CONTROLLABILITY AND STABILIZABILITY OF THE LINEARIZED COMpressible NAVIER–STOKES SYSTEM IN ONE DIMENSION∗
S. CHOWDHURY†, M. RAMASWAMY‡, AND J.-P. RAYMOND§

Abstract. In this paper we consider the one-dimensional compressible Navier–Stokes system linearized about a constant steady state \((Q_0, 0)\) with \(Q_0 > 0\). We study the controllability and stabilizability of this linearized system. We establish that the linearized system is null controllable for regular initial data by an interior control acting everywhere in the velocity equation. We prove that this result is sharp by showing that the null controllability cannot be achieved by a localized interior control or by a boundary control acting only in the velocity equation. On the other hand, we show that the system is approximately controllable. We also show that the system is not stabilizable with a decay rate \(e^{-\omega t}\) for \(\omega > \omega_0\), where \(\omega_0\) is an accumulation point of the real eigenvalues of the linearized operator.

Key words. linearized compressible Navier–Stokes system, null controllability, boundary control, localized interior control, approximate controllability, stabilizability

AMS subject classifications. 93C20, 93B05, 35B37

DOI. 10.1137/110846683

1. Introduction. Control of fluid flow has been an important area of research in recent years. While there has been considerable work on the incompressible model \([8, 9, 3, 14, 15, 21]\), there has been much less work on the compressible fluid flow model \([18, 19, 11]\). One of the reasons for this is that the compressible Navier–Stokes system is much less tractable theoretically since it is a coupled system of hyperbolic and parabolic equations \([13]\).

Setting \(I_\pi = (0, \pi)\), let us consider the full system in \(I_\pi \times (0, T)\) for the density \(\rho(x, t)\) and velocity field \(u(x, t)\) of a compressible isothermal barotropic fluid

\[
\begin{align*}
\partial_t \rho(x, t) + (\rho u)_x(x, t) &= 0, \\
\partial_t (\rho u)(x, t) + (\rho u^2)_x(x, t) + (p(\rho))_x(x, t) - \nu u_{xx}(x, t) &= 0.
\end{align*}
\]

Here \(\nu > 0\) is the fluid viscosity and the pressure \(p\) is assumed to satisfy the constitutive law

\[
p(\rho) = a \rho^\gamma \quad \text{for } a > 0, \gamma > 0.
\]

The one-dimensional Navier–Stokes system is a model for a fluid flow in a thin tube or a narrow channel and it is interesting to study the properties of the fluid, like velocity,
density, and energy change along the tube or the channel. These models can be viewed as one-dimensional approximations of two-dimensional or three-dimensional models.

In this paper we are interested in control and stabilization of the one-dimensional linearized compressible Navier–Stokes system. This is the first step in studying the full nonlinear system and getting some local or global controllability and stabilizability results near a stationary solution. For that purpose, in this paper we consider the control system linearized about a constant steady state \((Q_0, v_0)\) with \(Q_0 > 0\),\n
\[
\begin{align*}
\partial_t \rho(x, t) + v_0 \rho_x(x, t) + Q_0 u_x(x, t) &= 0, \\
\partial_t u(x, t) - \frac{\nu}{Q_0} u_{xx}(x, t) + v_0 u_x(x, t) + a\gamma Q_0^{-2} \rho_x(x, t) &= f \chi_O,
\end{align*}
\]

where \(\chi_O\) is the characteristic function of an open subset \(O \subset I_\pi\). When \(v_0 = 0\), system (1.1) may be completed by the following boundary and initial conditions:

\[
\begin{align*}
u(0, t) &= q_0(t), \quad u(\pi, t) = q_\pi(t), \quad \forall t > 0, \\
\rho(x, 0) &= \rho_0(x) \quad \text{and} \quad u(x, 0) = u_0(x).
\end{align*}
\]

In (1.1)–(1.2), \(f\) is a distributed control and \(q_0\) and \(q_\pi\) are boundary controls.

When \(v_0 \neq 0\), for example, if \(v_0 > 0\), an additional boundary condition has to be specified for \(\rho\).

By setting \(U(t) = (\rho(\cdot, t), u(\cdot, t))^T\), the above linearized system (1.1)–(1.2), with an additional homogeneous boundary condition for \(\rho\) if \(v_0 \neq 0\), may be written in the form

\[
U'(t) + AU(t) = 0, \quad U(0) = U_0 = (\rho_0, u_0)^T
\]

in \(Z = L^2(I_\pi) \times L^2(I_\pi)\) or in \(Z_0 = L^2_m(I_\pi) \times L^2_m(I_\pi)\), when \(f = 0\), \(q_0 = 0\), \(q_\pi = 0\). (Here \(L^2_m(I_\pi)\) denotes the space of functions in \(L^2(I_\pi)\) with mean value \(0\).)

When \(v_0 \neq 0\), the resolvent of \(-A\), that is, \(R(\lambda, A) = (\lambda I + A)^{-1}\) with \(\lambda > 0\), is a compact operator in \(Z\) (see Proposition IV.13 in chapter 4 of [10]). This is no longer true if \(v_0 = 0\). Moreover, if \(v_0 = 0\), \(-A\) is a sectorial operator in \(Z\) (it is a consequence of Lemma 2.5). This seems to be not the case when \(v_0 \neq 0\).

Thus we see that the properties of the two semigroups \((e^{-tA})_{t \geq 0}\) (the one when \(v_0 = 0\) and the one when \(v_0 \neq 0\)) are completely different.

In this paper we only study the case when \(v_0 = 0\).

The main results of the paper are stated in sections 5 and 6 where we obtain positive and negative results for the null controllability and approximate controllability of system (1.1)–(1.2) and in section 7 where we obtain negative stabilizability results for the same system. In Theorem 5.1, we state a null controllability for initial data in \((H^1(I_\pi) \cap L^2_m(I_\pi)) \times L^2(I_\pi)\) by a distributed control \(f\) acting everywhere in \(I_\pi\) in the velocity equation. We prove that this result is sharp by showing that the null controllability cannot be achieved by a localized interior control (see Theorem 5.10) or by a boundary control (see Theorems 5.6 and 5.8). On the other hand, the system is approximately controllable (see Theorem 6.2). In Theorem 7.11, we show that system (1.1)–(1.2) is not stabilizable with a decay rate \(e^{-\omega t}\) if \(\omega > \omega_0 = a\gamma Q_0^2/\nu\) and \(\omega\) is in the resolvent set of \(A\). (The stabilizability for a decay rate \(e^{-\omega t}\) with \(\omega < \omega_0\) was already established in [2].) To the authors’ knowledge, these results are totally new.

Our method is very much based on explicit eigenfunctions and the behavior of the spectrum. These techniques seem to extend to some other cases of fluid models, like the nonbarotropic case (i.e., \(p = p(\rho, \theta)\)) in one dimension involving an additional...
equation for temperature $\theta$ and the barotropic case in a two-dimensional rectangular domain. Studying the spectrum of the linearized operator and the behavior of the semigroup in these cases, for certain special boundary conditions, leads to similar eigenfunctions but involves more tedious computations. This is because in both cases, the dimension of the invariant spaces (which are two-dimensional in our case; see section 2.1) are greater than 2 and hence calculations are much more complicated, at least for the analysis of interior controllability as in our case. Some of the other results, like approximate controllability and boundary null controllability (Theorem 6.2 and Theorem 5.8) for the linearized operator in one dimension for the perfect gas (with an extra equation for temperature $\theta$ and $p = \gamma p \theta$), follow by similar techniques using only one boundary control for velocity. It also seems possible to extend Theorem 5.8 and Theorem 6.2 to two dimensions by similar methods. These works are in progress.

There have been some results regarding control of different fluid models in recent years. Amosova in [1] considers compressible viscous fluid in one dimension in Lagrangian coordinates with zero boundary condition for the velocity on the boundaries of the interval $(0,1)$ and an interior control on the velocity equation. She proves local exact controllability to trajectories for the velocity, provided that the initial density is already on the “targeted trajectory” in [1]. Ervedoza et al. consider the compressible Navier–Stokes system in one space dimension in a bounded domain $(0,L)$ in [7]. They prove in [7] local exact controllability to constant states $(\bar{\rho}, \bar{u})$ with $\bar{\rho} > 0, \bar{u} \neq 0$ using two boundary controls (both for density and velocity).

Renardy in [16] proves exact controllability results for the linear viscoelastic fluids of the Maxwell kind using interior control in one dimension for a bounded interval $(0,L)$ when time is sufficiently large. Douboue, Fernández-Cara, and González-Burgos in [6] prove large time null and approximate controllability results for the linear viscoelastic fluids of the Maxwell kind and for any time an approximate controllability result for fluids of the Jeffreys kind using interior control in a bounded domain of $\mathbb{R}^N$ for $N = 2, 3$. See also [5] for more results in this direction.

The plan of the paper is as follows. In section 2, we study the properties of the operator $A$ and of the semigroup $(e^{-tA})_{t \geq 0}$. We establish a controllability result of finite dimensional projection in section 3, and we estimate the control of minimal norm in section 4. As mentioned above, the positive and negative results on controllability and stabilizability are proved in sections 5, 6, and 7.

The authors acknowledge the financial support from the Indo-French Centre for the Promotion of Advanced Research, Delhi, under project 3701-1.

2. Linearized operator. Let us recall that $Z = L^2(I_\pi) \times L^2(I_\pi)$ and introduce the positive constants

\begin{equation}
(2.1) \quad b := a\gamma Q_0^{-2}, \quad \nu_0 := \frac{\nu}{Q_0}.
\end{equation}

Let $Z$ be endowed with the inner product

$$\left\langle \begin{pmatrix} \rho \\ u \end{pmatrix}, \begin{pmatrix} \sigma \\ v \end{pmatrix} \right\rangle_Z := b \int_0^\pi \rho(x)\sigma(x) \, dx + Q_0 \int_0^\pi u(x)v(x) \, dx.$$ 

The Lebesgue space $L^2_m(I_\pi)$ contains all the square integrable functions with zero mean value

$$L^2_m(I_\pi) = \left\{ f \in L^2(I_\pi), \quad \int_0^\pi f(x) \, dx = 0 \right\}.$$
We also introduce the space
\[ \mathbf{Z}_0 = L^2_m(I_\pi) \times L^2(I_\pi), \]
where the space \( \mathbf{Z}_0 \) will be equipped with the \( \mathbf{Z} \)-scalar product defined above. We set
\[ H^1_m(I_\pi) = H^1(I_\pi) \cap L^2_m(I_\pi), \]
where \( H^1(I_\pi) \) is the standard Sobolev space. We define \( H^1_{(\alpha)} \) as the space of functions in \( H^1(I_\pi) \) that vanish at \( x = 0 \)
\[ H^1_{(\alpha)}(I_\pi) = \left\{ f \in H^1(I_\pi), \quad f(0) = 0 \right\}. \]

We now define the unbounded operator \( (A,\mathcal{D}(A)) \) in \( \mathbf{Z} \) by
\[ \mathcal{D}(A) = \{U = (\rho, u)^T \in \mathbf{Z} \mid u \in H^1_0(I_\pi), \quad (b\rho - \nu_0 u') \in H^1(I_\pi)\} \]
and
\[ A = \begin{bmatrix} 0 & Q_0 \frac{d}{dx} \\ a\gamma Q_0^{-2} \frac{d}{dx} & \frac{\nu}{Q_0} \frac{d^2}{dx^2} \end{bmatrix} = \begin{bmatrix} 0 & Q_0 \frac{d}{dx} \\ b \frac{d}{dx} & -\nu \frac{d^2}{dx^2} \end{bmatrix}. \]

Setting \( U(t) = (\rho(\cdot, t), u(\cdot, t))^T \), the system (1.1) with homogeneous boundary conditions in (1.2) and \( f = 0 \) can be written as
\[ (2.3) \quad U'(t) + AU(t) = 0, \quad U(0) = U_0 \in \mathbf{Z}. \]

We now show that \( (-A,\mathcal{D}(A)) \) is the infinitesimal generator of a \( C^0 \) semigroup of contractions on \( \mathbf{Z} \).

