right)^2 + \frac{\partial \tilde{v}}{\partial x} \frac{\partial^2 X}{\partial x^2} \frac{\partial Y}{\partial y} + \frac{\partial Y}{\partial x} \frac{\partial^2 X}{\partial x \partial y^2} = 0.\]

Using Lemma 2.6, we get
\[\left\|\frac{\partial^2 X}{\partial y^2}\right\|_{L^\infty(0,\infty;L^2(\Omega_x))} \leq 8 \left\|\frac{\partial^2 Y}{\partial x^2}\right\|_{L^\infty(0,\infty;L^2(\Omega_x))} \leq \frac{4\sqrt{2}}{\sqrt{\omega}} \left\|\tilde{v}\right\|_{L^2(0,\infty;H^2(\Omega_x))} \leq \frac{2\sqrt{2}}{s_0}, \tag{2.24}\]

Now we choose
\[C_\omega = \frac{\sqrt{\omega}}{8(\sqrt{2}+1)s_0}.\]

Thus using (2.23) and (2.24) we get
\[\left\|\tilde{v}\right\|_{L^2(0,\infty;H^2(\Omega_x))} \leq \left(4 + 2\sqrt{2}\right) \left\|\tilde{v}\right\|_{L^2(0,\infty;H^2(\Omega_x))} \leq \frac{\sqrt{\omega} \sqrt{2}}{2\sqrt{2}s_0}.\]

This completes the proof. \(\square\)

3. Stabilization of the linearized system

Now, we shall make the abuse of notation consisting in writing $\Omega$ for $\Omega_x$.

3.1. Linearized system

We define the unbounded operator $(A, \mathcal{D}(A; Z))$ in $Z$ by
\[\mathcal{D}(A; Z) = \{(\sigma, v)^T \in Z \mid v \in H^1_0(\Omega), (-b\sigma + v_0v') \in H^1(\Omega)\},\]
and
\[A = \begin{bmatrix} 0 & -\bar{\rho}\frac{d}{dx} \\ -b \frac{d}{dx} & v_0 \frac{d^2}{dx^2} \end{bmatrix}. \tag{3.1}\]
Since $Z_m$ is invariant under $(e^{tA})_{t \geq 0}$, the operator $A$ may be restricted to $Z_m$, and the domain of $A$ in $Z_m$ is $\mathcal{D}(A; Z_m) = \mathcal{D}(A; Z) \cap Z_m$. Let us first recall some properties of the unbounded operator $A$. See [6] for details of the proof using semigroup theory.

**Lemma 3.1.** The operator $A$ is maximal dissipative in $Z$. Thus, $(A, \mathcal{D}(A; Z))$ is the infinitesimal generator of a strongly continuous semigroup of contractions on $Z$. Moreover $(A, \mathcal{D}(A; Z_m))$ is also the infinitesimal generator of a strongly continuous semigroup of contractions on $Z_m$.

We introduce the control operator $B \in \mathcal{L}(L^2(\Omega), Z)$ defined by

$$B\tilde{g}(\cdot, t) := (0, \chi_{\Omega} \tilde{g}(\cdot, t))^T.$$

By setting $z(t) = (\tilde{\sigma}(\cdot, t), \tilde{v}(\cdot, t))^T$ and $z(0) = (\sigma_0, v_0)^T$, the linear control system (1.11) can be written as

$$z'(t) = (A + \omega I)z(t) + B\tilde{g}(\cdot, t), \quad z(0) = z_0 \in Z_m. \quad (3.2)$$

### 3.2. Spectral analysis of the linearized operator

To study regularity properties of linearized system, we consider the space

$$V = H^1_m(\Omega) \times H^1_0(\Omega),$$

endowed with the inner product

$$\left\langle \left(\begin{array}{c} \rho \\ u \end{array}\right), \left(\begin{array}{c} \sigma \\ v \end{array}\right) \right\rangle_V := b \int_0^\pi \rho'(x)\sigma'(x) \, dx + \rho \int_0^\pi u'(x)v'(x) \, dx.$$

Due to Poincaré–Wirtinger and Poincaré inequalities, the associated norm, denoted by $\|\cdot\|_V$ is equivalent to the usual norm of $H^1_m(\Omega) \times H^1_0(\Omega)$. We now consider the unbounded operator $(A, \mathcal{D}(A; V))$ in $V$ with

$$\mathcal{D}(A; V) = \{ (\sigma, v)^T \in V \mid v \in H^2(\Omega), \, (-b\sigma' + v_0 v'') \in H^1_0(\Omega) \}.$$

We have

**Lemma 3.2.** The operator $(A, \mathcal{D}(A; V))$ is maximal dissipative in $V$. Thus, it is the infinitesimal generator of a strongly continuous semigroup of contractions on $V$. In addition $(A, \mathcal{D}(A; V))$ is the infinitesimal generator of an analytic semigroup on $V$.

**Proof.** Let us consider the subspace

$$W = H^1_m(\Omega) \times (H^2(\Omega) \cap H^1_0(\Omega))$$

of $V$ with usual norm of $H^1_m(\Omega) \times H^2(\Omega)$. We define the bilinear form $B$ on $W \times W$ associated with the operator $A$ as:
\[
B\left(\begin{pmatrix} \rho \\ u \end{pmatrix}, \begin{pmatrix} \sigma \\ v \end{pmatrix}\right) = b \int_0^\pi \tilde{\rho} u_{xx} \sigma_x \, dx - \bar{\rho} \int_0^\pi (b \rho x - \nu_0 u_{xx}) v_{xx} \, dx.
\]

Clearly \( B \) is a continuous bilinear form on \( W \times W \). Notice that
\[
B\left(\begin{pmatrix} \sigma \\ v \end{pmatrix}, \begin{pmatrix} \sigma \\ v \end{pmatrix}\right) = \bar{\rho} \nu_0 \pi \int_0^\pi v_{xx}^2 \, dx.
\]

And so there exist \( \lambda_0 > 0 \) and \( \alpha > 0 \) such that
\[
B\left(\begin{pmatrix} \sigma \\ v \end{pmatrix}, \begin{pmatrix} \sigma \\ v \end{pmatrix}\right) + \lambda_0 \left\| \begin{pmatrix} \sigma \\ v \end{pmatrix} \right\|^2_V \geq \alpha \left\| \begin{pmatrix} \sigma \\ v \end{pmatrix} \right\|^2_W \quad \text{for all} \quad \begin{pmatrix} \sigma \\ v \end{pmatrix} \in W.
\]

Hence the lemma follows by using Theorem 2.12, Part II, of [5].

Let us also recall an abstract existence and regularity result for weak solutions of the Cauchy problem when \( A \) generates an analytic semigroup. This result is required in the next section.

**Theorem 3.3.** Let \( A \) be the infinitesimal generator of an analytic semigroup \( \{e^{tA}\}_{t \geq 0} \) of negative type defined on a domain \( \mathcal{D}(A) \) in the Hilbert space \( Z \). Then, if \( f \in L^2(0, \infty; Z) \), the Cauchy problem

\[
z'(t) = Az(t) + f(t), \quad z(0) = 0 \in Z,
\]

admits a unique weak solution

\[
z(t) = \int_0^t e^{(t-s)A} f(s) \, ds.
\]

This solution belongs to \( L^2(0, \infty; \mathcal{D}(A)) \cap H^1(0, \infty; Z) \).

**Proof.** See [5], Proposition 3.7 in Section 3.6 of Part II, Chapter 1.

We shall use this theorem for \( Z = Z_s \cap V \), where \( Z_s \) is a subspace of \( Z_m \).

Now we recall the spectral properties of \((A, \mathcal{D}(A; Z_m))\). We define a Fourier basis \( \{\Phi_n\}_{n \geq 1} \) in \( Z_m \) as follows

\[
\Phi_{2n-1}(x) = \sqrt{\frac{2}{b\pi}} (\cos(nx), 0)^T, \quad \Phi_{2n}(x) = \sqrt{\frac{2}{b\pi}} (0, \sin(nx))^T \quad \text{for} \quad n \geq 1. \tag{3.3}
\]

For all \( n \geq 1 \), let us define the following two dimensional space

\[
V_n = \text{span} \{\Phi_{2n}, \Phi_{2n-1}\}. \tag{3.4}
\]
Notice that $\Phi_n \in \mathcal{D}(A; Z_m)$ and $\Phi_n \in \mathcal{D}(A; V)$ for all $n \in \mathbb{N}$. Moreover the space $Z_m$ is the orthogonal sum of the finite dimensional spaces $(V_n)_{n \geq 1}$ (orthogonal for the inner product in $Z$), and $V$ is also the orthogonal sum of the spaces $(V_n)_{n \geq 1}$, but now the orthogonality is with respect to the inner product in $V$:

$$Z_m = \bigoplus_{n \geq 1} V_n, \quad V = \bigoplus_{n \geq 1} V_n.$$  

By writing $\sum$ we emphasize that the corresponding series converges in the $Z$-topology, while we write $\sum$ to mention that the corresponding series converges in the $V$-topology, even if the finite dimensional spaces are the same ones in both the decompositions.

We now summarize the properties of $A$, see [6] for details.

(i) The spectrum $\sigma(A)$ of $A$ except zero, lies in the left half side of the complex plane. It consists of a finite number of pairs of complex eigenvalues and an infinite number of pairs of real eigenvalues, $\{-\lambda_n, -\mu_n\}_{n \geq n_0}$ where $\lambda_n > 0$, $\mu_n > 0$ for $n \geq n_0$. For $1 \leq k < n_0$, the complex eigenvalues $\{-\lambda_k, -\mu_k = -\bar{\lambda}_k\}$ satisfy

$$|\text{Re}(\lambda_k)| = \frac{v_0 k^2}{2} \geq \frac{v_0}{2}, \quad |\text{Im}(\lambda_k)| \leq 2\omega_0.$$  

For $n \geq n_0$, the eigenvalues are real and they satisfy

$$\lim_{n \to \infty} \lambda_n = \omega_0, \quad \lim_{n \to \infty} \frac{\mu_n}{n^2} = v_0. \quad (3.5)$$

(ii) The linearized system (3.2) is stable with $\tilde{g} = 0$ if

$$0 < \omega < \min\left\{ \frac{v_0}{2}, \omega_0 \right\}.$$  

In the case when $\omega_0 \leq v_0/2$, according to (ii), the linearized system (3.2) is stable with $\tilde{g} = 0$ and $0 < \omega < \omega_0$. If $\omega > \omega_0$, the linearized system (3.2) is unstable with $\tilde{g} = 0$, and it is not stabilizable by controls $\tilde{g} \in L^2(0, \infty; L^2(\Omega))$. Thus, the only interesting case is when $\frac{v_0}{2} < \omega_0$. This is why, in this paper, we only consider that case. From now on, we assume that $\omega$ is such that

$$\frac{v_0}{2} < \omega < \omega_0. \quad (3.6)$$

### 3.3. Feedback control for the linear system

Notice that the eigenvalues of

$$A_\omega := A + \omega I$$

are $\{-\lambda_n + \omega\}$ where $-\lambda_n$ is an eigenvalue of $A$. By our choice of $\omega$, made in (3.6), a finite number of eigenvalues of $A_\omega$, say $N$, are with nonnegative real part. Let us set
We have the following orthogonal decomposition

\[ Z_m = Z_u \oplus Z_s, \]

where \( Z_u \) is the unstable subspace of \( A_\omega \) of dimension \( 2N \), and \( Z_s \) is the stable subspace. Both are invariant under \( A_\omega \).