**Lemma 2.1** (see [2]). The operator \( A \) is maximal monotone in \( \mathbf{Z} \). Thus, \( (-A,\mathcal{D}(A)) \) is the infinitesimal generator of a strongly continuous semigroup of contractions on \( \mathbf{Z} \), denoted by \( (S(t))_{t \geq 0} \). For every \( U_0 \in \mathbf{Z} \), there is a unique solution \( U \) of (2.3) in \( C([0, \infty); \mathbf{Z}) \) and
\[ \|U(t)\|_\mathbf{Z} \leq \|U_0\|_\mathbf{Z}, \quad \forall t \geq 0. \]

**Proof.** For all \( (\mu^T) \in \mathcal{D}(A) \), we have
\[ \left\langle A \begin{pmatrix} \rho \\ u \end{pmatrix}, \begin{pmatrix} \rho \\ u \end{pmatrix} \right\rangle_\mathbf{Z} = \left\langle \begin{pmatrix} Q_0 u' \\ (b\rho - \nu_0 u')' \end{pmatrix}, \begin{pmatrix} \rho \\ u \end{pmatrix} \right\rangle_\mathbf{Z} = \nu \int_0^\pi |u'(x)|^2 dx \geq 0. \]
Thus \( (A,\mathcal{D}(A)) \) is monotone. It is also maximal. Indeed for \( (g,h)^T \in \mathbf{Z} \), the equation
\[ (I + A) \begin{pmatrix} \rho \\ u \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix} \]
is equivalent to
\[ \text{find } \rho \in L^2(I_\pi), \quad u \in H^1_0(I_\pi) \text{ such that } \]
\[ \rho + Q_0 u' = g \quad \text{and} \quad u + b\rho' - \nu_0 u'' = h \text{ in } (0, \pi). \]
Simplification leads to an equation for \( u \in H^1_0(I_\pi) \),
\[ u - Q_0 bu'' - \nu_0 u'' = (h - bg'), \]
where the right-hand side is in \( H^{-1}(I_\pi) \) and hence the equation admits a solution \( u \in H^1_0(I_\pi) \). From the equality \( (b\rho - \nu_0 u')' = h - u \), it follows that \( (\rho, u) \) is in \( \mathcal{D}(A) \).
Thus \( (A,\mathcal{D}(A)) \) is maximal monotone. \( \square \)
We easily verify that $\text{Ker}(A) = \{ \mathbf{U} = (e, 0)^T \mid e \in \mathbb{R} \}$.

**Remark 2.2.** The adjoint $(A^*, D(A^*))$ of the operator $(A, D(A))$ in $\mathbf{Z}$ is defined by

$$D(A^*) = \{ (\sigma, v)^T \in \mathbf{Z} \mid v \in H^1_0(\pi), (b\sigma + \nu_0 v') \in H^1(\pi) \}$$

and

$$A^* \begin{pmatrix} \sigma \\ v \end{pmatrix} = \begin{pmatrix} 0 & -Q_0 \frac{d}{dx} \\ -b \frac{d}{dx} & -\nu_0 \frac{d^2}{dx^2} \end{pmatrix} \begin{pmatrix} \sigma \\ v \end{pmatrix}$$

for $(\sigma, v)^T \in D(A^*)$. Moreover $(-A^*, D(A^*))$ is the infinitesimal generator of a strongly continuous semigroup of contractions $(S^*(t))_{t \geq 0}$ on $\mathbf{Z}$.

**2.1. Spectrum of the linearized operator.** In the case of the stationary solution $(Q_0, v_0)$ with velocity $v_0 > 0$, Girinon [10] has shown that the resolvent $R(\lambda, A)$ is a compact operator from $\mathbf{Z}$ to $\mathbf{Z}$ and that the semigroup $(S(t))_{t \geq 0}$ is exponentially stable on $\mathbf{Z}$, i.e., there exists $M > 0$ such that

$$\|S(t)\|_{L(\mathbf{Z})} \leq Me^{-\omega t}$$

for some $\omega > 0$. In our case, $v_0 = 0$, and the resolvent is no more compact. Yet, we can show the exponential stability by following the approach of [2], using Fourier analysis. For that we define a Fourier basis $\{\Phi_n\}_{n \geq 0}$ in $\mathbf{Z}$ as follows:

$$\Phi_0(x) = \frac{1}{\sqrt{b\pi}}(1, 0)^T, \quad \Phi_{2n}(x) = \sqrt{\frac{2}{b\pi}}(\cos(nx), 0)^T, \quad \Phi_{2n-1}(x) = \sqrt{\frac{2}{Q_0\pi}}(0, \sin(nx))^T$$

for $n \geq 1$.

Let us define the following spaces of one and two dimensions:

$$\mathbf{V}_0 = \text{span}\{\Phi_0\}, \quad \mathbf{V}_n = \text{span}\{\Phi_{2n}, \Phi_{2n-1}\}, \quad n \geq 1,$$

where “span” stands for the vector space generated by those functions. One can verify that $\mathbf{Z}$ is the orthogonal sum of all these subspaces $\{\mathbf{V}_n\}_{n \geq 0}$. Observe that

$$\mathbf{V}_0 = \text{Ker}(A).$$

This motivates us to introduce the space

$$\mathbf{Z}_0 := \{ (\rho, u)^T \in \mathbf{Z} \mid \int_0^\pi \rho(x)dx = 0 \}.$$ 

Then it follows that the space $\mathbf{Z}_0$ is the orthogonal sum of the subspaces $\{\mathbf{V}_n\}_{n \geq 1}$. 

**Lemma 2.3.** For all $n \geq 1$, $\mathbf{V}_n$ is invariant under $A$ and $A_n = A|_{\mathbf{V}_n} \in L(\mathbf{V}_n)$ has the matrix representation

$$\begin{pmatrix} 0 & \sqrt{bQ_0n} \\ -\sqrt{bQ_0n} & \nu_0 n^2 \end{pmatrix}$$

in the basis $\{\Phi_{2n}, \Phi_{2n-1}\}$ of $\mathbf{V}_n$. 

Proof. Notice that
\[ A\Phi_{2n} = \sqrt{\frac{2}{b\pi}}(0, -bn\sin(nx))^T = -\sqrt{Q_0 b} \Phi_{2n-1}, \]
\[ A\Phi_{2n-1} = \sqrt{\frac{2}{Q_0\pi}}(nQ_0\cos(nx), \nu_0 n^2 \sin(nx))^T = \sqrt{b Q_0} \Phi_{2n} + \nu_0 n^2 \Phi_{2n-1}. \]
Thus the matrix representation of \( A|_{\mathcal{V}_n} \) in this basis is
\[
(2.5) \quad A_n = \begin{pmatrix} 0 & \sqrt{b Q_0 n} \\ -\sqrt{Q_0 b} n & \nu_0 n^2 \end{pmatrix}. \]

Remark 2.4. In a similar manner, we can check that for all \( n \geq 1 \), \( \mathcal{V}_n \) is invariant under \( A^* \) and
\[ A_n^* = A^*|_{\mathcal{V}_n} \in L(\mathcal{V}_n) \]
has the matrix representation
\[
\begin{pmatrix} 0 & -\sqrt{b Q_0 n} \\ \sqrt{b Q_0} n & \nu_0 n^2 \end{pmatrix}
\]
in the basis \( \{ \Phi_{2n}, \Phi_{2n-1} \} \) of \( \mathcal{V}_n \).

Lemma 2.5. The spectrum \( \sigma(A) \) of \( A \) except zero, i.e., \( \sigma(A) \setminus \{0\} \), lies on the right side of the complex plane. It consists of a finite number of pairs of complex eigenvalues and an infinite number of pairs of real eigenvalues, \( \{\lambda_n, \mu_n\}_{n \geq n_0} \). For \( 1 \leq k < n_0 \), the complex eigenvalues \( \{\lambda_k, \mu_k = \bar{\lambda}_k\} \) satisfy
\[
|\text{Re}(\lambda_k)| \geq \frac{\nu_0}{2}, \quad |\text{Im}(\lambda_k)| \leq 2\omega_0, \quad \text{where } \omega_0 := \frac{bQ_0}{\nu_0}.
\]
For \( n \geq n_0 \), the real eigenvalues satisfy
\[
(2.6) \quad \lim_{n \to \infty} \lambda_n = \omega_0, \quad \mu_n \to \infty \quad \text{as } n \to \infty.
\]
The eigenfunctions of \( A \) in \( \mathbb{Z} \), corresponding, respectively, to \( \lambda_n \) and \( \mu_n \), are
\[
(2.7) \quad \xi_n(x) = \begin{pmatrix} \cos(nx) \\ \frac{\lambda_n}{Q_0 n} \sin(nx) \end{pmatrix}^T \quad \text{and} \quad \zeta_n(x) = \begin{pmatrix} \cos(nx) \\ \frac{\mu_n}{Q_0 n} \sin(nx) \end{pmatrix}^T.
\]
Proof. Let us set \( \alpha := bQ_0 \). The eigenvalues of \( A_n \) are then given by the roots of
\[ \lambda^2 - \nu_0 n^2 + \alpha n^2 = 0, \]
and they are
\[
\lambda_n = \frac{\nu_0 n^2}{2} \left( 1 - \sqrt{1 - \frac{4\alpha}{\nu_0 n^2}} \right), \quad \mu_n = \frac{\nu_0 n^2}{2} \left( 1 + \sqrt{1 - \frac{4\alpha}{\nu_0 n^2}} \right).
\]
If \( \frac{4\alpha}{\nu_0} \) is an integer, then define
\[
n_0 := \frac{2\sqrt{\alpha}}{\nu_0} = \frac{2\sqrt{a\gamma Q_0 \frac{\mu}{\nu}}}{\nu}.
\]
If not, define \( n_0 \) to be the smallest integer bigger than \( \frac{2\sqrt{\alpha}}{\nu_0} \). For \( n \geq n_0 \), eigenvalues are real. In that case, since \( \alpha = bQ_0, \omega_0 = \frac{\alpha}{\nu_0} \), we have

\[
\omega_0 - \lambda_n = -\left( \frac{\omega_0^2}{\nu_0 n^2} \right) + o \left( \frac{1}{n^2} \right),
\]
and hence

\[
\omega_0 < \lambda_n < 2\omega_0, \quad \frac{\nu_0 n^2}{2} < \nu_0 n^2 - 2\omega_0 < \mu_n < \nu_0 n^2.
\]

For \( 1 \leq n < n_0 \), the two complex eigenvalues are

\[
\lambda_n = \delta_n - i\eta_n, \quad \mu_n = \delta_n + i\eta_n,
\]
where

\[
\eta_n = \frac{\nu_0 n^2}{2} \sqrt{\frac{4\alpha}{\nu_0^2 n^2} - 1} \leq \frac{\nu_0 n^2}{2} \frac{2\sqrt{\alpha}}{\nu_0} \leq \frac{2\alpha}{\nu_0} = 2\omega_0
\]
and

\[
\delta_n = \frac{\nu_0 n^2}{2} \geq \frac{\nu_0}{2}.
\]

The limiting behavior can be determined using the above expressions and estimates. Eigenfunctions can be calculated using the operator \( A \).

Remark 2.6. Observe that the spectrum of \( A^* \) is the same as that of \( A \). The eigenfunctions of \( A^* \) can be calculated as

\[
\xi_n^*(x) = \left( \cos(nx), -\frac{\lambda_n}{Q_0 n} \sin(nx) \right)^T, \quad \xi_n^*(x) = \left( \cos(nx), -\frac{\mu_n}{Q_0 n} \sin(nx) \right)^T,
\]
corresponding to \( \lambda_n \) and \( \mu_n \), respectively.

2.2. The behavior of the semigroup. The semigroup \( (S(t))_{t \geq 0} \) is exponentially stable on the subspace \( \mathcal{V}_0 \), defined in (2.4). For details, see Corollary 1 in [2]. We now explore if the system is null controllable using interior control for the velocity component. For that we plan to take the finite dimensional projection of the controlled system onto \( \mathcal{V}_n \), using the semigroup generated by \( -A_n \)

\[
e^{-A_nt} := S_n(t) = S(t)|_{\mathcal{V}_n}, \quad n \geq 0.
\]

Clearly \( S_0(t) = I_{\mathcal{V}_0} \). For \( n \in [1, n_0) \), the matrix representation for \( S_n(t) \) in the basis \( \{ \Phi_{2n}, \Phi_{2n-1} \} \) is

\[
S_n(t) = \frac{1}{\eta_n} e^{-\delta_n t} \begin{pmatrix} \eta_n \cos(\eta_n t) + \delta_n \sin(\eta_n t) & -\sqrt{\alpha} n \sin(\eta_n t) \\ \sqrt{\alpha} n \sin(\eta_n t) & \eta_n \cos(\eta_n t) - \delta_n \sin(\eta_n t) \end{pmatrix},
\]
and for \( n \in [n_0, \infty) \), we have

\[
S_n(t) = \frac{1}{\mu_n - \lambda_n} \begin{pmatrix} \mu_n e^{-\lambda_n t} - \lambda_n e^{-\mu_n t} & -\sqrt{\alpha} n (e^{-\lambda_n t} - e^{-\mu_n t}) \\ \sqrt{\alpha} n (e^{-\lambda_n t} - e^{-\mu_n t}) & \mu_n e^{-\mu_n t} - \lambda_n e^{-\lambda_n t} \end{pmatrix}.
\]
For null controllability, we will need the behavior of the semigroup $S_n(t)$ as $n$ tends to $\infty$.

**Lemma 2.7.** For $n$ large, the semigroup $S_n(t)$ has the form

$$S_n(t) = \frac{1}{(\mu_n - \lambda_n)} \begin{pmatrix} s_{n,1}(t) & -s_n(t) \\ s_n(t) & s_{n,2}(t) \end{pmatrix},$$

where

$$s_{n,1}(t) = O(\nu_0 n^2 e^{-\omega_0 t}), \quad s_{n,2}(t) = O(2\omega_0 e^{-\omega_0 t}), \quad s_n(t) = O(\sqrt{\alpha} n e^{-\omega_0 t}),$$

uniformly for all $t \in [\varepsilon, T]$, for any $0 < \varepsilon < T$.

**Proof.** Using the previous estimates, we get

$$\frac{\nu_0 n^2}{2} e^{-2\omega_0 t} \leq \mu_n e^{-\lambda_n t} \leq \nu_0 n^2 e^{-\omega_0 t}, \quad \omega_0 e^{-\nu_0 n^2 t} \leq \lambda_n e^{-\mu_n t} \leq 2\omega_0 e^{-\nu_0 n^2 t}.$$  

From (2.11), we have $s_{n,1}(t) = \mu_n e^{-\lambda_n t} - \lambda_n e^{-\mu_n t}$ and hence

$$\frac{\nu_0 n^2}{2} e^{-2\omega_0 t} - 2\omega_0 e^{-\nu_0 n^2 t} \leq s_{n,1}(t) \leq \nu_0 n^2 e^{-\omega_0 t} - \omega_0 e^{-\nu_0 n^2 t}.$$  

Again from (2.11), $s_n(t) = \sqrt{\alpha} n (e^{-\lambda_n t} - e^{-\mu_n t})$ implies that

$$\sqrt{\alpha} n (e^{-2\omega_0 t} - e^{-\nu_0 n^2 t}) \leq s_n(t) \leq \sqrt{\alpha} n (e^{-\omega_0 t} - e^{-\nu_0 n^2 t}).$$  

Similarly we have

$$\frac{\nu_0 n^2}{2} e^{-\nu_0 n^2 t} \leq \mu_n e^{-\mu_n t} \leq \nu_0 n^2 e^{-\nu_0 n^2 t}, \quad \omega_0 e^{-2\omega_0 t} \leq \lambda_n e^{-\lambda_n t} \leq 2\omega_0 e^{-2\omega_0 t}.$$  

Thus for $s_{n,2}(t) = (\mu_n e^{-\mu_n t} - \lambda_n e^{-\lambda_n t})$, we obtain

$$\frac{\nu_0 n^2}{2} e^{-\nu_0 n^2 t} - 2\omega_0 e^{-\omega_0 t} \leq s_{n,2}(t) \leq \nu_0 n^2 e^{-2\omega_0 t} - \omega_0 e^{-2\omega_0 t}.$$  

Hence (2.12) follows.  

**Remark 2.8.** The semigroup $(S_n^*(t))_{t \geq 0}$, generated by $-A_n^*$, is defined by

$$S_n^*(t) = \frac{1}{(\mu_n - \lambda_n)} \begin{pmatrix} s_{n,1}(t) & s_n(t) \\ -s_n(t) & s_{n,2}(t) \end{pmatrix}.$$  

**3. Controllability of finite dimensional projections.** Now we look for null controllability in $Z_0$ of the system below using interior control $f \in L^2(0, \infty; L^2(I))$

$$U'(t) + AU(t) = F(t), \quad U(0) = U_0 \in Z_0,$$

with $F(t) = (0, f(\cdot, t))^T$. Recall that the weak solution in $Z_0$ of this system is given by

$$U(t) = S(t)U_0 + \int_0^t S(t-s)F(s)ds \quad \forall t \geq 0.$$
Using the Fourier basis in $Z_0$, we set

$$F(t) = \sum_{n=1}^{\infty} g_n(t) \Phi_{2n} + \sum_{n=1}^{\infty} f_n(t) \Phi_{2n-1}. $$

For all $n$, the Fourier coefficients are given by

$$g_n(t) = \langle F(t), \Phi_{2n} \rangle_{Z_0} = 0, $$

$$f_n(t) = \langle F(t), \Phi_{2n-1} \rangle_{Z_0} = \sqrt{2Q_0 \pi} \int_0^{\pi} f(x,t) \sin(nx) dx.$$ 

Thus the projection of the control $F(t)$ on the two-dimensional space $V_n$ is $F_n(x,t) = f_n(t) \Phi_{2n-1}(x)$. Defining $U_n = U|V_n$ and $U_{0,n} = U(0)|V_n$, the finite dimensional system obtained by projecting (3.1) on $V_n$ is

$$U_n'(t) = -A_n U_n(t) + f_n(t) B, \quad U_n(0) = U_{0,n} \quad \text{with} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. $$

**Lemma 3.1.** For any given $T > 0$, the finite dimensional system (3.2) is controllable.

**Proof.** One can verify that the Kalman rank condition holds and hence the finite dimensional system is controllable. Indeed, the matrix $[B \mid A_n B]$ is of rank two. \[\square\]

**4. Estimates for minimum norm control.** For a given $T > 0$, the finite dimensional system (3.2) is controllable and hence there exists a control which brings this system to rest in time $T$. Among all such controls, the one of minimal norm is given by

$$\tilde{f}_n(t) = -(B^* e^{-(T-t)A_n^*} W_{n,T}^{-1} e^{-T A_n}) U_{0,n}, $$

where

$$W_{n,T} = \int_0^{T} e^{-t A_n} BB^* e^{-t A_n^*} dt.$$ 

(See [22, Part IV, Chapter 2, Theorem 2.3].) Let us set

$$D_n(t) = -B^* e^{-(T-t)A_n^*} W_{n,T}^{-1} e^{-T A_n}. $$

We will need a few estimates for $W_{n,T}$ and its inverse.

**Lemma 4.1.** If $A_n, B$ are as in (2.5), (3.2) and if we set

$$W_{n,T} := (\mu_n - \lambda_n)^{-2} \overline{W}_{n,T}, \quad \text{where} \quad \overline{W}_{n,T} = \begin{bmatrix} w_{n,1} & w_n \\ w_n & w_{n,2} \end{bmatrix}, $$

then

$$w_{n,1} = \frac{n^2 \alpha}{2 \lambda_n} (1 - e^{-2 \lambda_n T}) + O(1), \quad w_n = -\left(\frac{n \sqrt{\alpha}}{2}\right) e^{-2 \lambda_n T} + o(1), $$

$$w_{n,2} = \frac{n^2 \nu_0}{2} + \frac{\lambda_n}{2} + O(1). $$
Furthermore the inverse of $W_{n,T}$ is given by

$$W_{n,T}^{-1} = \frac{(\mu_n - \lambda_n)^2}{\det(W_{n,T})} \begin{bmatrix} w_{n,2} & -w_n \\ -w_n & w_{n,1} \end{bmatrix},$$

and the determinant of $\tilde{W}_{n,T}$ satisfies the estimate

$$\frac{\alpha \sqrt{n^4}}{8\lambda_n} (1 - e^{-2\lambda_n T}) + \frac{n^2\alpha}{2} (e^{-2\lambda_n T} - 3) + O(1) \leq \det(\tilde{W}_{n,T}) \leq \frac{\alpha \sqrt{n^4}}{4\lambda_n} (1 - e^{-4\lambda_n T}) + O(1).$$

(4.5)

**Proof.** Using the expression of $S_n(t)$ and Lemma 2.7, we have

$$S_n(t)B = \frac{1}{(\mu_n - \lambda_n)} \begin{bmatrix} s_{n,1}(t) & -s_n(t) \\ s_n(t) & s_{n,2}(t) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{(\mu_n - \lambda_n)} \begin{bmatrix} -s_n(t) \\ s_{n,2}(t) \end{bmatrix}.$$

Similarly, we have

$$B^* S_n^*(t) = \frac{1}{(\mu_n - \lambda_n)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} s_{n,1}(t) & s_n(t) \\ -s_n(t) & s_{n,2}(t) \end{bmatrix} = \frac{1}{(\mu_n - \lambda_n)} \begin{bmatrix} -s_n(t) & s_{n,2}(t) \end{bmatrix}.$$

Putting these two together yields

$$S_n(t)BB^* S_n^*(t) = (\mu_n - \lambda_n)^{-2} \begin{bmatrix} s_n^2(t) & -s_n(t)s_{n,2}(t) \\ -s_n(t)s_{n,2}(t) & s_{n,2}^2(t) \end{bmatrix}.$$

Using the definition of $W_{n,T}$ given by (4.4), we find

$$w_n = -\int_0^T (s_n s_{n,2})(t) dt = -\int_0^T \sqrt{\alpha n}(e^{-\lambda_n t} - e^{-\mu_n t})(\mu_n e^{-\mu_n t} - \lambda_n e^{-\lambda_n t}) dt$$

$$= -\sqrt{\alpha n} \int_0^T ((\mu_n + \lambda_n)e^{-(\lambda_n + \mu_n)t} - \mu_n e^{-2\mu_n t} - \lambda_n e^{-2\lambda_n t}) dt$$

$$= -\sqrt{\alpha n} \left[ -e^{-(\lambda_n + \mu_n)t} + \frac{e^{-2\mu_n t}}{2} + \frac{e^{-2\lambda_n t}}{2} \right]_0^T$$

$$= -\sqrt{\alpha n} \left(\frac{1}{2}(e^{-2\lambda_n T} - e^{-2\mu_n T}) - e^{-(\lambda_n + \mu_n)T}\right).$$

Thus the behavior of $w_n$ for large $n$ is given by

$$-\frac{n\sqrt{\alpha}}{2} e^{-2\lambda_n T} + o(1) < w_n < -\frac{n\sqrt{\alpha}}{2} (e^{-2\lambda_n T} - e^{-(\lambda_n + \mu_n)T}) + o(1).$$

(4.6)