We define the orthogonal projection \( \pi_u \) from \( Z_m \) onto \( Z_u \) and the orthogonal projection \( \pi_s \) from \( Z_m \) onto \( Z_s \), and we set

\[ \pi_u z_u = z_u, \quad \pi_s z_s = z_s. \]

Since \( Z_u \) and \( Z_s \) are both invariant under \( A_\omega \), we have

\[ \pi_u A_\omega = A_\omega \pi_u = \pi_u A_\omega \pi_u, \quad \pi_s A_\omega = A_\omega \pi_s = \pi_s A_\omega \pi_s. \]

Now by our choice, \( \omega \in \left( \frac{\nu_0}{2}, \omega_0 \right) \), eigenvalues of \( \pi_s (A_\omega) \) remain on the left half of the complex plane. It consists of a finite number of pairs of complex eigenvalues \( \{ -\lambda_k + \omega, -\mu_k + \omega = -\lambda_k + \omega \} \), for \( N < k < n_0 \), and an infinite number of pairs of real eigenvalues, \( \{ -\lambda_n + \omega, -\mu_n + \omega \} \). Then, proceeding as in Corollary 1 of [2], we obtain

\[ \| e^{t \pi_s A_\omega} \| \leq C e^{-\gamma_1 t}, \quad (3.7) \]

where \( \gamma_1 = (\text{Re}(\lambda_{n+1}) - \omega) > 0. \)

**Proposition 3.4.** \((A_\omega, B)\) is stabilizable in \( Z_m \).

**Proof.** We want to use Proposition 3.3, Part V of [5]. For that, we are going to verify the following condition

\[ \text{Ker}(\lambda I - A_\omega^* \pi_u) \cap \text{Ker}(B^*) = \{0\} \text{ for all } \lambda \in \sigma(A_\omega) \text{ with Re} \lambda \geq 0. \quad (3.8) \]

The adjoint \((A_\omega^*, D(A^*; Z_m))\) of the operator \((A_\omega, D(A; Z_m))\) in \( Z_m \) is defined by

\[ D(A^*; Z_m) = \{ (\sigma, v)^T \in Z_m \mid v \in H^1_0(\Omega), (b \sigma + \nu_0 v') \in H^1(\Omega) \} \]

and

\[ A_\omega^* = \begin{pmatrix} \omega & \tilde{\rho} \frac{d}{dx} \\ b \frac{d}{dx} & \omega + \nu_0 \frac{d^2}{dx^2} \end{pmatrix}. \]

Observe that \( A_\omega^* \) has the same spectrum as that of \( A_\omega \). The eigenfunctions of \( A_\omega^* \) corresponding to \(-\lambda_n + \omega\) and \(-\mu_n + \omega\) are respectively of the form
\[ \xi_n^*(x) = C \left( \cos(nx), \frac{-\lambda_n}{n \bar{\rho}} \sin(nx) \right)^T, \quad \zeta_n^*(x) = D \left( \cos(nx), \frac{-\mu_n}{n \bar{\rho}} \sin(nx) \right)^T, \]

where \(-\lambda_n, -\mu_n\) are the eigenvalues of \(A\), and \(C\) and \(D\) are arbitrary constants.

The operator \(B^*\) from \(Z\) to \(L^2(\Omega)\) is defined by

\[ B^* (f_1, f_2)^T = \chi_{\Omega} f_2. \]

Now observe that \(B^* \xi_n^* = 0\) implies that

\[ C \frac{\lambda_n}{n \bar{\rho}} \sin(nx) = 0 \text{ for all } x \in \Omega. \]

Thus \(C = 0\). Similarly we can get \(D = 0\). Therefore, we have checked (3.8). Hence \((A_\omega, B)\) is stabilizable in \(Z_m\).

Applying the projections \(\pi_u\) and \(\pi_s\) to (3.2), we obtain a finite dimensional system in \(Z_u\)

\[ z_u' = \pi_u A_\omega z_u + \pi_u B \tilde{g}, \quad z_u(0) = \pi_u z_0 = z_{0,u}, \quad (3.9) \]

and an infinite dimensional system in \(Z_s\)

\[ z_s' = \pi_s A_\omega z_s + \pi_s B \tilde{g}, \quad z_s(0) = \pi_s z_0 = z_{0,s}. \quad (3.10) \]

As \((A_\omega, B)\) is stabilizable in \(Z_m\), \((\pi_u A_\omega, \pi_u B)\) is stabilizable in \(Z_u\). Therefore there exists a feedback \(K_u \in \mathcal{L}(Z_u; L^2(\Omega))\) such that \((\pi_u A_\omega + \pi_u B K_u)\) is stable. Let us define \(K_m \in \mathcal{L}(Z_m; L^2(\Omega))\) by

\[ K_m := K_u \pi_u. \quad (3.11) \]

By choosing \(\tilde{g}(\cdot, t)\) of the form \(K_m z(t)\) in (3.2), we obtain the system

\[ z'(t) = A_\omega z(t) + B K_m z(t), \quad z(0) = z_0. \quad (3.12) \]

**Theorem 3.5.** There exist positive constants \(\gamma_0\) and \(C\), independent of \(t\) such that for all \(z_0 \in Z_m\), the solution of (3.12) satisfies

\[ \|z(t)\|_{Z_m} \leq C e^{-\gamma_0 t} \|z_0\|_{Z_m}. \quad (3.13) \]

**Proof.** The proof follows from [15, Theorem 6.1]. \(\square\)

**Remark 3.6.** The operator \(K_u\) and \(K_m\) can be obtained by solving a Riccati equation (see e.g. [10]). In that case, \(K_m\) can be expressed by its kernel (see e.g. [11]). Thus, there exist \(k_\sigma \in L^2(\Omega_x \times \Omega_x)\) and \(k_v \in L^2(\Omega_x \times \Omega_x)\) such that

\[ (K_m(\sigma, v)^T)(x) = \int_0^\pi k_\sigma(x, \xi) \sigma_m(\xi) d\xi + \int_0^\pi k_v(x, \xi) v(\xi) d\xi, \quad (3.14) \]
for all $\sigma_m \in L^2_m(\Omega_x)$ and all $v \in L^2(\Omega_x)$. Notice that $k_\sigma$ is such that the mappings $x \mapsto k_\sigma(x, \xi)$ and $\xi \mapsto k_\sigma(x, \xi)$ belong to $L^2_m(\Omega_x)$.

**Theorem 3.7.** Let $z_0 = (\sigma_0, v_0)$ belong to $H^1_m(\Omega) \times H^1_0(\Omega)$. Then the system (3.12) has a unique solution $z \in C([0, \infty); H^1_m(\Omega) \times H^1_0(\Omega))$ and the following estimate holds

$$\left\| \tilde{\sigma} \right\|_{H^1(0, \infty; H^1(\Omega))} + \left\| \tilde{v} \right\|_{L^2(0, \infty; H^2(\Omega))} + \left\| \tilde{v} \right\|_{H^1(0, \infty; L^2(\Omega))} \leq C_1 \left\| (\sigma_0, v_0)^T \right\|_{H^1_m(\Omega) \times H^1_0(\Omega)}. \tag{3.15}$$

**Proof.** Applying the projection $\pi_u$ to (3.12) we first obtain

$$z'_u(t) = (\pi_u A_\omega + \pi_u B K_u) z_u(t), \quad z_u(0) = z_{0,u}. \tag{3.16}$$

We have

$$\left\| z'_u(t) \right\|_{Z_m} = \left\| (\pi_u A_\omega + \pi_u B K_u) z_u(t) \right\|_{Z_m} \leq C e^{-\gamma_0 t} \left\| z_0 \right\|_{Z_m}.$$

Combining the above estimate with (3.13), we get

$$\left\| z_u \right\|_{H^1(0, \infty; Z_m)} \leq C \left\| z_0 \right\|_{Z_m}.$$

Since the norms $\| \cdot \|_{L^2(\Omega) \times L^2(\Omega)}$ and $\| \cdot \|_{H^1_m(\Omega) \times H^2(\Omega)}$ are equivalent in the finite dimensional space $Z_u$, the solution of system (3.16) satisfies

$$\left\| z_u \right\|_{H^1(0, \infty; H^1_m(\Omega) \times H^2(\Omega))} \leq C \left\| z_0 \right\|_V. \tag{3.17}$$

Applying the projection $\pi_s$ to (3.12) we obtain

$$z'_s(t) = \pi_s A_\omega z_s(t) + \pi_s B K_m z(t), \quad z_s(0) = z_{0,s}. \tag{3.18}$$

Let us notice that

$$\pi_s B K_m z(t) = \pi_s (0, \chi \circ K_u z_u(t))^T.$$

Using (3.17), we have

$$\| \chi \circ K_u z_u \|_{L^2(0, \infty; L^2(\Omega))} \leq C \| z_u \|_{L^2(0, \infty; Z_m)} \leq C \| z_0 \|_V.$$

We also see that $\pi_s A_\omega$ generates a $C_0$-semigroup of negative type (see (3.7)). Let us set $(\tilde{\sigma}_s, \tilde{v}_s)^T := z_s$. With Proposition 3 of [2], we have

$$\left\| \tilde{\sigma}_s \right\|_{H^1(0, \infty; H^1_m(\Omega))} + \left\| \tilde{v}_s \right\|_{L^2(0, \infty; L^2(\Omega))} + \left\| \tilde{v}_s \right\|_{H^1(0, \infty; H^2(\Omega))} \leq C \| z_0 \|_V. \tag{3.19}$$

Thus, (3.15) follows from (3.17) and (3.19).
3.4. Nonhomogeneous closed loop linear system

In order to study the local stabilization of the nonlinear system (1.10) by the feedback control law defined in the previous section, we are going to study a nonhomogeneous closed loop linear system. As already explained in Remark 2.5, due to the change of variable, the nonlinear term $F_1$ in (2.8) and also $\tilde{\sigma}$ do not satisfy a zero average condition. Thus we look for $\tilde{\sigma}$ in the form

$$\tilde{\sigma}(x,t) = \tilde{\sigma}_m(x,t) + \tilde{\sigma}_\Omega(t)$$

where $\int_{\Omega} \tilde{\sigma}_m(x,t) dx = 0$ and $\tilde{\sigma}_\Omega(t) = \frac{1}{\pi} \int_{\Omega} \tilde{\sigma}(x,t) dx$. (3.20)

We also introduce the following space

$$L^\infty(0,\infty; e^{-\omega(\cdot)}) = \{h : e^{-\omega(\cdot)}h \in L^\infty(0,\infty)\}.$$ 

Since the nonlinear term $F_1(\tilde{\sigma}, \tilde{\nu}, t)$ does not belong to $L^2(0,\infty; H^1_m(\Omega))$, we have to consider right hand sides of the form

$$f_1 = f_{1,m} + f_{1,\Omega},$$

where $f_{1,m} \in L^2(0,\infty; H^1_m(\Omega))$, $f_{1,\Omega}(t) := \frac{1}{\pi} \int_{\Omega} f_1(x,t) \, dx$ and

$$f_{1,\Omega} \in L^1(0,\infty; e^{-\omega(\cdot)}) = \{h : e^{-\omega(\cdot)}h \in L^1(0,\infty)\}.$$

We are going to study the feedback stabilization of the following nonhomogeneous closed loop linear system

$$\tilde{\sigma}_t + \tilde{\rho} \tilde{\nu}_x - \omega \tilde{\sigma} = f_{1,m} + f_{1,\Omega}, \text{ in } \Omega \times (0,\infty),$$

$$\tilde{\sigma}(t) = \tilde{\sigma}_m(t) + \tilde{\sigma}_\Omega(t) \text{ with } \tilde{\sigma}_\Omega(t) := \frac{1}{\pi} \int_{\Omega} \tilde{\sigma}(x,t) \, dx, \text{ \forall } t > 0,$$

$$\tilde{\nu}_t + b \tilde{\sigma}_x - {\nu}_0 \tilde{\nu}_{xx} - \omega \tilde{\nu} = f_2 + \chi_{\Omega} K_m(\tilde{\sigma}_m(\cdot,t), \tilde{\nu}(\cdot,t))^T, \text{ in } \Omega \times (0,\infty),$$

$$\tilde{\sigma}(0) = \tilde{\nu}(0) = 0, \text{ in } \Omega,$$

$$\tilde{\nu}(0,t) = 0, \text{ in } \Omega, \text{ \forall } t > 0.$$ (3.21)

Let us notice that, since $K_m \in \mathcal{L}(Z_m, L^2(\Omega))$, the operator $K_m$ must be applied to $(\tilde{\sigma}_m, \tilde{\nu})$ and not to $(\tilde{\sigma}, \tilde{\nu})$. It is convenient to define $K \in \mathcal{L}(Z, L^2(\Omega))$ by

$$K(\sigma, \nu)(x) = \chi_{\Omega} K_m \left( \sigma - \frac{1}{\pi} \int_{\Omega} \sigma(\xi) d\xi, \nu \right)^T(x),$$ (3.22)

for all $\sigma \in L^2(\Omega_x)$ and all $\nu \in L^2(\Omega_x)$. 

Remark 3.8. From Remark 3.6, it follows that if $K_m$ is determined by (3.14), then we have

$$K(\sigma, v)(x) = \int_0^\pi k_\sigma(x, \xi) \sigma(\xi) d\xi + \int_0^\pi k_v(x, \xi) v(\xi) d\xi,$$

for all $\sigma \in L^2(\Omega_x)$ and all $v \in L^2(\Omega_x)$, where $k_\sigma$ and $k_v$ are the kernels appearing in (3.14).

Theorem 3.9. Let $f_{1,m} \in L^2(0, \infty; H^1_m(\Omega))$, $f_{1,\Omega} \in L^1(0, \infty; e^{-\omega(\cdot)})$ and $f_2 \in L^2(0, \infty; L^2(\Omega))$. Then the system (3.21) admits a unique solution satisfying

$$\|\tilde{\sigma}_m\|_{L^1(0, \infty; H^1_m(\Omega))} + \|\tilde{\sigma}\|_{L^\infty(0, \infty; e^{-\omega(\cdot)})} + \|\hat{v}\|_{L^2(0, \infty; H^2(\Omega))} + \|\hat{v}\|_{H^1(0, \infty; L^2(\Omega))} \leq C_2(\|f_{1,m}\|_{L^2(0, \infty; H^1_m(\Omega))} + \|f_{1,\Omega}\|_{L^1(0, \infty; e^{-\omega(\cdot)})} + \|f_2\|_{L^2(0, \infty; L^2(\Omega))}).$$

(3.23)

Proof. Let us define $F(t) = (f_{1,m}(\cdot, t), f_2(\cdot, t))$. Then the system (3.21) with $f_{1,\Omega} \equiv 0$ can be written as

$$z'(t) = A_{\omega}z(t) + F(t) + (0, \chi \sigma Kz(t))^T, \quad z(0) = 0,$$

with $z = (\tilde{\sigma}_m, \hat{v})^T$. Applying the projection $\pi_u$ we have

$$z_u'(t) = (\pi_u A_{\omega} + \pi_u B K) z_u(t) + \pi_u F(t), \quad z_u(0) = 0.$$

Proceeding as in Theorem 3.7, we see that the solution $z_u$ satisfies

$$\|z_u\|_{H^1(0, \infty; H^1_m(\Omega) \times H^2(\Omega))} \leq C(\|f_{1,m}\|_{L^2(0, \infty; H^1_m(\Omega))} + \|f_2\|_{L^2(0, \infty; L^2(\Omega))}).$$

(3.25)

Now applying $\pi_s$ to Eq. (3.21) we have

$$z_s'(t) = \pi_s A_{\omega}z_s(t) + \pi_s F(t) + \pi_s (0, Kz(t))^T, \quad z_s(0) = 0.$$

We notice that

$$\|\pi_s (0, Kz(t))^T\|_{L^2(0, \infty; L^2(\Omega))} \leq C \|z_u\|_{L^2(0, \infty; Z_m)}$$

$$\leq C(\|f_{1,m}\|_{L^2(0, \infty; H^1_m(\Omega))} + \|f_2\|_{L^2(0, \infty; L^2(\Omega))}).$$

(3.26)

Let us set $z_s = (\tilde{\sigma}_{m,s}, \hat{v}_s)^T$. Writing $\pi_s F(t) = (f_{1,m,s}(\cdot, t), f_{2,s}(\cdot, t))$, let us first consider the system

$$z_s'(t) = \pi_s A_{\omega}z_s(t) + (0, f_{2,s}(\cdot, t))^T + \pi_s (0, Kz(t))^T, \quad z_s(0) = 0.$$
Then by Proposition 3 of [2] and (3.26), we have

\[
\|\tilde{\sigma}_{m,s}\|_{H^1(0,\infty; H^1_m(\Omega))} + \|\tilde{v}_s\|_{L^2(0,\infty; H^2(\Omega))} + \|\tilde{v}_s\|_{H^1(0,\infty; L^2(\Omega))}
\leq C(\|f_2\|_{L^2(0,\infty; L^2(\Omega))} + \|\pi_s(0, Kz(t))T\|_{L^2(0,\infty; L^2(\Omega))}).
\]

(3.27)

Now consider the system

\[
z'_s(t) = \pi_s A \omega z_s(t) + (f_{1,m,s}, 0)^T, \quad z_s(0) = 0.
\]

Notice that we can make use of Theorem 3.3 for this equation with \( Z = Z_s \cap V \), because the decomposition \( V = \bigoplus_{n \geq 1} V_n \) is an orthogonal decomposition in \( V \) into invariant subspaces for \( (e^{tA})_{t \geq 0} \). As \( f_{1,m,s} \in L^2(0, \infty; Z_s \cap V) \), from Lemma 3.2 and Theorem 3.3, it follows that \( z_s \in L^2(0, \infty; D(A; V) \cap Z_s) \cap H^1(0, \infty; Z_s \cap V) \), and

\[
\|\tilde{\sigma}_{m,s}\|_{H^1(0,\infty; H^1_m(\Omega))} + \|\tilde{v}_s\|_{L^2(0,\infty; H^2(\Omega))} + \|\tilde{v}_s\|_{H^1(0,\infty; L^2(\Omega))} \leq C \|f_1\|_{L^2(0,\infty; H^1_m(\Omega))}.
\]

(3.28)

Now we consider the following equation

\[
\tilde{\sigma}_i - \omega \tilde{\sigma} = f_{1,\Omega}(t), \quad \text{in} \ \Omega \times (0, \infty), \quad \tilde{\sigma}(0) = 0 \quad \text{in} \ \Omega.
\]

(3.29)

The above equation has unique solution \( \tilde{\sigma}_\Omega \) defined by

\[
\tilde{\sigma}_\Omega(t) = e^{\omega t} \int_0^t e^{-\omega s} f_{1,\Omega}(s) \, ds,
\]

and we have

\[
\|\tilde{\sigma}_\Omega\|_{L^\infty(0,\infty; e^{-\omega(\cdot)})} \leq \|f_{1,\Omega}\|_{L^1(0,\infty; e^{-\omega(\cdot)})}.
\]

(3.31)

Thus (3.23) follows from (3.25), (3.27), (3.28) and (3.31).

\( \square \)

4. Nonlinear system

4.1. Estimates of nonlinear terms

Now we collect some estimates required later on. Let us recall that \( \tilde{\sigma}(x,t) = \tilde{\sigma}_m(x,t) + \tilde{\sigma}_\Omega(t) \), with \( \tilde{\sigma}_\Omega(t) = \frac{1}{\pi} \int_\Omega \tilde{\sigma}(x,t) \, dx \).