Similarly, we find

$$w_{n,1} = \int_0^T s_n^2(t) dt = n^2 \alpha \int_0^T (e^{-2\lambda_n t} + e^{-2\mu_n t} - 2e^{-(\lambda_n + \mu_n)t}) dt$$

$$= n^2 \alpha \left[ \frac{e^{-2\lambda_n t}}{-2\lambda_n} + \frac{e^{-2\mu_n t}}{-2\mu_n} + \frac{2e^{-(\lambda_n + \mu_n)t}}{(\lambda_n + \mu_n)} \right]_0^T$$

$$= n^2 \alpha \left( \frac{1}{2\lambda_n} + \frac{1}{2\mu_n} - \frac{2}{\lambda_n + \mu_n} - \frac{1}{2} \left( \frac{e^{-2\lambda_n T}}{\lambda_n} + \frac{e^{-2\mu_n T}}{\mu_n} \right) + \frac{2e^{-(\lambda_n + \mu_n)T}}{\lambda_n + \mu_n} \right).$$
Further, we have

\begin{equation}
\frac{n^2\alpha}{2\lambda_n}(1 - e^{-2\lambda_n T}) + O(1) \leq w_{n,1} \leq \frac{n^2\alpha}{2\lambda_n} \left(1 - e^{-2\lambda_n T} + \frac{\lambda_n}{\mu_n}\right) + O(1)
\end{equation}

and

\begin{equation}
w_{n,2} = \int_0^T s_{n,2}(t)\,dt = \int_0^T \left(\mu_n^2 e^{-2\mu_n t} + \lambda_n^2 e^{-2\lambda_n t} - 2\mu_n \lambda_n e^{-(\lambda_n + \mu_n)t}\right)\,dt
\end{equation}

\begin{align*}
&= \left[-\frac{\mu_n e^{-2\mu_n t}}{2} - \frac{\lambda_n e^{-2\lambda_n t}}{2} + 2\mu_n \lambda_n e^{-(\lambda_n + \mu_n)t}\right]_0^T \\
&= \frac{\mu_n + \lambda_n}{2} - 2\mu_n \lambda_n \frac{1}{(\mu_n + \lambda_n)} - \frac{1}{2}(\mu_n e^{-2\mu_n T} + \lambda_n e^{-2\lambda_n T}) + 2\mu_n \lambda_n e^{-(\lambda_n + \mu_n)T}.
\end{align*}

Hence, we can write

\begin{equation}
\frac{\nu_0 n^2}{2} + \frac{\lambda_n}{2}(1 - e^{-2\lambda_n T}) + O(1) \leq w_{n,2} \leq \frac{\nu_0 n^2}{2} + \frac{\lambda_n}{2} + O(1).
\end{equation}

Further, we have

\begin{equation}
W_{n,T}^{-1} = \frac{(\mu_n - \lambda_n)^2}{w_{n,1} w_{n,2} - w_n^2} \begin{bmatrix} w_{n,2} & -w_n \\ -w_n & w_{n,1} \end{bmatrix}.
\end{equation}

From the expression of $w_{n,1}, w_{n,2},$ and $w_n,$ it follows that

\begin{align*}
\frac{\alpha\nu_0 n^4}{8\lambda_n}(1 - e^{-2\lambda_n T}) + \frac{n^2\alpha}{4}(e^{-4\lambda_n T} + 2e^{-2\lambda_n T} - 6) + O(1) \\
\leq w_{n,1} w_{n,2} \leq \frac{\alpha\nu_0 n^4}{4\lambda_n} + \frac{n^2\alpha}{4} + O(1),
\end{align*}

\begin{align*}
n^2\alpha - e^{-4\lambda_n T} + O(1) \leq w_n^2 \leq \frac{n^2\alpha}{4}(e^{-4\lambda_n T} + e^{-2(\lambda_n + \mu_n)T}) + o(1).
\end{align*}

Thus $\det(W_{n,T})$ satisfies (4.5).

Now we can prove the following.

**Theorem 4.2.** If system (3.2) is driven to rest in time $T > 0$ by using the minimal norm control $\hat{f}_n$ determined in (4.1), setting $\hat{f}_n(t) = D_n(t) \mathbf{U}_{0,n}$ with $D_n(t) = [d_{n,1}(t) \quad d_{n,2}(t)]$ as defined in (4.3), then for $n$ large we have

\begin{equation}
d_{n,1}(t) = O(ne^{-\omega_0(T-t)}) \quad \text{and} \quad d_{n,2}(t) = O(e^{-\omega_0(T-t)})
\end{equation}

uniformly with respect to $t \in [0, T - \epsilon]$ for any $0 < \epsilon < T$.

**Proof.** Using the expression for the control of minimum norm, we have

\begin{equation}
D_n(t) = -B^* e^{-(T-t)A^*_n}e^{-TA_n}W_{n,T}^{-1}
\end{equation}

\begin{align*}
&= -\frac{1}{\det(W_{n,T})} \begin{bmatrix} -s_{n}(T - t) & s_{n,2}(T - t) \\ -w_n & -w_n \end{bmatrix} \begin{bmatrix} w_{n,2} & -w_n \\ -w_n & w_{n,1} \end{bmatrix} \begin{bmatrix} s_{n,1}(T) & -s_n(T) \\ s_n(T) & s_{n,2}(T) \end{bmatrix} \\
&= \frac{1}{\det(W_{n,T})} \begin{bmatrix} \hat{d}_{n,1}(t) & \hat{d}_{n,2}(t) \end{bmatrix},
\end{align*}
Recall from Lemma 2.7 that for all \( n \) and all \( t \in (0, T] \), \( s_n(T-t) \) is bounded. Hence for large \( n \), the dominant terms in these two expressions are

\[
\tilde{d}_{n,1}(t) := w_{n,2}s_n(T-t)s_n(T) - s_n(T)s_n(T-t)w_n, \quad \tilde{d}_{n,2}(t) := -w_{n,2}s_n(T-t)s_n(T).
\]

Using all the earlier estimates from Lemma 2.7, we see that

\[
\frac{\nu_0^2 n^5 \sqrt{\alpha}}{4} e^{-2\omega_0(T-t)} e^{-2\omega_0 T} + O(n^3) \leq w_{n,2}s_n(T-t)s_n(T)
\]

Thus, we can write

\[
\frac{\nu_0^2 n^5 \sqrt{\alpha}}{4} e^{-2\omega_0(T-t)} e^{-2\omega_0 T} + O(n^3) \leq \frac{\nu_0^2 n^5 \sqrt{\alpha}}{2} e^{-\omega_0 T} e^{-\omega_0(T-t)} + O(n^3).
\]

Similarly, the other terms satisfy

\[
\frac{\alpha \sqrt{n^3}}{2} e^{-2\omega_0 T} e^{-2\omega_0 T} e^{-2\lambda_n T} + o(1) \leq -w_{n,2}s_n(T-t)s_n(T)
\]

From (4.5) for sufficiently large \( n \), we have

\[
\frac{\alpha \nu_0 n^4}{32\omega_0 (1 - e^{-2\omega_0 T})} \leq \text{det} W_n, T \leq \frac{\alpha (\nu_0 + \omega_0)n^4}{2\omega_0}.
\]

From these estimates, we see that

\[
\frac{\nu_0^2 \omega_0 n}{2 \sqrt{\alpha (\omega_0 + \nu_0)}} e^{-2\omega_0 T} e^{-2\omega_0(T-t)} + o(1) \leq \frac{\tilde{d}_{n,1}}{\text{det}(W_n, T)} \leq \frac{32 n \nu_0 n}{\sqrt{\alpha (e^{\omega_0 T} - e^{-\omega_0 T})}} e^{-\omega_0(T-t)} + o(1),
\]

\[
\frac{\nu_0 \omega_0}{(\nu_0 + \omega_0)} e^{-2\omega_0 T} e^{-2\omega_0(T-t)} + o(1) \leq \frac{\tilde{d}_{n,2}}{\text{det}(W_n, T)} \leq \frac{16 \omega_0}{(e^{\omega_0 T} - e^{-\omega_0 T})} e^{-\omega_0(T-t)} + o(1),
\]
so that
\[ D_n(t) = [d_{n,1}(t) \quad d_{n,2}(t)] = [O(ne^{-\omega_0(T-t)}) \quad O(e^{-\omega_0(T-t)})] \]
uniformly with respect to \( t \in [0, T - \varepsilon] \) for any \( 0 < \varepsilon < T \). □

5. Null controllability. We will explore first interior and then boundary and localized interior null controllability.

5.1. Interior control. Now we are in a position to answer the interior null controllability question for our system, using the earlier estimates.

**Theorem 5.1.** Let us denote \( Y := H^1_m(I_\pi) \times L^2(I_\pi) \), where \( H^1_m(I_\pi) = \{ \rho \in H^1(I_\pi) \mid \int_0^\pi \rho(x)dx = 0 \} \). Assume that \( U_0 = (\rho_0, u_0)^T \in Z_0 \) and that the system (3.1) is null controllable in time \( T > 0 \) by an interior control \( f \in L^2(0, T; L^2(I_\pi)) \) for the velocity. Then \( U_0 = (\rho_0, u_0)^T \in Y \). Conversely, assume that \( U_0 = (\rho_0, u_0)^T \in Y \). Then, for every \( T > 0 \), the system (3.1) is null controllable in time \( T \) by an interior control \( f \in L^2(0, T; L^2(I_\pi)) \) for the velocity.

**Proof.** If the system (3.1) were null controllable in time \( T > 0 \), using an interior control for the velocity, then there will exist a minimum norm control \( F(x,t) = (0, f(x,t))^T \) which brings the system to rest in time \( T \).

Then the projections of this control \( F \) into the space \( V_n \), say, \( F_n(t) = (0, f_n(t))^T \), will bring the finite dimensional system on \( V_n \) to rest in time \( T \). Since

\[ \|F\|_{L^2(0,T;Z_0)}^2 = \sum_{n=1}^{\infty} \|f_n\|_{L^2(0,T)}^2, \]

the controls \( f_n \) are also with minimal norm. Moreover, we have

\[ f_n(t) = D_n(t)U_{0,n} = d_{n,1}(t)U_{0,n,1} + d_{n,2}(t)U_{0,n,2} \tag{5.1} \]

and

\[ \sum_{n=1}^{\infty} \int_0^T |d_{n,1}(t)U_{0,n,1} + d_{n,2}(t)U_{0,n,2}|^2 dt < \infty. \tag{5.2} \]

Since \( U_0 = (\rho_0, u_0)^T \in Z_0 \), we also know that

\[ \sum_{n=1}^{\infty} |U_{0,n,1}|^2 < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |U_{0,n,2}|^2 < \infty. \]

From Theorem 4.2 and (5.2), it follows that

\[ \sum_{n=1}^{\infty} n^2 |U_{0,n,1}|^2 < \infty. \]

Since \( Y = H^1_m(I_\pi) \times L^2(I_\pi) \) is the orthogonal sum of the spaces \( \{V_n\}_{n \geq 1} \), one can verify that

\[ \|(\rho_0, u_0)^T\|^2_{Y} = \sum_{n=1}^{\infty} n^2 U_{0,n,1}^2 + \sum_{n=1}^{\infty} U_{0,n,2}^2. \tag{5.3} \]

Thus if \( U_0 \) belongs to \( Z_0 \) and the system (3.1) is null controllable in time \( T > 0 \), by \( f \in L^2(0, T; L^2(I_\pi)) \), then \( U_0 \) belongs to \( Y \).
Conversely, let us assume that $U_0 = (\rho_0, u_0)^T \in Y$. The control $f_n$ given in (5.1) brings the finite dimensional system on $V_n$ to zero in time $T > 0$. Our estimates derived from Theorem 4.2 for the coefficients $d_{n,1}$ and $d_{n,2}$, together with the condition

$$
\| (\rho_0, u_0)^T \|_Y^2 = \sum_{n=1}^{\infty} n^2 U_{0,n,1}^2 + \sum_{n=1}^{\infty} U_{0,n,2}^2 < \infty,
$$

imply that

$$
\sum_{n=1}^{\infty} \| f_n \|_{L^2(0,T)}^2 < \infty.
$$

Thus the control

$$
F = \sum_{n=1}^{\infty} f_n \Phi_{2n-1}
$$

belongs to $L^2(0,T; \mathbf{Z}_0)$ and it brings the system to zero in time $T$. This completes the proof. □

5.2. Boundary control. Here we will answer in the negative the question of null controllability in time $T$ of the system

$$
\begin{align*}
\partial_t [ \begin{bmatrix} \rho \\ u \end{bmatrix} ] + \begin{bmatrix} 0 & Q_0 \frac{d}{dx} \\ b \frac{d}{dx} & -\nu_0 \frac{d^2}{dx^2} \end{bmatrix} \begin{bmatrix} \rho \\ u \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
\rho(x,0) &= \rho_0(x), \quad u(x,0) = u_0(x), \quad x \in I_\pi, \\
u(0,t) &= 0, \quad u(\pi, t) = q(t), \quad t > 0,
\end{align*}
$$

(5.4)

using a regular boundary control $q \in H^1(0,T)$ at $\pi$ for the velocity component. The initial condition $(\rho_0, u_0)$ is in the space $H^1_m(I_\pi) \times H^1(I_\pi)$. (Here $H^1(I_\pi)$ is the space of functions in $H^1(I_\pi)$ vanishing at 0.) First we recall that the solution of (5.4) is also regular enough. The following proposition can be proved on the same lines as in [2, Theorem 3.1].