**Lemma 4.1.** If \( \tilde{\sigma}(x,t) = \tilde{\sigma}_m(x,t) + \tilde{\sigma}_\Omega(t) \) with \( \tilde{\sigma}_m \in H^1(0,\infty; H^1_m(\Omega)) \) and \( \tilde{\sigma}_\Omega \in L^\infty(0,\infty; e^{-\omega(\cdot)}) \) and if
\[ \| \tilde{s}_m \|_{H^1(0, \infty; H^1_m(\Omega))} + \| \tilde{\sigma}_{\Omega} \|_{L^\infty(0, \infty; e^{-\omega t}(\cdot))} \leq \min \left\{ \frac{\tilde{\rho}}{4s_0s_1}, \frac{\tilde{\rho}}{4} \right\}, \]

(4.1)

where \( s_1 \) is the constant of the Sobolev embedding \( H^1(0, \infty) \hookrightarrow L^\infty(0, \infty) \), then we have

\[ |e^{-\omega t} \tilde{s}(x, t)| \leq \frac{\tilde{\rho}}{2} \quad \text{for all} \quad (x, t) \in \Omega \times (0, \infty). \]

(4.2)

**Proof.** Using (4.1), we have

\[ |e^{-\omega t} \tilde{s}(x, t)| \leq s_0s_1 \| \tilde{s}_m \|_{H^1(0, \infty; H^1_m(\Omega))} + \| \tilde{\sigma}_{\Omega} \|_{L^\infty(0, \infty; e^{-\omega t}(\cdot))} \leq \frac{\tilde{\rho}}{2}. \]

\[ \square \]

Conditions (4.1), (2.18) and Lemma 2.12 motivate us to introduce the two constants

\[ \mu_1 = \min \left\{ \frac{\tilde{\rho}}{4s_0s_1}, \frac{\tilde{\rho}}{4} \right\} \quad \text{and} \quad \mu_2 = \min \left\{ \frac{\sqrt{\omega}}{2s_0}, \frac{\sqrt{2\omega(\ell_2 - \ell_1)}}{8s_0}, C_\omega \right\}. \]

(4.3)

Now we estimate the nonlinear terms \( F_1(\tilde{s}, \tilde{v}, t) \) and \( F_2(\tilde{s}, \tilde{v}, t) \) given by (2.8) and also derive the Lipschitz estimates for these terms when \( (\tilde{s}, \tilde{v}) \) belongs to the ball \( D_{\bar{\mu}} \) defined by

\[ D_{\bar{\mu}} = \left\{ (\tilde{s}, \tilde{v}) \mid \tilde{s} = \tilde{s}_m + \tilde{\sigma}_{\Omega}, \tilde{\sigma}_{\Omega} \in L^\infty(0, \infty; e^{-\omega t}(\cdot)), \tilde{s}_m \in H^1(0, \infty; H^1_m(\Omega_x)), \right. \]
\[ \left. \tilde{v} \in H^1(0, \infty; L^2(\Omega_x)) \cap L^2(0, \infty; H^2(\Omega_x) \cap H^1_m(\Omega_x)), \right. \]
\[ \left. \| \tilde{s}_m \|_{H^1(0, \infty; H^1_m(\Omega_x))} + \| \tilde{\sigma}_{\Omega} \|_{L^\infty(0, \infty; e^{-\omega t}(\cdot))} \leq \bar{\mu}, \right. \]
\[ \left. \| \tilde{v} \|_{H^1(0, \infty; L^2(\Omega_x))} + \| \tilde{v} \|_{L^2(0, \infty; H^2(\Omega_x))} \leq \bar{\mu} \right\}. \]

(4.4)

From now on we assume that

\[ \bar{\mu} \leq \min\{\mu_1, \mu_2\}. \]

(4.5)

Let us set

\[ F_{1, \Omega}(\tilde{s}, \tilde{v}, t) = \frac{1}{\pi} \int_\Omega F_1(\tilde{s}, \tilde{v}, t) \, dx \quad \text{and} \quad F_{1, m} = F_1 - F_{1, \Omega}. \]

(4.6)

We need to estimate \( \| F_{1, m} \|_{L^2(0, \infty; H^1(\Omega))}, \| F_{1, \Omega} \|_{L^1(0, \infty; e^{-\omega t}(\cdot))} \) and \( \| F_2 \|_{L^2(0, \infty; L^2(\Omega))} \). Notice that by Poincaré–Wirtinger inequality \( \| F_{1, m} \|_{L^1(0, \infty; H^1(\Omega))} \) is equivalent to \( \left\| \frac{\partial F_1}{\partial x} \right\|_{L^2(0, \infty; L^2(\Omega))}. \)

Now we list the required estimates needed to prove the existence of solution to system (1.10). Proofs of these estimates are given in Appendix A.

In the following lemmas we give estimates for the nonlinear terms under the assumption (4.5).

**Lemma 4.2.** There exists a positive constant \( C_3 = C_3(s_0, s_1, \tilde{\rho}, \omega) \) such that, for all \( (\tilde{s}, \tilde{v}) \in D_{\bar{\mu}}, \) we have
\[ \| F_{1,m} (\tilde{\sigma}, \tilde{v}, \cdot) \|_{L^2(0, \infty; H^1(\Omega))} \leq C_3 (\| \tilde{\sigma}_m \|_{H^1(0, \infty; H^1(\Omega))}^2 + \| \tilde{\sigma}_\Omega \|_{L^\infty(0, \infty; e^{-\omega(t)})}^2) + \| \tilde{v} \|_{L^2(0, \infty; H^2(\Omega))}^2. \] 

(4.7)

Further there exists a positive constant \( C_4 = C_4(s_0, s_1, \tilde{\rho}, \omega) \) such that, for \( (\tilde{\sigma}^1, \tilde{v}^1) \in D_{\tilde{\mu}} \) and \( (\tilde{\sigma}^2, \tilde{v}^2) \in D_{\tilde{\mu}} \), we have

\[ \| F_{1,m} (\tilde{\sigma}^1, \tilde{v}^1, \cdot) - F_{1,m} (\tilde{\sigma}^2, \tilde{v}^2, \cdot) \|_{L^2(0, \infty; H^1(\Omega))} \leq C_4 (\| \tilde{\sigma}_m \|_{H^1(0, \infty; H^1(\Omega))}^2 + \| \tilde{\sigma}_\Omega \|_{L^\infty(0, \infty; e^{-\omega(t)})}^2) + \| \tilde{v}^1 - \tilde{v}^2 \|_{L^2(0, \infty; H^2(\Omega))}^2. \] 

(4.8)

**Lemma 4.3.** There exists a positive constant \( C_5 = C_5(\tilde{\rho}, \omega) \) such that, for all \( (\tilde{\sigma}, \tilde{v}) \in D_{\tilde{\mu}} \), we have

\[ \| F_{1,\Omega} (\tilde{\sigma}, \tilde{v}, \cdot) \|_{L^1(0, \infty; e^{-\omega(t)})} \leq C_5 (\| \tilde{\sigma}_m \|_{H^1(0, \infty; H^1(\Omega))}^2 + \| \tilde{\sigma}_\Omega \|_{L^\infty(0, \infty; e^{-\omega(t)})}^2) + \| \tilde{v} \|_{L^2(0, \infty; H^2(\Omega))}^2. \] 

(4.9)

Further there exists a positive constant \( C_6 = C_6(\tilde{\rho}, \omega) \) such that, for \( (\tilde{\sigma}^1, \tilde{v}^1) \in D_{\tilde{\mu}} \) and \( (\tilde{\sigma}^2, \tilde{v}^2) \in D_{\tilde{\mu}} \), we have

\[ \| F_{1,\Omega} (\tilde{\sigma}^1, \tilde{v}^1, \cdot) - F_{1,\Omega} (\tilde{\sigma}^2, \tilde{v}^2, \cdot) \|_{L^1(0, \infty; e^{-\omega(t)})} \leq C_6 (\| \tilde{\sigma}_m \|_{H^1(0, \infty; H^1(\Omega))}^2 + \| \tilde{\sigma}_\Omega \|_{L^\infty(0, \infty; e^{-\omega(t)})}^2) + \| \tilde{v}^1 - \tilde{v}^2 \|_{L^2(0, \infty; H^2(\Omega))}^2. \] 

(4.10)

**Lemma 4.4.** There exists a positive constant \( C_7 = C_7(s_0, s_1, \tilde{\rho}, \omega, \gamma) \) such that, for all \( (\tilde{\sigma}, \tilde{v}) \in D_{\tilde{\mu}} \), we have

\[ \| F_2 (\tilde{\sigma}, \tilde{v}, \cdot) \|_{L^2(0, \infty; L^2(\Omega))} \leq C_7 (\| \tilde{\sigma}_m \|_{H^1(0, \infty; H^1(\Omega))}^2 + \| \tilde{\sigma}_\Omega \|_{L^\infty(0, \infty; e^{-\omega(t)})}^2) + \| \tilde{v} \|_{L^2(0, \infty; H^2(\Omega))}^2. \] 

(4.11)

Further there exists a positive constant \( C_8 = C_8(s_0, s_1, \tilde{\rho}, \omega, \gamma) \) such that, for \( (\tilde{\sigma}^1, \tilde{v}^1) \in D_{\tilde{\mu}} \) and \( (\tilde{\sigma}^2, \tilde{v}^2) \in D_{\tilde{\mu}} \), we have

\[ \| F_2 (\tilde{\sigma}^1, \tilde{v}^1, \cdot) - F_2 (\tilde{\sigma}^2, \tilde{v}^2, \cdot) \|_{L^2(0, \infty; L^2(\Omega))} \leq C_8 (\| \tilde{\sigma}_m \|_{H^1(0, \infty; H^1(\Omega))}^2 + \| \tilde{\sigma}_\Omega \|_{L^\infty(0, \infty; e^{-\omega(t)})}^2) + \| \tilde{v}^1 - \tilde{v}^2 \|_{L^2(0, \infty; H^2(\Omega))}^2. \] 

(4.12)

**Theorem 4.5.** There exists a positive constant \( C_9 = C_9(s_0, s_1, \tilde{\rho}, \omega, \gamma) \) such that, for all \( (\tilde{\sigma}, \tilde{v}) \in D_{\tilde{\mu}} \) and we have
\[ \|F_{1,m}(\tilde{\sigma}, \tilde{v}, \cdot)\|_{L^2(0,\infty; H^1(\Omega))} + \|F_{1,\Omega}(\tilde{\sigma}, \tilde{v}, \cdot)\|_{L^1(0,\infty; e^{-\omega t})} + \|F_{2}(\tilde{\sigma}, \tilde{v}, \cdot)\|_{L^2(0,\infty; L^2(\Omega))} \leq C_9(\|\tilde{\sigma}_m\|_{H^1(0,\infty; H^1(\Omega))} + \|\tilde{\sigma}_\Omega\|_{L^2(0,\infty; e^{-\omega t})} + \|\tilde{v}\|_{L^2(0,\infty; H^2(\Omega))}). \] (4.13)