**Proposition 5.2.** Given $(\rho_0, u_0)$ in $H^1_m(I_\pi) \times H^1(I_\pi)$ and $q \in H^1(0,\infty)$, satisfying the compatibility condition $u_0(\pi) = q(0)$, the system (5.4) has a unique solution $(\rho, u)$ with $\rho \in H^1(0,\infty; H^1(I_\pi))$ and $u \in H^1(0,\infty; L^2(I_\pi)) \cap L^2(0,\infty; H^2(I_\pi))$ satisfying

$$
\| \rho \|_{H^1(0,\infty; H^1)} + \| u \|_{H^1(0,\infty; L^2)} + \| u \|_{L^2(0,\infty; H^2)} \leq C(\| U_0 \|_{H^1_m(I_\pi) \times H^1(I_\pi)} + \| q \|_{H^1(0,\infty)}).
$$

To discuss the boundary null controllability, the idea is, as in [20] and [17], to use the adjoint equation to derive certain identity equivalent to the boundary null controllability. For that, we consider the adjoint problem

$$
\begin{align*}
-\partial_t [ \begin{bmatrix} \sigma \\ v \end{bmatrix} ] + \begin{bmatrix} 0 & -Q_0 \frac{d}{dx} \\ -b \frac{d}{dx} & -\nu_0 \frac{d^2}{dx^2} \end{bmatrix} \begin{bmatrix} \sigma \\ v \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
v(0,t) &= 0, \quad v(\pi, t) = 0, \quad t > 0,
\end{align*}
$$

(5.5)

$\sigma(x,T) = \sigma_T(x), \quad v(x,T) = v_T(x), \quad x \in I_\pi,$

with the terminal condition $(\sigma_T, v_T)$ in $H^1_m(I_\pi) \times H^1(I_\pi)$. Existence and regularity of
the solutions of (5.5) are given in the proposition below. For details, see Theorem 2.3 in [2].

**Proposition 5.3.** The family of operators \((S^*(t))_{t \geq 0}\) determines a strongly continuous semigroup on \(H^1_m(I_{\pi}) \times H^0_0(I_{\pi})\). For all \(V_T = (\sigma_T, v_T)^T \in H^1_m(I_{\pi}) \times H^0_0(I_{\pi})\), the solution \(V(t) = S^*(t)V_T\) satisfies

\[
\|V(t)\|_{H^1_m \times H^0_0} \leq C\|V_T\|_{H^1_m \times H^0_0}.
\]

In addition

\[
\|\sigma\|_{H^1(0,\infty;H^m_m)} + \|v\|_{H^1(0,\infty;L^2)} + \|v\|_{L^2(0,\infty;H^2)} \leq C\|V_T\|_{H^1_m \times H^0_0}.
\]

We also need a regularity result for the nonhomogeneous adjoint system

\[
-\partial_t \begin{bmatrix} \sigma \\ v \end{bmatrix} + \begin{bmatrix} 0 & -Q_0 \frac{d}{dx} \\ -b \frac{d}{dx} & -\nu_0 \frac{d^2}{dx^2} \end{bmatrix} \begin{bmatrix} \sigma \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix},
\]

(5.6)

with \(v(0,t) = 0, v(\pi,t) = 0, t > 0\), \(\sigma(x,T) = 0, v(x,T) = 0, x \in I_{\pi}\).

**Proposition 5.4.** If \((f,g)^T \in L^2(0,T;Z)\), then the solution \(V\) to (5.6) belongs to \(L^2(0,T;D(A^*)\)) and it satisfies

\[
\|V\|_{L^2(0,T;D(A^*))} \leq C\|(f,g)^T\|_{L^2(0,T;Z)}.
\]

In particular, we have

\[
\|b\sigma(\pi, \cdot) + \nu_0 v_x(\pi, \cdot)\|_{L^2(0,T)} \leq C\|(f,g)^T\|_{L^2(0,T;Z)}.
\]

This proposition follows from Theorem 3.1, Part II, in [4].

Let \((\rho, u)\) and \((\sigma, v)\) be the solutions of (5.4) and (5.5), respectively, as mentioned in the above propositions. Taking the inner product in \(Z\) of (5.4) with \((\sigma, v)^T\) and integrating, we obtain

\[
\int_0^T \left\langle \begin{bmatrix} \partial_t \rho + Q_0 u_x \\ \partial_t u + b \rho_x - \nu_0 v_{xx} \end{bmatrix}, \begin{bmatrix} \sigma \\ v \end{bmatrix} \right\rangle_{Z} \, dt = 0.
\]

An integration by parts and the use of (5.5) leads to

\[
Q_0 \int_0^T (b\sigma(\pi,t) + \nu_0 v_x(\pi,t)) q(t) \, dt
\]

\[
= b \int_0^\pi [\rho_0(x)\sigma(x,0) - \rho(x,T)\sigma_T(x)] \, dx + Q_0 \int_0^\pi [u_0(x)v(x,0) - u(x,T)v_T(x)] \, dx.
\]

The above relation will lead us to the identity equivalent to the boundary null controllability.

**Proposition 5.5.** For each initial state \((\rho_0, u_0) \in H^1_m(I_{\pi}) \times H^1_0(I_{\pi})\), the solution of the system (5.4) can be brought to rest in time \(T\) by a control \(q \in H^1(0,T)\) satisfying \(q(0) = u_0(\pi)\) if and only if

\[
Q_0 \int_0^T [b\sigma(\pi,t) + \nu_0 v_x(\pi,t)] q(t) \, dt = \left\langle \begin{bmatrix} \rho_0 \\ u_0 \end{bmatrix}, \begin{bmatrix} \sigma(\cdot,0) \\ v(\cdot,0) \end{bmatrix} \right\rangle_{Z}
\]

for all \((\sigma_T, v_T)^T \in H^1_m(I_{\pi}) \times H^0_0(I_{\pi})\), where \((\sigma, v)^T\) is the solution of the adjoint system (5.5).
Using this proposition, we now rule out the boundary null controllability.

**Theorem 5.6.** For any $T > 0$, no nontrivial finite linear combination of eigenvectors can be driven to rest by a boundary control $q \in H^1(0,T)$ at $\pi$ for the velocity.

**Proof.** Recall that $H_0^1(I_\pi) \times H_0^1(I_\pi)$ is the orthogonal sum of $\{V_n\}_{n \geq 1}$. Thus we can write

\[
(\rho_0, u_0)^T = \sum_{n=1}^{\infty} c_n \Phi_{2n} + \sum_{n=1}^{\infty} d_n \Phi_{2n-1}.
\]

Since $\Phi_{2n}, \Phi_{2n-1}$ are linear multiples of $(\cos(nx), 0)^T$ and $(0, \sin(nx))^T$, respectively, $(\rho_0, u_0)^T \in H_0^1(I_\pi) \times H_0^1(I_\pi)$ can be written as

\[
\rho_0(x) = \sum_{n \geq 1} c_n \cos(nx), \quad u_0(x) = \sum_{n \geq 1} d_n \sin(nx).
\]

Let us choose $(\sigma_T, \nu_T)^T = a_n \xi_n^* + b_n \chi_n^*$ for some $a_n$ and $b_n$. With this terminal condition the solution of the adjoint system (5.5) is

\[
(\sigma, v)^T = a_n e^{-\lambda_n(T-t)} \xi_n^* + b_n e^{-\mu_n(T-t)} \chi_n^*.
\]

In particular, we have

\[
v_x(x, t) = [a_n \lambda_n e^{-\lambda_n (T-t)} + b_n \mu_n e^{-\mu_n (T-t)}] \frac{(-1) \cos(nx)}{Q_0}.
\]

Now applying Proposition 5.5, we get

\[
Q_0 \int_0^T \left[ \left( a_n e^{-\lambda_n (T-t)} + b_n e^{-\mu_n (T-t)} \right) b(-1)^n 
+ \left( a_n \lambda_n e^{-\lambda_n (T-t)} + b_n \mu_n e^{-\mu_n (T-t)} \right) \nu_0(-1)^{n+1} \right] q(t) dt
\]

\[
= \int_0^\pi \left( b c_n \cos^2(nx) \left( a_n e^{-\lambda_n T} + b_n e^{-\mu_n T} \right) \right) dx
\]

\[- \int_0^\pi \left( \frac{d_n}{n} \sin^2(nx) \left( a_n \lambda_n e^{-\lambda_n T} + b_n \mu_n e^{-\mu_n T} \right) \right) dx.
\]

After simplification this reduces to

\[
(-1)^n (bQ_0 - \nu_0 \lambda_n) a_n e^{-\lambda_n T} \int_0^T e^{\lambda_n t} q(t) dt
+ (-1)^n (bQ_0 - \nu_0 \mu_n) b_n e^{-\mu_n T} \int_0^T e^{\mu_n t} q(t) dt
= \frac{(c_n b\pi n - \pi d_n \lambda_n)}{2n} a_n e^{-\lambda_n T} + \frac{(c_n b\pi n - \pi d_n \mu_n)}{2n} b_n e^{-\mu_n T}.
\]

A choice of $b_n = 0$ and then $a_n = 0$ in this identity gives

\[
(-1)^n (bQ_0 - \nu_0 \lambda_n) \int_0^T e^{\lambda_n t} q(t) dt = \frac{(c_n b\pi n - \pi d_n \lambda_n)}{2n},
\]

\[
(-1)^n (bQ_0 - \nu_0 \mu_n) \int_0^T e^{\mu_n t} q(t) dt = \frac{(c_n b\pi n - \pi d_n \mu_n)}{2n}.
\]
This can be done for each \( n \in \mathbb{N} \). Thus the null controllability is equivalent to the existence of a function \( q \in H^1(0, T) \) such that for each \( n \geq 1 \), the identities (5.11) and (5.12) hold. Now we propose to show that no nontrivial finite linear combination of eigenvectors can be driven to zero in finite time. Let us take as initial conditions

\[
(q_0, u_0)^T = \sum_{n=1}^{N} c_n \xi_n + \sum_{n=1}^{N} \hat{d}_n \xi_n.
\]

This can be rewritten in the form (5.9) with the following relation between the coefficients:

\[
c_n = c_n + \hat{d}_n, \quad d_n = \frac{\lambda_n}{n \zeta_0} c_n + \frac{\mu_n}{n \zeta_0} \hat{d}_n.
\]

If the system starting from these initial conditions were null controllable, then there would exist a function \( q \in H^1(0, T) \) such that (5.11)–(5.12) holds true. For \( z \) in the complex plane, let us set

\[
F(z) = \int_0^T e^{izt}\, q(t) \, dt.
\]

Then using the Paley–Wiener theorem, \( F \) is an entire function of \( z \). Since \( c_n = d_n = 0 \) for \( n > N \), we have

\[
F(-\lambda_n i) = 0 = F(-\mu_n i) \quad \forall n > N.
\]

As \(-\lambda_n i \to -\omega_0 i\) when \( n \to \infty \), \(-\omega_0 i\) is an accumulation point of zeros of the entire function \( F \). Hence \( F \equiv 0 \). Then for \( n \leq N \) also, \( F(-\lambda_n i) = 0 = F(-\mu_n i) \), which from (5.11) and (5.12) implies that

\[
\frac{(c_n b \pi n - \pi d_n \lambda_n)}{2n} = 0 = \frac{(c_n b \pi n - \pi d_n \mu_n)}{2n}.
\]

Hence \( c_n = d_n = 0 \) for each \( n \geq 1 \). As \( \lambda_n \) and \( \mu_n \) are different, we conclude from (5.13) that

\[
c_n = \hat{d}_n = 0, \quad n \geq 1.
\]

This is a contradiction since the initial condition is nontrivial. Hence the theorem follows. \( \square \)

**Remark 5.7.** The previous theorem rules out null controllability using regular boundary control \( q \in H^1(0, T) \). We can extend this negative result to less regular controls \( q \in L^2(0, T) \). For that we need to interpret the solution of (5.4) by transposition. With Propositions 5.3 and 5.4, we can show that (5.4) has a unique solution \((\rho, u)^T \in L^2(0, T; \mathbb{Z})\) in the sense of transposition when initial condition \((\rho_0, u_0)^T \in \mathbb{Z}\) and boundary value \( q \in L^2(0, T) \). Then as in Proposition 5.5, we can conclude that the system (5.4) is null controllable in \( \mathbb{Z} \) in time \( T \) if and only if for each initial state \((\rho_0, u_0)^T \in \mathbb{Z}\) there exists some control \( q \in L^2(0, T) \) such that for any pair \((\sigma_T, v_T)^T \in H^1_m(I_n) \times H^1_m(I_n)\), the solution \((\sigma, v)^T \in C([0, T]; H^1_m(I_n) \times H^1_m(I_n))\) of (5.5) satisfies

\[
Q_0 \int_0^T (b \sigma(\pi, t) + v_0 \nu x(\pi, t)) \, q(t) \, dt = b \int_0^\pi \rho_0(x) \sigma(x, 0) \, dx + Q_0 \int_0^\pi u_0(x) \nu v(x, 0) \, dx.
\]
Thus using Remark 5.7 and the proof of Theorem 5.6, we will get the following generalization of Theorem 5.6.

**Theorem 5.8.** For any $T > 0$, no nontrivial finite linear combination of eigenvectors can be driven to rest using a boundary control $q \in L^2(0, T)$ at $\pi$ for the velocity. Thus the system (5.4) is not null controllable in $Z$ in time $T$, using boundary control $q \in L^2(0, T)$, for the velocity component $u$ at $\pi$.

**Remark 5.9.** The above results established for system (5.4), stated in the interval $(0, \pi)$, can also be proved in any arbitrary bounded open interval $(L_1, L_2)$, with $-\infty < L_1 < L_2 < +\infty$, by a suitable analysis.

### 5.3. Localized interior control

We will use the negative boundary null controllability result of the previous section to rule out null controllability using a localized interior control. This will show that Theorem 5.1 is optimal in the sense that the system is not null controllable using any boundary control or an interior control acting on a subset of $I_\pi$.

Let us analyze the null controllability issue for the following system using localized interior control $f \in L^2(0, \infty; L^2(O))$ for the velocity component, where $O$ is an open interval of $I_\pi$:

\begin{equation}
U'(t) + AU(t) = F(t), \quad U(0) = U_0,
\end{equation}

where $F(t) = (0, f(\cdot, t)\chi_O)^T$, $U_0 = (\rho_0, u_0)^T$, and $\chi_O$ is the characteristic function of $O$.

**Theorem 5.10.** For $T > 0$ and $O$ an open interval of $I_\pi$, there exists initial condition $U_0 \in H^1_0(I_\pi) \times H^1_0(I_\pi)$ such that for all $f \in L^2(0, \infty; L^2(O))$, the system (5.15) starting from $U_0$ cannot be brought to rest in time $T$.

**Proof.** We will prove this by contradiction. Let us assume that the system (5.15) is null controllable using localized interior control. Let us fix an initial condition $(\rho_0, u_0)^T \in H^1_0(I_\pi) \times H^1_0(I_\pi)$. Using Proposition 3 of [2], the solution $U = (\rho, u)^T$ of system (5.15) is in $H^1(0, \infty; H^1_0(I_\pi)) \times [L^2(0, \infty; H^2(I_\pi)) \cap H^1(0, \infty; L^2(I_\pi))]$. Hence the trace of $u$ at $x = L$, for any $L, 0 < L < \pi$ lies in $L^2(0, \infty)$. Let us choose $L \in (0, \pi)$ such that $O \subset (L, \pi)$. By our assumption, the solution $U$ vanishes in time $T$ on $I_\pi$. Hence if we take as the boundary control $q(t) = u(L, t)$, the system (5.4) on $(0, L)$ is null controllable in time $T$ for this initial condition $(\rho_0, u_0)^T$. By Remark 5.9, after a change of variable, this will contradict Theorem 5.8, since the initial condition is arbitrary. Therefore the theorem follows. 

### 6. Approximate controllability

As the system is not null controllable, except in the particular case considered in Theorem 5.1, we explore here if it is at least approximately controllable using a boundary control $q \in H^1(0, T)$ for the velocity component at $\pi$.

**Definition 6.1.** The system (5.4) is approximately controllable in time $T > 0$ when for any $U_0 = (\rho_0, u_0)^T \in Z_0$ and any other $U_T = (\rho_T, u_T)^T \in Z$ and any $\varepsilon > 0$, there exists a control $q \in H^1(0, T)$ such that the solution to system (5.4) satisfies

\[ \|U(T) - U_T\|_Z \leq \varepsilon. \]

The following theorem positively answers the question of approximate controllability.

**Theorem 6.2.** The system (5.4) is approximately controllable in time $T > 0$ by a control $q \in H^1(0, T)$. 

Proof. Setting
\[ L_T(q) = \int_0^T S(T - s) B q(s) \, ds, \]
it is well known that the system \((5.4)\) is approximately controllable when \(\text{Im} L_T\) is dense in \(\mathbb{Z}\). (See [22, Part IV, Chapter 2, Theorem 2.5].)

Then using the identity \((5.7)\), it is enough to prove that if the solution to the adjoint system \((5.5)\), with terminal condition \((\sigma_T, v_T) \in H^1_n(I_n) \times H^1_n(I_n)\), obeys
\[ (6.1) \quad \int_0^T (b\sigma(x,t) + \nu_0 v_x(x,t)) \, q(t) \, dt = 0 \]
for all \(q \in H^1(0, T)\), then \((\sigma_T, v_T) = (0, 0)\). Let us decompose the solution \((\sigma, v)\) to system \((5.5)\) as follows:
\[
(\sigma, v)^T(x, t) = \sum_{n \geq 1} c_{n, T} e^{-\lambda_n(T-t)} \xi_n^*(x) + \sum_{n \geq 1} d_{n, T} e^{-\mu_n(T-t)} \xi_n^*(x),
\]
where \(c_{n, T}\) and \(d_{n, T}\) are defined by \((\sigma_T, v_T)^T = \sum_{n \geq 1} c_{n, T} \xi_n^* + \sum_{n \geq 1} d_{n, T} \xi_n^*\). Hence, it follows that
\[
\sigma(x, t) = \sum_{n \geq 1} (-1)^n (c_{n, T} e^{-\lambda_n(T-t)} + d_{n, T} e^{-\mu_n(T-t)}),
\]
\[
v_x(x, t) = \sum_{n \geq 1} (-1)^n \left( c_{n, T} \left( -\frac{\lambda_n}{Q_0} \right) e^{-\lambda_n(T-t)} \right)
+ \sum_{n \geq 1} (-1)^n \left( d_{n, T} \left( -\frac{\mu_n}{Q_0} \right) e^{-\mu_n(T-t)} \right).
\]
Inserting these expressions in identity \((6.1)\), we obtain
\[
\int_0^T \left( \sum_{n \geq 1} (-1)^n \left( c_{n, T} (bQ_0 - \nu_0 \lambda_n) e^{-\lambda_n(T-t)}
+ d_{n, T} (bQ_0 - \nu_0 \mu_n) e^{-\mu_n(T-t)} \right) \right) q(t) \, dt = 0
\]
for every \(q \in H^1(0, T)\) and hence for every \(q \in L^2(0, T)\) by density. It follows that
\[ (6.2) \quad \sum_{n \geq 1} (-1)^n \left( c_{n, T} (bQ_0 - \nu_0 \lambda_n) e^{-\lambda_n(T-t)} + d_{n, T} (bQ_0 - \nu_0 \mu_n) e^{-\mu_n(T-t)} \right) = 0 \]
for \(0 < t < T\). Let us set
\[
p_n = (-1)^n \left\{ c_{n, T} (bQ_0 - \nu_0 \lambda_n) e^{-\frac{\lambda_n T}{2}} \right\} \quad \text{and} \quad q_n = (-1)^n \left\{ d_{n, T} (bQ_0 - \nu_0 \mu_n) e^{-\frac{\mu_n T}{2}} \right\}.
\]
Notice that the sequences \(\{p_n\}\) and \(\{q_n\}\) belong to \(\ell^1(\mathbb{C})\). Indeed, we have in view of \((2.6)\) and \((2.9)\),
\[
\sum_{n \geq 1} \left| c_{n, T} (bQ_0 - \nu_0 \lambda_n) e^{-\frac{\lambda_n T}{2}} \right| \leq c \left( \sum_{n \geq 1} (c_{n, T} n^2) \right)^{1/2} \left( \sum_{n \geq 1} \frac{1}{n^2} \right)^{1/2} < \infty,
\]
\[
\sum_{n \geq 1} \left| d_{n, T} (bQ_0 - \nu_0 \mu_n) e^{-\frac{\mu_n T}{2}} \right| \leq c \left( \sum_{n \geq 1} |d_{n, T}|^2 \right)^{1/2} \left( \sum_{n \geq 1} \mu_n^2 e^{-\mu_n T} \right)^{1/2} < \infty.
\]
Now, we consider the function
\[ \varphi(s) = \sum_{n \geq 1} p_n e^{-\lambda_n s} + q_n e^{-\mu_n s}. \]

In view of (6.2), \( \varphi(s) = 0 \) for \( 0 < s < \frac{T}{2} \). Since this function is analytic in the half complex plane \( \text{Re} \, s > 0 \), we have \( \varphi(s) = 0 \) for \( \text{Re} \, s > 0 \). The Laplace transform of \( \varphi \),
\[ L\varphi(z) = \int_0^\infty e^{-sz} \varphi(s) \, ds, \]
is well defined for \( \text{Re} \, z > -\min(v_0/2, \omega_0) \), and by a direct calculation, we have
\[ L\varphi(z) = \sum_{n=1}^{\infty} \left( \frac{p_n}{z + \lambda_n} + \frac{q_n}{z + \mu_n} \right) = 0. \]

For \( 1 \leq m < n_0 \), \( -\lambda_m \) and \( -\mu_m \) are isolated poles of \( L\varphi \) and \( L\varphi \) is identically zero in a neighborhood of each of those poles, thus \( p_m = 0 \) and \( q_m = 0 \) for \( 1 \leq m < n_0 \). Now, applying Lemma 1 in [17] to the function
\[ \sum_{n \geq n_0} p_n e^{-\lambda_n s} + q_n e^{-\mu_n s} = 0, \]
for \( 0 < s < \frac{T}{2} \), we obtain
\[ (3.3) \quad c_{n,T}(bQ_0 - \nu_0 \lambda_n) = 0 \quad \text{and} \quad d_{n,T}(bQ_0 - \nu_0 \mu_n) = 0 \]
for all \( n \geq n_0 \). The proof is complete. \( \square \)

7. Stabilizability. From the results of section 5, the systems (3.1) and (5.4) are not null controllable, except in the case of (3.1), with interior control everywhere in \( I_\pi \) and only for regular initial conditions. Thus it is natural to ask if we can at least stabilize the system for all initial values with a prescribed decay rate. This question has already been explored for the case of boundary control by Arfaoui et al. in [2]. They show that a similar system is stabilizable using boundary control with decay rate \( e^{-\omega t} \) for \( 0 < \omega < \omega_0 \), where \( \omega_0 = \frac{\text{dist}(A)}{\text{dist}(\sigma(A))} \) is the accumulation point for the real eigenvalues of \( A \) (see Theorem 4.1 in [2]).