Further there exists a positive constant \( C_{10} = C_{10}(s_0, s_1, \tilde{\rho}, \omega, \gamma) \) such that, for \( (\tilde{\sigma}^1, \tilde{v}^1) \in D_{\tilde{\mu}} \) and \( (\tilde{\sigma}^2, \tilde{v}^2) \in D_{\tilde{\mu}}, \) we have

\[ \|F_{1,m}(\tilde{\sigma}^1, \tilde{v}^1, \cdot) - F_{1,m}(\tilde{\sigma}^2, \tilde{v}^2, \cdot)\|_{H^1(0,\infty; H^1(\Omega))} + \|F_{1,\Omega}(\tilde{\sigma}^1, \tilde{v}^1, \cdot) - F_{1,\Omega}(\tilde{\sigma}^2, \tilde{v}^2, \cdot)\|_{L^1(0,\infty; e^{-\omega t})} + \|F_{2}(\tilde{\sigma}^1, \tilde{v}^1, \cdot) - F_{2}(\tilde{\sigma}^2, \tilde{v}^2, \cdot)\|_{L^2(0,\infty; L^2(\Omega))} \leq C_{10} (\tilde{\mu}^2 + \tilde{\mu})(\|\tilde{\sigma}_m^1 - \tilde{\sigma}_m^2\|_{H^1(0,\infty; H^1(\Omega))} + \|\tilde{\sigma}_\Omega^1 - \tilde{\sigma}_\Omega^2\|_{L^\infty(0,\infty; e^{-\omega t})} + \|\tilde{v}^1 - \tilde{v}^2\|_{L^2(0,\infty; H^2(\Omega))}). \] (4.14)

**Proof.** This follows from Lemmas 4.2, 4.3 and 4.4. \( \square \)

### 4.2. Proof of **Theorem 1.1**

Now we consider the following nonlinear closed loop system

\[ \begin{align*}
\bar{\sigma}_t + \tilde{\rho} \bar{v}_x - \omega \tilde{\sigma} &= F_1(\tilde{\sigma}, \bar{v}, t), & \text{in } Q_x^\infty, \\
\bar{v}_t + b \bar{\sigma}_x - \nu_0 \bar{v}_{xx} - \omega \bar{v} &= F_2(\bar{\sigma}, \bar{v}, t) + K(\tilde{\sigma}(\cdot, t), \tilde{v}(\cdot, t)), & \text{in } Q_x^\infty, \\
\tilde{\sigma}(x, 0) = \sigma_0(x), & \bar{v}(x, 0) = \nu_0(x) & \text{in } \Omega_x, & \int_{\Omega_x} \sigma_0(x) dx = 0, \\
\tilde{v}(0, t) = 0, & \bar{v}(\pi, t) = 0, & \forall t > 0, \\
Y(x, t) = x + \int_0^t e^{-\omega s} \bar{v}(x, s) ds, & t > 0, & x \in \Omega_x, \\
X(Y(x, t), t) = x, & X(Y(y, t), t) = y, & y \in \Omega_y, \ t > 0, \\
\tilde{\ell}_j(t) = X(\tilde{\ell}_j, t), & \text{for } j = 1, 2, & (4.15)
\end{align*} \]

where \( F_1(\tilde{\sigma}, \bar{v}, t) \) and \( F_2(\bar{\sigma}, \bar{v}, t) \) are defined in (2.8) and \( K \in \mathcal{L}(Z, L^2(\Omega_x)) \) is defined by (3.22).

**Theorem 1.1** may be rewritten as follows.

**Theorem 4.6.** Let \( \omega \) belong to \( (0, \omega_0) \). There exists a constant \( \mu_0 > 0 \) such that if

\[ 0 < \tilde{\mu} \leq \mu_0, \quad \|\sigma_0, \nu_0\|_{H^1_0(\Omega) \times H^1_0(\Omega)} \leq \frac{\mu}{2C_1}, \]

then the system (4.15) admits a unique solution \( (\tilde{\sigma}, \tilde{v}, X, Y) \) such that \( (\tilde{\sigma}, \tilde{v}) \) belongs to \( D_{\tilde{\mu}}, \) \( X \) belongs to \( C^1_b([0, \infty); H^1(\Omega_x)) \cap C_b([0, \infty); H^2(\Omega_y)) \), \( Y \) belongs to \( C^1_b([0, \infty); H^1(\Omega_x)) \cap C_b([0, \infty); H^2(\Omega_x)) \).
Proof. We prove this theorem by Banach fixed point theorem. We define

\[
\| (\tilde{\sigma}, \tilde{v}) \|_D := \| \tilde{\sigma}_m \|_{H^1(0,\infty; H^1(\Omega))} + \| \tilde{\sigma}_\Omega \|_{L^\infty(0,\infty; e^{-\omega(\cdot)})} + \| \tilde{v} \|_{L^2(0,\infty; H^2(\Omega))} \\
+ \| \tilde{v} \|_{H^1(0,\infty; L^2(\Omega))}.
\]  

(4.16)

Let us choose \((\psi, \phi) \in D_{\bar{\mu}}\). We define \((\tilde{\sigma}^{(\psi, \phi)}, \tilde{v}^{(\psi, \phi)})\) as the solution to the linear system

\[
\tilde{\sigma}_t^{(\psi, \phi)} + \rho \tilde{v}_x^{(\psi, \phi)} - \omega \tilde{\sigma}^{(\psi, \phi)} = F_1(\psi, \phi, t), \quad \text{in } \Omega \times (0, \infty),
\]

\[
\tilde{v}_t^{(\psi, \phi)} + b \tilde{\sigma}_x^{(\psi, \phi)} - v_0 \tilde{v}_{xx}^{(\psi, \phi)} - \omega \tilde{v}^{(\psi, \phi)} = F_2(\psi, \phi, t) + K(\tilde{\sigma}^{(\psi, \phi)}(\cdot, t), \tilde{v}^{(\psi, \phi)}(\cdot, t)),
\]

in \(\Omega \times (0, \infty),\)

\[
\tilde{\sigma}^{(\psi, \phi)}(0) = \sigma_0(x), \quad \tilde{v}^{(\psi, \phi)}(0) = v_0(x) \quad \text{in } \Omega, \quad \int_0^\pi \sigma_0(x) dx = 0,
\]

\[
\tilde{v}^{(\psi, \phi)}(0, t) = 0, \quad \tilde{v}^{(\psi, \phi)}(\pi, t) = 0, \quad \forall t > 0,
\]

\[
Y^{(\psi, \phi)}(x, t) = x + \int_0^t e^{-\omega s} \phi(x, s) ds.
\]  

(4.17)

First we choose

\[
\| (\sigma_0, v_0) \|_{H^1_m(\Omega) \times H^1_0(\Omega)} \leq \frac{\bar{\mu}}{2C_1}.
\]  

(4.18)

Now we show that \((\tilde{\sigma}^{(\psi, \phi)}, \tilde{v}^{(\psi, \phi)})\) belongs to \(D_{\bar{\mu}}\). Since \((\psi, \phi) \in D_{\bar{\mu}},\) we can apply Theorems 3.7, 3.9 and 4.5 to system (4.17) and using (4.18), we obtain

\[
\| (\tilde{\sigma}^{(\psi, \phi)}, \tilde{v}^{(\psi, \phi)}) \|_D \leq C_1 \| (\sigma_0, v_0) \|_{H^1_m(\Omega) \times H^1_0(\Omega)} + C_2 \| (F_1, m(\psi, \phi, \cdot)) \|_{L^1(0,\infty; e^{-\omega(\cdot)})} + \| F_2(\psi, \phi, \cdot) \|_{L^2(0,\infty; L^2(\Omega))} \\
\leq C_1 \| (\sigma_0, v_0) \|_{H^1_m(\Omega) \times H^1_0(\Omega)} + C_2 C_9 \| (\psi, \phi) \|_D^2 \\
\leq \frac{\bar{\mu}}{2} + 4C_2 C_9 \bar{\mu}^2.
\]  

(4.19)

If, in addition to (4.5), \(\bar{\mu}\) satisfies

\[
\bar{\mu} \leq \frac{1}{8C_2 C_9},
\]  

(4.20)

then we see that \(4C_2 C_9 \bar{\mu}^2 \leq \frac{\bar{\mu}}{2}\). Hence from (4.19), it follows that

\[
\| (\tilde{\sigma}^{(\psi, \phi)}, \tilde{v}^{(\psi, \phi)}) \|_D \leq \bar{\mu}.
\]
Thus

\[(\tilde{\sigma}(\psi, \phi), \tilde{v}(\psi, \phi)) \in D_{\mu}.\]  

(4.21)

Therefore \((\psi, \phi) \mapsto (\tilde{\sigma}(\psi, \phi), \tilde{v}(\psi, \phi))\) is a mapping from \(D_{\mu}\) to itself. Now we will show that it is a contraction.

Let \((\psi^j, \phi^j) \in D_{\mu}\), for \(j = 1, 2\). We set \(\tilde{V} = \tilde{v}(\psi^1, \phi^1) - \tilde{v}(\psi^2, \phi^2)\) and \(\tilde{\Sigma} = \tilde{\sigma}(\psi^1, \phi^1) - \tilde{\sigma}(\psi^2, \phi^2)\).

The system satisfied by \((\tilde{\Sigma}, \tilde{V})\) is

\[
\begin{align*}
\tilde{\Sigma}_t + \tilde{\rho} \tilde{\Sigma}_x - \omega \tilde{\Sigma} &= \mathcal{F}_1(\psi^1, \phi^1, t) - \mathcal{F}_1(\psi^2, \phi^2, t), \quad \text{in} \ \Omega \times (0, \infty), \\
\tilde{V}_t + b \tilde{\Sigma}_x - \nu \tilde{V}_{xx} - \omega \tilde{V} &= \mathcal{F}_2(\psi^1, \phi^1, t) - \mathcal{F}_2(\psi^2, \phi^2, t) + K(\tilde{\sigma}(\psi^1, \phi^1)(\cdot, t), \tilde{v}(\psi^1, \phi^1)(\cdot, t)) \\
&\quad - K(\tilde{\sigma}(\psi^2, \phi^2)(\cdot, t), \tilde{v}(\psi^2, \phi^2)(\cdot, t)), \quad \text{in} \ \Omega \times (0, \infty), \\
\tilde{\Sigma}(0) &= 0, \quad \tilde{V}(0) = 0 \quad \text{in} \ \Omega, \\
\tilde{V}(0, t) &= 0, \quad \tilde{V}(\pi, t) = 0, \quad \forall t > 0.
\end{align*}
\]

(4.22)

Notice that

\[
K(\tilde{\sigma}(\psi^1, \phi^1)(\cdot, t), \tilde{v}(\psi^1, \phi^1)(\cdot, t)) - K(\tilde{\sigma}(\psi^2, \phi^2)(\cdot, t), \tilde{v}(\psi^2, \phi^2)(\cdot, t)) = K\left(\tilde{\Sigma}(\cdot, t), \tilde{V}(\cdot, t)\right).
\]

(4.23)

Therefore, using Theorems 3.9 and 4.5, we obtain

\[
\| (\tilde{\sigma}(\psi^1, \phi^1), \tilde{v}(\psi^1, \phi^1)) - (\tilde{\sigma}(\psi^2, \phi^2), \tilde{v}(\psi^2, \phi^2)) \|_D \\
\leq C_2C_{10}(\bar{\mu}^2 + \bar{\mu})\| (\psi^1, \phi^1) - (\psi^2, \phi^2) \|_D.
\]

(4.24)

If \(\bar{\mu}\) satisfies in addition to (4.5) and (4.20),

\[
\bar{\mu} \leq \min \left\{ \frac{1}{6C_2C_{10}}, \frac{1}{\sqrt{6C_2C_{10}}} \right\},
\]

(4.25)

then we see that

\[
C_2C_{10}(\bar{\mu}^2 + \bar{\mu}) \leq \frac{1}{2}.
\]

Therefore with (4.24), we have

\[
\| (\tilde{\sigma}(\psi^1, \phi^1), \tilde{v}(\psi^1, \phi^1)) - (\tilde{\sigma}(\psi^2, \phi^2), \tilde{v}(\psi^2, \phi^2)) \|_D \leq \frac{1}{2} \| (\psi^1, \phi^1) - (\psi^2, \phi^2) \|_D.
\]

(4.26)

Now we set

\[
\mu_0 := \min \left\{ \mu_1, \mu_2, \frac{1}{8C_2C_9}, \frac{1}{6C_2C_{10}}, \frac{1}{\sqrt{6C_2C_{10}}} \right\}.
\]

(4.27)
Thus, if $0 < \bar{\mu} \leq \mu_0$ and if $(\sigma_0, v_0)$ satisfies (4.18), the mapping $(\psi, \phi) \mapsto \left( \tilde{\sigma}(\psi, \phi), \tilde{v}(\psi, \phi) \right)$ is a contraction in $D_{\mu}$ and the theorem is proved. □

**Proof of Theorems 1.1, 1.2 and 1.3.** Theorem 1.1 has been rewritten in a more precise form in Theorem 4.6, and Theorem 4.6 is proved. Theorem 1.2 may be deduced from Theorem 4.6 by using the change of variables introduced in Section 2. Theorem 1.3 is a direct consequence of Theorem 1.2. □

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**Appendix A**

**Proof of Lemma 4.2 (i).** The derivative of the function $F_1(\tilde{\sigma}, \tilde{v}, t)$ with respect to $x$ is

$$
\frac{\partial F_1}{\partial x}(\tilde{\sigma}, \tilde{v}, t) = \tilde{\rho} \tilde{v}_{xx} \left( 1 - \frac{1}{\frac{\partial Y}{\partial x}} \right) + \tilde{\rho} \tilde{v}_{x} \frac{\partial^2 Y}{\partial x^2} - e^{-\omega t} \tilde{\sigma}_m \frac{\partial Y}{\partial x} - e^{-\omega t} \tilde{\sigma}_\omega \frac{\partial^2 Y}{\partial x^2} \left( \frac{\partial Y}{\partial x} \right)^2
$$

$$
- e^{-\omega t} \frac{\partial Y}{\partial x} + e^{-\omega t} (\tilde{\sigma}_m + \tilde{\sigma}_\omega) \tilde{v}_{x} \frac{\partial^2 Y}{\partial x^2} \left( \frac{\partial Y}{\partial x} \right)^2.
$$

(A.1)

We estimate the different terms in the $L^2(0, \infty; L^2(\Omega))$-norm. The first term in (A.1) can be estimated as follows

$$
\left\| \tilde{\rho} \tilde{v}_{xx} \left( 1 - \frac{1}{\frac{\partial Y}{\partial x}} \right) \right\|_{L^2(0, \infty; L^2(\Omega))} = \tilde{\rho} \left\| \tilde{v}_{xx} \left( \frac{\partial Y}{\partial x} - 1 \right) \right\|_{L^2(0, \infty; L^2(\Omega))}
$$

$$
\leq 2\tilde{\rho} \left\| \frac{\partial Y}{\partial x} - 1 \right\|_{L^\infty(Q^\infty_\omega)} \left\| \tilde{v}_{xx} \right\|_{L^2(0, \infty; L^2(\Omega))}
$$

$$
\leq 2\tilde{\rho} s_0 \left\| \frac{\partial Y}{\partial x} - 1 \right\|_{L^\infty(0, \infty; H^1(\Omega))} \left\| \tilde{v} \right\|_{L^2(0, \infty; H^2(\Omega))}
$$

$$
\leq \frac{2\tilde{\rho} s_0}{\sqrt{\omega}} \left\| \tilde{v} \right\|_{L^2(0, \infty; H^2(\Omega))}^2 \quad (\text{using} \ (2.12)).
$$

With (2.13) and (2.11), we estimate the second term as follows

$$
\left\| \tilde{\rho} \tilde{v}_{x} \frac{\partial^2 Y}{\partial x^2} \left( \frac{\partial Y}{\partial x} \right)^2 \right\|_{L^2(0, \infty; L^2(\Omega))} \leq 4\tilde{\rho} \left\| \tilde{v}_{x} \frac{\partial^2 Y}{\partial x^2} \right\|_{L^2(0, \infty; L^2(\Omega))}
$$

$$
\leq 4\tilde{\rho} \left\| \tilde{v}_{x} \right\|_{L^2(0, \infty; L^\infty(\Omega))} \left\| \frac{\partial^2 Y}{\partial x^2} \right\|_{L^\infty(0, \infty; L^2(\Omega))}.
$$
\[
\leq 4\rho_0\|v_x\|_{L^2(0,\infty;H^1(\Omega))} \left\| \frac{\partial^2 Y}{\partial x^2} \right\|_{L^\infty(0,\infty;L^2(\Omega))}
\leq \frac{4\rho_0}{\sqrt{2\omega}} \|v\|_{L^2(0,\infty;H^2(\Omega))}^2.
\]

Using (2.13), we can estimate the third term as follows
\[
\left\| e^{-ot} \left( \frac{\partial (\sigma_m)}{\partial x} \right)_x v_x \right\|_{L^2(0,\infty;L^2(\Omega))} \leq 2\left\| \left( \frac{\partial (\sigma_m)}{\partial x} \right)_x v_x \right\|_{L^2(0,\infty;L^2(\Omega))}
\leq 2\|\sigma_m\|_{L^\infty(0,\infty;L^2(\Omega))} \|v_x\|_{L^2(0,\infty;L^\infty(\Omega))}
\leq 2s_0\|\sigma_m\|_{H^1(0,\infty;L^2(\Omega))} \|v_x\|_{L^2(0,\infty;H^1(\Omega))}
\leq 2s_0\|\sigma_m\|_{H^1(0,\infty;H^1(\Omega))} \|v\|_{L^2(0,\infty;H^2(\Omega))}.
\]

Using (2.13), the fourth term can be estimated as follows
\[
\left\| e^{-ot} \sigma_m v_{xx} \right\|_{L^2(0,\infty;L^2(\Omega))} \leq 2\|\sigma_m\| \|v_{xx}\|_{L^2(0,\infty;L^2(\Omega))}
\leq 2\|\sigma_m\|_{L^\infty(Q^\infty)} \|v_{xx}\|_{L^2(0,\infty;L^2(\Omega))}
\leq 2s_0s_1\|\sigma_m\|_{H^1(0,\infty;H^1(\Omega))} \|v\|_{L^2(0,\infty;H^2(\Omega))}.
\]

Using (2.13), the fifth term can be estimated as follows
\[
\left\| e^{-ot} \left( \frac{\partial Y}{\partial x} \right)_x v_{xx} \right\|_{L^2(0,\infty;L^2(\Omega))} \leq 2\left\| e^{-ot} \left( \frac{\partial Y}{\partial x} \right)_x v_{xx} \right\|_{L^2(0,\infty;L^2(\Omega))}
\leq 2\|\sigma_m\|_{L^\infty(0,\infty;e^{-\omega t}(\sigma m))} \|v\|_{L^2(0,\infty;H^2(\Omega))}.
\]

Using Lemma 4.1, Eqs. (2.13) and (2.11), the estimate for the last term is
\[
\left\| e^{-ot} \left( \sigma_m + \sigma\Omega \right) v_x \frac{\partial^2 Y}{\partial x^2} \right\|_{L^2(0,\infty;L^2(\Omega))} \leq 2\rho \left\| v_x \frac{\partial^2 Y}{\partial x^2} \right\|_{L^2(0,\infty;L^2(\Omega))}
\leq \frac{2\rho_0}{\sqrt{2\omega}} \|v\|^2_{L^2(0,\infty;H^2(\Omega))}.
\]

Therefore there exists \( C_3 = C_3(\rho, s_0, s_1, \omega) > 0 \) such that
\[
\|F_{1,m}(\sigma, v, \cdot\|_{L^2(0,\infty;H^1(\Omega))} \leq C_3(\|\sigma_m\|_{H^1(0,\infty;H^1(\Omega))}) + \|\sigma\Omega\|_{L^\infty(0,\infty;e^{-\omega t}(\sigma m))}
+ \|v\|^2_{L^2(0,\infty;H^2(\Omega))}.
\]