Here we explore further the stabilizability for the system (5.4) with decay rate \( e^{-\omega t} \), with \( \omega > \omega_0 \), using boundary control for the velocity component. Since we are looking for decay rate \( e^{-\omega t} \), it is convenient to consider the shifted system
\[ \partial_t \begin{bmatrix} \rho \\ u \end{bmatrix} + \begin{bmatrix} 0 & Q_0 \frac{d}{dx} \\ b \frac{d}{dx} & -\nu_0 \frac{d^2}{dx^2} \end{bmatrix} \begin{bmatrix} \rho \\ u \end{bmatrix} - \omega \begin{bmatrix} \rho \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \]
\[ \rho(x, 0) = \rho_0(x), \quad u(x, 0) = u_0(x), \quad x \in I_\pi, \]
\[ u(0, t) = 0, \quad u(\pi, t) = q(t), \quad t > 0. \]

Since \( (\rho, u, q) \) obeys system (7.1) if and only if \( (\tilde{\rho}, \tilde{u}, \tilde{q}) = e^{-\omega t}(\rho, u, q) \) solves system (5.4), system (7.1) is stabilizable by a boundary feedback control \( q_{\rho, u_0} \in L^2(0, \infty) \) if and only if system (5.4) is stabilizable with the exponential decay rate \( e^{-\omega t} \) by the corresponding feedback boundary control. Thus studying the stabilizability of system (5.4) with the exponential decay rate \( e^{-\omega t} \) is equivalent to studying the stabilizability of system (7.1).

Throughout this section, \( \omega > \omega_0 \) is in the resolvent set of \( A \) and \( \omega \) is fixed.
7.1. The shifted system. Here we establish the existence and regularity of a solution for (7.1) and then rewrite the system as an operator equation. This system is the same as the one considered in (5.4), except for the new term $\omega (\rho, u)^T$ appearing in (7.1). Then as in Proposition 5.2, we have the following existence result.

**Proposition 7.1.** For any $T > 0$, $\mathbf{U}_0 = (\rho_0, u_0)^T \in H^1_m(I_\pi) \times H^1_{(0)}(I_\pi)$ and $q \in H^1(0, \infty)$ satisfying the compatibility condition $q(0) = u_0(\pi)$, there exists a unique solution $\mathbf{U}_{U_0,q} = (\rho, u)^T$ of (7.1), with $\rho \in H^1(0, T; H^1(I_\pi))$ and $u \in H^1(0, T; L^2(I_\pi)) \cap L^2(0, T; H^2(I_\pi)) \cap H^1_{(0)}(I_\pi)$, satisfying

$$
\|\rho\|_{H^1(0, T; H^1)} + \|u\|_{H^1(0, T; L^2)} + \|u\|_{L^2(0, T; H^2)} \\
\leq C(T)(\|\mathbf{U}_0\|_{H^1_m(I_\pi) \times H^1_{(0)}} + \|q\|_{H^1(0, T)}).
$$

**Proof.** The proof is easy and is left to the reader. \qed

Now we show the existence of a solution with less regular initial conditions.

**Proposition 7.2.** For $\mathbf{U}_0 = (\rho_0, u_0)^T \in \mathbf{Z}_0$ and $q \in H^1(0, \infty)$, there exists a unique solution $(\rho, u)$ of (7.1) in $C([0, \infty); \mathbf{Z})$.

**Proof.** Let us set

$$
\tilde{u}_0(x) = \frac{q(0)}{\pi}x, \quad \forall \ x \in [0, \pi].
$$

So $\tilde{u}_0(\pi) = q(0)$. Then with Proposition 7.1 we get $\mathbf{U}_{(0, \tilde{u}_0),q} \in C([0, \infty); \mathbf{Z})$. With Lemma 2.1, we already have $\mathbf{U}_{(\rho_0, u_0 - \tilde{u}_0),0} \in C([0, \infty); \mathbf{Z})$. Therefore using linearity we conclude that $\mathbf{U}_{(0, \tilde{u}_0),q} + \mathbf{U}_{(\rho_0, u_0 - \tilde{u}_0),0}$, the solution of (7.1), belongs to $C([0, \infty); \mathbf{Z})$. \qed

In order to find the projections of (7.1) on the finite dimensional subspaces, we need to write this evolution equation with inhomogeneous boundary condition as an inhomogeneous operator equation using the usual lifting procedure and the Dirichlet operator corresponding to $A$. For that, for each $t > 0$, we consider the stationary problem with inhomogeneous boundary condition for the second component at $\pi$

$$
\begin{bmatrix}
-\omega & Q_0 \frac{d}{dx} \\
\frac{d}{dx} & -(v_0 \frac{d}{dx} + \omega)
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2
\end{bmatrix} = \begin{bmatrix} 0 \\
0
\end{bmatrix},
$$

$$
w_2(0, t) = 0, \quad w_2(\pi, t) = q(t).
$$

**Proposition 7.3.** Assume that $q \in H^1(0, \infty)$. Then, for all $t > 0$, (7.4) admits a unique solution $\mathbf{W}(t) = (w_1(t), w_2(t))^T$ belonging to $H^1(I_\pi) \times H^2(I_\pi) \cap H^1_{(0)}(I_\pi)$. Moreover, as a function of $t \in (0, \infty)$, $\mathbf{W}$ satisfies the following estimates:

$$
\|w_1\|_{H^1(0, \infty; H^1)} + \|w_2\|_{H^1(0, \infty; H^2)} \leq C\|q\|_{H^1(0, \infty)}
$$

and

$$
\|w_1\|_{L^2(0, \infty; H^1)} + \|w_2\|_{L^2(0, \infty; H^2)} \leq C\|q\|_{L^2(0, \infty)}.
$$

**Proof.** The proof is standard using the fact that $\omega$ belongs to the resolvent set of $A$. Also it relies on the estimate

$$
\|w_1(t)\|_{H^1(I_\pi)} + \|w_2(t)\|_{H^2(I_\pi)} \leq C|q(t)|
$$

for the solution $\mathbf{W}(t) = (w_1(t), w_2(t))^T$ to (7.4). \qed
Let $H$ denote the Banach space $D(A^*)$ endowed with the graph norm of $A^*$. Let $H'$ denote the dual of $H$ with $Z$ as pivot space. Set $A_\omega := A - \omega J$. Observe that the unbounded operator $A_\omega^*$ can be viewed as an isomorphism from $H$ to $Z$ since $\omega (> \omega_0)$ belongs to the resolvent set of $A$. Thus $\hat{A}_\omega = (A_\omega^*)^*$, the adjoint of $A_\omega^* \in \text{isom}(H, Z)$, belongs to $\text{isom}(Z, H')$. But it can also be viewed as an unbounded operator in $(D(A^*))'$ with domain $Z$, and therefore it is also an extension to $(D(A^*))'$ of the unbounded operator $A_\omega$. We equip $H'$ with the inner product
\begin{equation}
(f, g)_{H'} = \langle (\hat{A}_\omega)^{-1} f, (\hat{A}_\omega)^{-1} g \rangle_Z.
\end{equation}

Now we write system (7.1) as an evolution equation. For that, we define the Dirichlet operator
\begin{equation}
D : \mathbb{R} \rightarrow Z,
D q(t) = W(t),
\end{equation}
where $W(t)$ is the solution of (7.4). We define the control operator $B \in L(\mathbb{R}, H')$ by
\begin{equation}
B := \hat{A}_\omega D,
\end{equation}
where $D \in L(\mathbb{R}, Z)$, as defined in (7.8).

**Theorem 7.4.** Assume that $q \in H^1(0, \infty)$ and $(\rho_0, u_0)^T \in Z$. Then $U \in C([0, \infty); Z)$ is the unique solution of system (7.1) if and only if it is the weak solution to the evolution equation
\begin{equation}
U'(t) + \hat{A}_\omega U(t) = Bq(t),
U(0) = U_0 = (\rho_0, u_0)^T.
\end{equation}

If $q \in L^2(0, \infty)$ and $(\rho_0, u_0)^T \in H'$, then (7.9) has a unique solution $U$ in $C([0, \infty); H')$.

**Proof.** Let $U$ be the solution of (7.1) and $W(t)$ be the solution of (7.4). Let us set $X = U - W$. Then, due to Propositions 7.2 and 7.3, $X \in C([0, \infty); Z)$. Considering $\hat{A}_\omega$ as the extension of $A_\omega$ to $H'$, we notice that $W(t)$ belongs to the domain of $\hat{A}_\omega$ and we have
\begin{align*}
X' + \hat{A}_\omega X &= -(W' + \hat{A}_\omega W) = -W', \\
X(0) &= U_0 - W(0),
\end{align*}
where $X_2$ stands for the second component of $X$. As $\hat{A}_\omega$ is also the infinitesimal generator of a semigroup on $H'$, by Duhamel’s formula we have
\begin{equation}
X(t) = e^{-t\hat{A}_\omega}(U_0 - W(0)) - \int_0^t e^{-(t-s)\hat{A}_\omega}W'(s)ds.
\end{equation}

Using integration by parts and the equation for $W$, we obtain
\begin{align*}
- \int_0^t e^{-(t-s)\hat{A}_\omega}W'(ds) = \int_0^t e^{-(t-s)\hat{A}_\omega} \hat{A}_\omega W(s)ds - W(t) + e^{-t\hat{A}_\omega}W(0).
\end{align*}

Using this in (7.10) yields
\begin{equation}
U(t) = e^{-t\hat{A}_\omega}U_0 + \int_0^t e^{-(t-s)\hat{A}_\omega} \hat{A}_\omega W(s)ds.
\end{equation}

This identity is equivalent to (7.9).
For \( q \in H^1(0, \infty) \) and \((\rho_0, u_0)^T \in H^1_0(I_x) \times H^1(I_x)\) with \( q(0) = u_0(\pi) \), the solution given by formula (7.11) coincides with the solution given by Proposition 7.1. Hence, by density arguments it follows that the solution defined by (7.11) and the one given by Proposition 7.2 also coincide when \( U_0 \in Z_0 \) and \( q \in H^1(0, \infty) \). So for \( q \in H^1(0, \infty) \) and \((\rho_0, u_0)^T \in Z_0\), (7.9) has a solution in \( C([0, T]; Z) \) and

\[
U'(t) = -\hat{A}_\omega U(t) + Bq(t) \in L^2(0, T; H')
\]

as \( \hat{A}_\omega \in \mathcal{L}(Z, H') \), \( B \in \mathcal{L}(\mathbb{R}, H') \).

For \( q \in L^2(0, \infty) \), \( Bq \in L^2(0, T; H') \) for each \( T, 0 < T < \infty \). Therefore using semigroup theory we conclude that for initial conditions \((\rho_0, u_0)^T \in H', (7.9) \) has a unique solution \( U \in C([0, T]; H') \) for each \( T, 0 < T < \infty \). \( \square \)

7.2. Estimates for stabilizing control. In this section we will show that if the system (7.9) is stabilizable by some control \( q \in L^2(0, \infty) \), then we can find a stabilizing control which depends continuously on the initial data of this system. First we recall the standard definition of stabilizability from [4, Part V, Chapter 1, section 2] when the control \( q \in L^2(0, \infty) \).

Definition 7.5. \((-\hat{A}_\omega, B) \) is said to be stabilizable in \( H' \), by controls in \( L^2(0, \infty) \), if for any \( U_0 \in H' \) there exists a control \( q \in L^2(0, \infty) \) such that the solution \( U_{U_0,q} \) of (7.9) satisfies

\[
\int_0^\infty \|U_{U_0,q}(t)\|^2_{H'} dt < \infty.
\]

The continuous dependence of the stabilizing control on the initial data is proved in the following theorem.

Theorem 7.6. Let \( \omega > \omega_0 \) belong to the resolvent set of \( A \). Assume that \((-\hat{A}_\omega, B) \) is stabilizable in \( H' \) by a \( L^2 \)-control. Then the Riccati equation

\[
X \in \mathcal{L}(H', H), \ X = X^* \geq 0,
\]

\[
-\hat{A}_\omega^* X - X \hat{A}_\omega - XB^* X + ((-\hat{A}_\omega)^*)^{-1}(-\hat{A}_\omega)^{-1} = 0
\]

admits a unique solution that we denote by \( X_{\min}^\infty \).