**Proof of Lemma 4.2 (ii).** We can proceed as in the proof of the previous part. To estimate \( \frac{\partial F_1}{\partial x}(\sigma^1, v^1, t) - \frac{\partial F_2}{\partial x}(\sigma^2, v^2, t) \), we decompose this difference into a sum of terms similar to
those obtained in the previous proof. We only establish the estimates for two terms. First, we consider
\[ e^{-ot} \frac{\partial^2 \tilde{\nu}^1_{m,x}}{\partial x^2} - e^{-ot} \frac{\partial^2 \tilde{\nu}^2_{m,x}}{\partial x^2}, \]
where \( Y^j(x, t) = x + \int_0^t e^{-os} \tilde{v}^j(x, s) \, ds, \ j = 1, 2. \) Now
\[ e^{-ot} \frac{\partial^2 \tilde{\nu}^1_{m,x}}{\partial x^2} - e^{-ot} \frac{\partial^2 \tilde{\nu}^2_{m,x}}{\partial x^2} = e^{-ot} \frac{\partial \tilde{\nu}^1_{m,x}}{\partial x} - e^{-ot} \frac{\partial \tilde{\nu}^2_{m,x}}{\partial x} + e^{-ot} \frac{\partial^2 \tilde{\nu}^1_{m,x}}{\partial x^2} - e^{-ot} \frac{\partial^2 \tilde{\nu}^2_{m,x}}{\partial x^2}. \]
For the first term we have
\[ \| e^{-ot} \frac{\partial \tilde{\nu}^1_{m,x}}{\partial x} - \tilde{\sigma}^1_m \|_{L^2(0, \infty; L^2(\Omega))} \leq 2s_0s_1 \| \tilde{\sigma}^1_m - \tilde{\sigma}^2_m \|_{H^1(0, \infty; H^1(\Omega))} \| \tilde{v}^1 \|_{L^2(0, \infty; H^2(\Omega))} \]
\[ \leq 2s_0s_1 \tilde{\mu} \| \tilde{\sigma}^1_m - \tilde{\sigma}^2_m \|_{H^1(0, \infty; H^1(\Omega))}. \]
The second term can be estimated as follows
\[ \| e^{-ot} \frac{\partial \tilde{\nu}^2_{m,x}}{\partial x} - \tilde{\sigma}^2_m \|_{L^2(0, \infty; L^2(\Omega))} \leq 2s_0s_1 \| \tilde{\sigma}^2_m \|_{H^1(0, \infty; H^1(\Omega))} \| \tilde{v}^1 - \tilde{v}^2 \|_{L^2(0, \infty; H^2(\Omega))} \]
\[ \leq 2s_0s_1 \tilde{\mu} \| \tilde{v}^1 - \tilde{v}^2 \|_{L^2(0, \infty; H^2(\Omega))}. \]
For the third term, we have
\[ \left\| e^{-ot} \frac{\partial^2 \tilde{\nu}^2_{m,x}}{\partial x^2} \left( \frac{1}{\partial x} - \frac{1}{\partial x} \right) \right\|_{L^2(0, \infty; L^2(\Omega))} \leq 4 \left\| e^{-ot} \frac{\partial \tilde{\nu}^1_{m,x}}{\partial x} - \frac{\partial \tilde{\nu}^2_{m,x}}{\partial x} \right\|_{L^2(0, \infty; L^2(\Omega))} \]
\[ \leq \tilde{\rho}s_0 \| \tilde{v}^1 \|_{L^2(0, \infty; H^2(\Omega))} \| \tilde{v}^1 - \tilde{v}^2 \|_{L^2(0, \infty; H^2(\Omega))} \leq \tilde{\rho}s_0 \| \tilde{v}^1 - \tilde{v}^2 \|_{L^2(0, \infty; H^2(\Omega))}. \]
Hence, we have
\[ \left\| e^{-ot} \frac{\partial \tilde{\nu}^1_{m,x}}{\partial x} - e^{-ot} \frac{\partial \tilde{\nu}^2_{m,x}}{\partial x} \right\|_{L^2(0, \infty; L^2(\Omega))} \]
\[ \leq \max \left\{ 2s_0s_1, \frac{\tilde{\rho}s_0}{\sqrt{\omega}} \right\} \tilde{\mu} \| \tilde{v}^1 - \tilde{v}^2 \|_{L^2(0, \infty; H^2(\Omega))}. \]
Consider another term of the difference $\frac{\partial F_1}{\partial x}(\bar{\sigma}^1, \bar{v}^1, t) - \frac{\partial F_2}{\partial x}(\bar{\sigma}^2, \bar{v}^2, t)$, namely

$$\bar{\rho} \bar{v}_x^1 \frac{\partial^2 Y^1}{\partial x^2} - \bar{\rho} \bar{v}_x^2 \frac{\partial^2 Y^2}{\partial x^2} \leq \bar{\rho} \bar{v}_x^2 \frac{\partial^2 Y^1}{\partial x^2} - \bar{\rho} \bar{v}_x^2 \frac{\partial^2 Y^2}{\partial x^2}.$$

Now write

$$\bar{\rho} \bar{v}_x^1 \frac{\partial^2 Y^1}{\partial x^2} - \bar{\rho} \bar{v}_x^2 \frac{\partial^2 Y^2}{\partial x^2} = \bar{\rho} \frac{\partial^2 Y^1}{\partial x^2} (\bar{v}_x^1 - \bar{v}_x^2) + \bar{\rho} \frac{\partial^2 Y^1}{\partial x^2} + \bar{\rho} \frac{\partial^2 Y^2}{\partial x^2} \left( \frac{1}{\partial x^2} - \frac{1}{\partial x^2} \right).$$

We can estimate the first term as follows

$$\left\| \bar{\rho} \frac{\partial^2 Y^1}{\partial x^2} (\bar{v}_x^1 - \bar{v}_x^2) \right\|_{L^2(0, \infty; L^2(\Omega))} \leq \frac{4\bar{\rho} \mu}{\sqrt{2\omega}} \| \bar{v}_1^1 - \bar{v}_1^2 \|_{L^2(0, \infty; H^2(\Omega))}.$$  

The estimate of the second term is

$$\left\| \bar{\rho} \frac{\partial^2 Y^1}{\partial x^2} \right\|_{L^2(0, \infty; L^2(\Omega))} \leq \frac{4\bar{\rho} \mu}{\sqrt{2\omega}} \| \bar{v}_1^1 - \bar{v}_1^2 \|_{L^2(0, \infty; H^2(\Omega))}.$$  

The estimate for last term is

$$\left\| \bar{\rho} \frac{\partial^2 Y^1}{\partial x^2} \right\|_{L^2(0, \infty; L^2(\Omega))} \leq \frac{4\bar{\rho} \mu}{\sqrt{2\omega}} \| \bar{v}_1^1 - \bar{v}_1^2 \|_{L^2(0, \infty; H^2(\Omega))}.$$  

$$\leq \frac{48\bar{\rho} \mu^2}{\omega \sqrt{2}} \| \bar{v}_1^1 - \bar{v}_1^2 \|_{L^2(0, \infty; H^2(\Omega))} \leq \frac{48\bar{\rho} \mu^2}{\omega \sqrt{2}} \| \bar{v}_1^1 - \bar{v}_1^2 \|_{L^2(0, \infty; H^2(\Omega))}.$$
Hence we have

\[
\| \hat{\rho} \, \hat{v}_x^1 \left( \frac{\partial^2 Y_1}{\partial x^2} \right) - \hat{\rho} \, \hat{v}_x^2 \left( \frac{\partial^2 Y_2}{\partial x^2} \right) \|_{L^2(0, \infty; L^2(\Omega))} \leq \max \left\{ \frac{48 \hat{\rho} \delta_0^2}{\omega \sqrt{2}}, \frac{4 \hat{\rho} \delta_0}{\sqrt{2} \omega} \right\} \left( \hat{\mu}^2 + \hat{\mu} \right) \| \hat{v}^1 - \hat{v}^2 \|_{L^2(0, \infty; H^2(\Omega))}. \]

The other terms can be estimated similarly. Therefore, there exists a positive constant \( C_4 \) such that

\[
\| \mathcal{F}_{1,m}(\hat{\sigma}^1, \hat{v}^1, \cdot) - \mathcal{F}_{1,m}(\hat{\sigma}^2, \hat{v}^2, \cdot) \|_{L^2(0, \infty; H^1(\Omega))} \leq C_4 \left( \| \hat{\sigma}_m^1 - \hat{\sigma}_m^2 \|_{H^1(0, \infty; H^1(\Omega))} + \| \tilde{\sigma}_\Omega - \hat{\sigma}_\Omega \|_{L^\infty(0, \infty; e^{-\omega t})} + \| \hat{v}^1 - \hat{\sigma}_m^2 \|_{L^2(0, \infty; H^2(\Omega))} \right). \]

Proof of Lemma 4.3.

\[
\mathcal{F}_{1, \Omega}(\hat{\sigma}, \hat{v}, t) = \frac{1}{\pi} \int_\Omega \hat{\rho} \hat{v}_x \left( 1 - \frac{1}{\frac{\partial Y}{\partial x}} \right) \, dx - \frac{1}{\pi} \int_\Omega e^{-\omega t} (\hat{\sigma}_m + \hat{\sigma}_\Omega) \hat{v}_x \frac{1}{\frac{\partial Y}{\partial x}} \, dx.
\]

Using (2.13), we can estimate the first term as follows

\[
\left\| \frac{1}{\pi} \int_\Omega \hat{\rho} \hat{v}_x \left( 1 - \frac{1}{\frac{\partial Y}{\partial x}} \right) \, dx \right\|_{L^1(0, \infty; e^{-\omega t})} = \frac{1}{\pi} \int_0^\infty \left\| e^{-\omega t} \int_\Omega \hat{\rho} \hat{v}_x \left( \frac{\partial Y}{\partial x} - \frac{1}{\frac{\partial Y}{\partial x}} \right) \, dx \right\| \, dt
\]

\[
\leq \frac{2 \hat{\rho}}{\pi} \int_0^\infty e^{-\omega t} \int_\Omega \left| \hat{v}_x \left( \frac{\partial Y}{\partial x} - 1 \right) \right| \, dx \, dt
\]

\[
\leq \frac{2 \hat{\rho}}{\pi} \int_0^\infty e^{-\omega t} \| \hat{v}_x (\cdot, t) \|_{L^2(\Omega)} \left\| \frac{\partial Y}{\partial x} (\cdot, t) - 1 \right\|_{L^2(\Omega)} \, dt
\]

\[
\leq \frac{2 \hat{\rho}}{\pi \sqrt{2} \omega} \| \hat{v}_x \|_{L^2(0, \infty; L^2(\Omega))} \int_0^\infty e^{-\omega t} \| \hat{v}_x (\cdot, t) \|_{L^2(\Omega)} \, dt
\]

\[
\leq \frac{\hat{\rho}}{\pi \omega} \| \hat{v} \|_{L^2(0, \infty; H^2(\Omega))}^2.
\]

Second term can be estimated as follows

\[
\left\| \frac{1}{\pi} \int_\Omega e^{-\omega t} (\hat{\sigma}_m + \hat{\sigma}_\Omega) \hat{v}_x \frac{1}{\frac{\partial Y}{\partial x}} \, dx \right\|_{L^1(0, \infty; e^{-\omega t})} = \frac{1}{\pi} \int_0^\infty e^{-\omega t} \int_\Omega e^{-\omega t} (\hat{\sigma}_m + \hat{\sigma}_\Omega) \hat{v}_x \frac{1}{\frac{\partial Y}{\partial x}} \, dx \, dt
\]
\[
\begin{align*}
&\leq \frac{2}{\pi} \int_0^\infty \int_{\Omega_x} |\tilde{\sigma}_m \tilde{v}_x| \, dx \, dt + \frac{2}{\pi} \|\tilde{\sigma}_\Omega\|_{L^\infty(0, \infty; e^{-\omega t})} \int_0^\infty e^{-\omega t} \int_{\Omega} |\tilde{v}_x| \, dx \, dt \\
&\leq \frac{2}{\pi} \int_0^\infty \|\tilde{\sigma}_m (\cdot, t)\|_{L^2(\Omega)} \|\tilde{v}_x (\cdot, t)\|_{L^2(\Omega)} \, dt \\
&+ \frac{2}{\pi} \sqrt{\pi} \|\tilde{\sigma}_\Omega\|_{L^\infty(0, \infty; e^{-\omega t})} \int_0^\infty e^{-\omega t} \|\tilde{v}_x (\cdot, t)\|_{L^2(\Omega)} \, dt \\
&\leq \frac{1}{\pi} \int_0^\infty \left( \|\tilde{\sigma}_m\|_{H^1(0, \infty; H^1(\Omega))}^2 + \|\tilde{\sigma}_\Omega\|_{L^\infty(0, \infty; e^{-\omega t})}^2 + \|\tilde{v}\|_{L^2(0, \infty; H^2(\Omega))}^2 \right) \, dt + \frac{2\sqrt{\pi}}{\sqrt{2\alpha}} \|\tilde{\sigma}_\Omega\|_{L^\infty(0, \infty; e^{-\omega t})} \|\tilde{v}_x\|_{L^2(0, \infty; L^2(\Omega))}.
\end{align*}
\]

Therefore there exists a positive constant \( C_5 = C_5 (\rho, \omega) > 0 \) such that

\[
\| F_{1, \Omega} (\tilde{\sigma}, \tilde{v}, \cdot) \|_{L^1(0, \infty; e^{-\omega t})} \leq C_5 \left( \|\tilde{\sigma}_m\|_{H^1(0, \infty; H^1(\Omega))}^2 + \|\tilde{\sigma}_\Omega\|_{L^\infty(0, \infty; e^{-\omega t})}^2 + \|\tilde{v}\|_{L^2(0, \infty; H^2(\Omega))}^2 \right). \quad \square
\]

**Proof of Lemma 4.4.** Let us recall that

\[
F_2 (\tilde{\sigma}, \tilde{v}, t) = (\tilde{\sigma}_m)_{\times} \left( b - a y (\rho + e^{-\omega t} \tilde{\sigma}) y^{-2} \frac{\partial Y}{\partial x} \right) + \frac{\gamma}{\rho + e^{-\omega t} \tilde{\sigma}} \frac{\partial^2 Y}{\partial x^2} - \tilde{v} \tilde{v}_{xx} \left( \frac{1}{\rho} - \frac{1}{\tilde{\rho} + e^{-\omega t} \tilde{\sigma}} \left( \frac{\partial Y}{\partial x} \right)^2 \right).
\]

To estimate \( F_2 (\tilde{\sigma}, \tilde{v}, t) \) we need a lower bound for \( \tilde{\rho} + e^{-\omega t} \tilde{\sigma} \). Notice that

\[
|\tilde{\rho} + e^{-\omega t} \tilde{\sigma}| \geq \tilde{\rho} - |e^{-\omega t} \tilde{\sigma}| \geq \frac{\tilde{\rho}}{2} \quad \text{(by Lemma 4.1).} \quad (A.2)
\]

By Mean Value Theorem, we have

\[
(\tilde{\rho} + e^{-\omega t} \tilde{\sigma}) y^{-2} - \tilde{\rho} y^{-2} = (y - 2)e^{-\omega t} \tilde{\sigma} \int_0^1 (\tilde{\rho} + s e^{-\omega t} \tilde{\sigma}) y^{-3} \, ds.
\]

Therefore we have

\[
|(\tilde{\rho} + e^{-\omega t} \tilde{\sigma}) y^{-2} - \tilde{\rho} y^{-2}| \leq C e^{-\omega t} |\tilde{\sigma}|. \quad (A.3)
\]
Using Lemma 4.1, (A.3), (2.12) and (2.13), the first term can be estimated as follows

\[
\left\| a\gamma (\tilde{\sigma}_m)\right\| \left( \hat{\rho}^{\gamma - 2} - (\hat{\rho} + e^{-\omega t} \tilde{\sigma})^{\gamma - 2} \frac{1}{\partial Y / \partial x} \right) \left\| L^2(0, \infty; L^2(\Omega)) \right\|
\]

\[
\leq \left\| a\gamma (\tilde{\sigma}_m)\right\| \left( \hat{\rho}^{\gamma - 2} - \tilde{\rho}^{\gamma - 2} + \tilde{\rho}^{\gamma - 2} - (\hat{\rho} + e^{-\omega t} \tilde{\sigma})^{\gamma - 2} \frac{1}{\partial Y / \partial x} \right) \left\| L^2(0, \infty; L^2(\Omega)) \right\|
\]

\[
\leq C \left( \left\| (\tilde{\sigma}_m) \partial Y / \partial x \right\| L^2(0, \infty; L^2(\Omega)) + \left\| (\tilde{\sigma}_m) e^{-\omega t} (\tilde{\sigma}_m + \tilde{\sigma}_m) \right\| L^2(0, \infty; L^2(\Omega)) \right)
\]

\[
\leq C \left( \left\| (\tilde{\sigma}_m) \partial Y / \partial x \right\| L^2(0, \infty; L^2(\Omega)) + \left\| (\tilde{\sigma}_m) \partial Y / \partial x \right\| L^2(0, \infty; L^2(\Omega)) \right)
\]

\[
\leq C \left( \left\| (\tilde{\sigma}_m) \partial Y / \partial x \right\| L^2(0, \infty; L^2(\Omega)) + \left\| (\tilde{\sigma}_m) \partial Y / \partial x \right\| L^2(0, \infty; L^2(\Omega)) \right)
\]

For the second term, we have the following estimate using (A.2) and (2.13)

\[
\left\| \frac{\gamma}{\rho} \left( \frac{\partial Y}{\partial x} \right)^2 \right\| L^2(0, \infty; L^2(\Omega)) \leq \frac{16\gamma}{\rho} \left\| \frac{\partial Y}{\partial x} \right\| L^2(0, \infty; L^2(\Omega))
\]

To estimate the third term, first notice that

\[
\frac{1}{\rho} - \frac{1}{\hat{\rho} + e^{-\omega t} \tilde{\sigma} \left( \frac{\partial Y}{\partial x} \right)^2}
\]

\[
= \frac{1}{\rho} - \frac{1}{\hat{\rho} + e^{-\omega t} \tilde{\sigma}} + \frac{1}{\rho} - e^{-\omega t} \tilde{\sigma} \frac{1}{\rho} + \frac{1}{\rho} - e^{-\omega t} \tilde{\sigma} \frac{1}{\rho} - \frac{1}{\rho} - e^{-\omega t} \tilde{\sigma} \frac{1}{\rho}
\]

\[
= \frac{e^{-\omega t} \tilde{\sigma}}{\hat{\rho} + e^{-\omega t} \tilde{\sigma}} + \frac{1}{\rho} - e^{-\omega t} \tilde{\sigma} \left( \frac{\partial Y}{\partial x} - 1 \right) \frac{1}{\rho} \frac{\partial Y}{\partial x} + \frac{1}{\rho} - e^{-\omega t} \tilde{\sigma} \left( \frac{\partial Y}{\partial x} - 1 \right) \frac{1}{\rho \left( \frac{\partial Y}{\partial x} \right)^2}
\]

Using Lemma 4.1 and estimate (2.12), (2.12), (2.13) we have the following estimate for third term

\[
\left\| \tilde{\nu}_{xx} \left( \frac{1}{\rho} - \frac{1}{\hat{\rho} + e^{-\omega t} \tilde{\sigma} \left( \frac{\partial Y}{\partial x} \right)^2} \right) \right\| L^2(0, \infty; L^2(\Omega))
\]

\[
\leq \frac{2\nu}{\rho^2} \left\| e^{-\omega t} (\tilde{\sigma}_m + \tilde{\sigma}_m) \tilde{\nu}_{xx} \right\| L^2(0, \infty; L^2(\Omega)) + \frac{4\nu}{\rho} \left\| \tilde{\nu}_{xx} \left( \frac{\partial Y}{\partial x} - 1 \right) \right\| L^2(0, \infty; L^2(\Omega))
\]

\[
+ \frac{8\nu}{\rho} \left\| \tilde{\nu}_{xx} \left( \frac{\partial Y}{\partial x} - 1 \right) \right\| L^2(0, \infty; L^2(\Omega))
\]
\[
\leq C(\|\overline{\sigma}_m\|_{H^1(0, \infty; H^1(\Omega))} + \|\overline{\sigma}_\Omega\|_{L^\infty(0, \infty; e^{-\omega t})})\|\overline{\nu}\|_{L^2(0, \infty; H^2(\Omega))} + \frac{12\nu s_0}{\bar{\rho} \sqrt{\omega}} \|\overline{\nu}\|_{L^2(0, \infty; H^2(\Omega))}^2
\]
\[
\leq C(\|\overline{\sigma}_m\|_{H^1(0, \infty; H^1(\Omega))} + \|\overline{\sigma}_\Omega\|_{L^\infty(0, \infty; e^{-\omega t})})\|\overline{\nu}\|_{L^2(0, \infty; H^2(\Omega))}^2.
\]

Therefore, there exists \( C_7 = C_7(s_0, s_1, \bar{\rho}, \omega, \gamma) > 0 \) such that
\[
\|F_2(\overline{\sigma}, \overline{\nu}, \cdot)\|_{L^2(0, \infty; L^2(\Omega))} \leq C_7(\|\overline{\sigma}_m\|_{H^1(0, \infty; H^1(\Omega))} + \|\overline{\sigma}_\Omega\|_{L^\infty(0, \infty; e^{-\omega t})})\|\overline{\nu}\|_{L^2(0, \infty; H^2(\Omega))}^2.
\]

\[ \square \]

References