The equation

\[
U'(t) = (-\hat{A}_\omega - BB^* X_{\min}^\infty) U(t) \text{ in } (0, \infty), \ U(0) = U_0,
\]

admits a unique solution in \( C([0, \infty); H') \) and this solution satisfies

\[
\|U(t)\|_{H'} \leq C_1 e^{-\theta t}\|U_0\|_{H'}, \ \forall \ U_0 \in H'
\]

for some positive constants \( C_1 \) and \( \theta \). Moreover, the stabilizing control \( \hat{f}(t) = -B^* X_{\min}^\infty U(t) \) satisfies

\[
\|\hat{f}\|_{L^2(0, \infty)} \leq C\|U_0\|_{H'}.
\]

Proof. Step 1. We set \( \hat{B} = (-\hat{A}_\omega)^{-1} B \in \mathcal{L}(\mathbb{R}, Z) \). We first show that \((-\hat{A}_\omega, B) \) is stabilizable in \( H' \) by a control \( q \) in \( L^2(0, \infty) \) if and only if \((-\hat{A}_\omega, \hat{B}) \) is stabilizable in \( Z \) by the same control \( q \).

Set \( V(t) = (-\hat{A}_\omega)^{-1} U(t) \), where \( U \in C([0, \infty); H') \) is the unique solution of (7.9) for \( U_0 \in H' \) and \( q \in L^2(0, \infty) \). Then \( V(t) \in C([0, \infty); Z) \) is the weak solution of

\[
V'(t) = -\hat{A}_\omega V(t) + \hat{B}q(t), \quad V(0) = V_0 = (-\hat{A}_\omega)^{-1} U_0 \in Z.
\]
Since $\hat{A}_ω$ is an isomorphism from $Z$ to $H'$, we have
\[ C_1|| - \hat{A}_ω V(t)||_{H'} \leq ||V(t)||_{Z} \leq C_2|| - \hat{A}_ω V(t)||_{H'}, \]
where $C_1, C_2$ are positive constants. So,
\[ C_1 \int_0^\infty ||U(t)||_{Z}^2 dt \leq \int_0^\infty ||V(t)||_{Z}^2 dt \leq C_2 \int_0^\infty ||U(t)||_{H'}^2 dt. \]
Hence the equivalence between the stabilizability of $(-\hat{A}_ω, B)$ in $H'$ and of $(-\hat{A}_ω, \hat{B})$ in $Z$ is proved.

Step 2. We can easily verify that $X ∈ L(H', H)$ is a solution to (7.13) if and only if $P = (−\hat{A}_ω)^{∗}X(−\hat{A}_ω)$ is a solution to the following equation:
\[ (7.18) \quad P ∈ L(Z), \quad P = P^{∗} \geq 0, \quad −\hat{A}_ω^{∗}P − P \hat{A}_ω − P \hat{B} \hat{B}^{∗}P + I = 0. \]
Due to Step 1, we know that $(-\hat{A}_ω, \hat{B})$ is stabilizable in $Z$. From [4, Part V, Chapter 1, section 3], it follows that (7.18) admits a unique solution $P_{\text{min}}^{∞}$. Thus, $X_{\text{min}} = ((−\hat{A}_ω)^{∗})^{-1}P_{\text{min}}^{∞}(−\hat{A}_ω)^{-1} ∈ L(H', H)$ is the unique solution of (7.13). Moreover, the semigroup $(e^{(−\hat{A}_ω − BB^{∗})P_{\text{min}}^{∞}})_{t≥0}$ is exponentially stable on $Z$ and the semigroup $(e^{(−\hat{A}_ω − BB^{∗}X_{\text{min}}^{∞})})_{t≥0}$ is exponentially stable on $H'$. Thus (7.15) is proved, and (7.16) follows from (7.15).

7.3. Projection on the eigenspace. Here we compute first the projection of (7.9) on the eigenspaces corresponding to the real eigenvalues of $\hat{A}_ω$. The eigenvalues of $\hat{A}_ω$ are $λ_n − ω$ and $μ_n − ω$. The corresponding eigenfunctions are respectively the functions $\xi_n$ and $\zeta_n$ defined in (2.7). The adjoint operator $\hat{A}_ω^{∗}$ has the same eigenvalues and its eigenvectors, after normalization, are respectively
\[ \xi_n^{∗}(x) = \theta_n \left( \cos(nx), \frac{λ_n}{Q_0 n} \sin(nx) \right)^T, \quad \zeta_n^{∗}(x) = \tilde{\theta}_n \left( \cos(nx), \frac{μ_n}{Q_0 n} \sin(nx) \right)^T, \]
where the normalizing constants $θ_n$ and $\tilde{θ}_n$ are chosen suitably later on.

As $ω > ω_0$ and the real eigenvalues $\{λ_n\}$ of $A$ accumulate at $ω_0$, there are infinitely many $λ_n$ which are less than $ω$. Our aim is to calculate the projection of (7.9) on the eigenspaces corresponding to such $λ_n$'s. Let us set
\[ E_n = \text{span}\{\xi_n\}. \]

Lemma 7.7. The projection $Q_n$ from $H'$ into $E_n$, for $n ≥ n_0$, is defined by
\[ Q_n(\zeta) = \langle \zeta, \xi_n^{∗} \rangle_{H'} \xi_n, \quad \zeta ∈ H'. \]

Proof. As $Z_0$ is the orthogonal sum of the spaces $V_n$, for $n ≥ 1$, we note that the eigenfunctions $\{\xi_n\}$, $\{\zeta_n\}$ corresponding to the eigenvalues of $\hat{A}_ω$ form a complete orthogonal system in $H'$ and similarly the eigenfunctions $\{\xi_n^{∗}\}$, $\{\zeta_n^{∗}\}$ for $\hat{A}_ω^{∗}$. We normalize these eigenfunctions in such a way that
\[ \langle \xi_n^{∗}, \xi_m \rangle_{H'} = δ_{m,n} \quad \text{and} \quad \langle \zeta_n^{∗}, \zeta_m \rangle_{H'} = δ_{m,n} \]
so that
\[ (7.19) \quad θ_n = \frac{2(λ_n − ω)[Q_0 τ_0 ω^2(ω_0 − ω)n^4 + Q_0 ω^4 n^2]}{π[bQ_0 τ_0 ω^2 n^4 + bQ_0 ω^2(2λ_n − ω)n^2 + ω^3 λ_n^2]} \]
in view of the inner product in $H'$. Thus these two sequences of eigenfunctions together form a biorthogonal system in $H'$. Hence we can take the projection from $H'$ onto $E_n$ as $Q_n$. \[\square\]

We will need the following lemma regarding the action of $B^*$.

**Lemma 7.8.** Let $B \in \mathcal{L}(\mathbb{R}, H')$ be the control operator defined as $B = \tilde{A}_\omega D$ with the Dirichlet operator $D \in \mathcal{L}(\mathbb{R}, Z)$ defined in (7.8). Then for the adjoint operator $B^* \in \mathcal{L}(H, \mathbb{R})$ and $\xi_n^*$, the eigenfunction of $\tilde{A}_\omega^*$ corresponding to $(\lambda_n - \omega)$ for $n \geq n_0$, we have

$$B^* \xi_n^* = (-1)^{n+1} \theta_n \{bQ_0 - \nu_0 \lambda_n\}.$$  

**Proof.** We have $B^* = D^* \tilde{A}_\omega^*$. Here $D^*: Z \to \mathbb{R}$ is the adjoint of $D$ given by

$$\langle Dq, \tilde{V} \rangle_z = \langle q, D^* \tilde{V} \rangle_\mathbb{R}.$$  

Let us calculate $D^* \tilde{V}$, for $\tilde{V} \in Z$ of the form

$$\tilde{V} = A_\omega^* \tilde{V} \quad \text{with} \quad V \in H^1(I_\pi) \times (H^2(I_\pi) \cap H^1_0(I_\pi)).$$

Then $\tilde{V} = (\tilde{v}_1, \tilde{v}_2)$ is given by

$$\begin{bmatrix}
-\omega & -Q_0 \frac{d}{dx} \\
-b \frac{d}{dx} & -\left(\nu_0 \frac{d^2}{dx^2} + \omega\right)
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} = \begin{bmatrix}
\tilde{v}_1 \\
\tilde{v}_2
\end{bmatrix},
\begin{align*}
v_2(0) &= 0, \\
v_2(\pi) &= 0.
\end{align*}$$  

If we take the inner product of (7.20) with $W(t)$ and integrate by parts we get

$$\langle W(t), \tilde{V} \rangle_z = -\left\langle q(t), bQ_0 v_1(\pi) + \nu \frac{dv_2}{dx}(\pi) \right\rangle_\mathbb{R}.$$  

Since $\langle W(t), \tilde{V} \rangle_z = \langle Dq(t), \tilde{V} \rangle_z$, it follows that

$$\langle q(t), D^* \tilde{V} \rangle_\mathbb{R} = -q(t) \left( bQ_0 v_1(\pi) + \nu \frac{dv_2}{dx}(\pi) \right).$$

Using these we calculate for $\xi_n^* = (\xi_{n,1}, \xi_{n,2})^T$

$$\begin{align*}
\langle q(t), D^* A_\omega^* \xi_n^* \rangle_\mathbb{R} &= -q(t) (\nu_0 Q_0 \xi_{n,2}(\pi) + bQ_0 \xi_{n,1}(\pi)) \\
&= -q(t) \theta_n \cos(n\pi) (bQ_0 - \nu_0 \lambda_n). \quad \square
\end{align*}$$

These now lead to the projected equation.

**Proposition 7.9.** The projection of system (7.9) on the eigenspace $E_n$ for $n \geq n_0$ is given by the scalar equation

$$(7.21) \quad \langle U(t), \xi_n^* \rangle_{H'} + (\lambda_n - \omega)(U(t), \xi_n^*)_{H'} = b_n q(t), \quad \langle U(0), \xi_n^* \rangle_{H'} = \langle U_0, \xi_n^* \rangle_{H'},$$

where

$$b_n := B^* \xi_n^* = (-1)^{n+1} \theta_n (bQ_0 - \nu_0 \lambda_n). \quad \begin{align*}
\text{Proof. Let us define the projection of } U(t) \text{ on } E_n \\
\text{as} \quad U_n(t) := Q_n U(t) = (U(t), \xi_n^*)_{H'} \xi_n.
\end{align*}$$
Observe that
\[ Q_n(\tilde{A_n}U(t)) = (\lambda_n - \omega)(U(t), \xi_n^*)_{H'} \xi_n. \]

Now the projection of the inhomogeneous term is, by the definition of \( Q_n \),
\[ Q_n(Bq(t)) = (Bq(t), \xi_n^*)_{H'} \xi_n = (q(t), B^* \xi_n)_{\mathbb{R}} \xi_n = b_n q(t) \xi_n. \]

Thus the projected system in the one-dimensional space \( E_n \) is
\[ (7.23) \quad U_n'(t) + (\lambda_n - \omega)U_n(t) = b_n q(t) \xi_n, \quad U_n(0) = Q_n(U_0). \]

After dropping \( \xi_n \), we get the scalar equation (7.21).

**7.4. Negative results for stabilizability.** Here we first get the expression for the minimum norm control \( q_n \) that stabilizes the one-dimensional projected system.

**Lemma 7.10.** If the system (7.9) is stabilizable in \( H' \) by boundary control \( q \in L^2(0, \infty) \), then the projected scalar equation (7.21) is also stabilizable in \( E_n \) and the minimum norm control, for \( n \geq n_0 \), is given by
\[ q_n(t) = -2e^{-t(\omega - \lambda_n)} \left( \frac{\omega - \lambda_n}{b_n} \right) (U_0, \xi_n^*)_{H'}. \]

**Proof.** We apply to our scalar equation (7.21) the results for infinite dimensional system from [12] (see Proposition 2, section 5), where the expression for minimum norm control is obtained using variational arguments. Denoting
\[ W_\infty := \int_0^\infty e^{-t(\omega - \lambda_n)} b_n^2 e^{-t(\omega - \lambda_n)} dt = \frac{b_n^2}{2(\omega - \lambda_n)}, \]
we can write the expression for the minimum norm control that stabilizes the scalar equation (7.21) as
\[ (7.24) \quad \tilde{q}_n(t) = -b_n 2(\omega - \lambda_n) e^{-t((\omega - \lambda_n) - (2(\omega - \lambda_n)) (U_0, \xi_n^*)_{H'}
= -2e^{-t(\omega - \lambda_n)} \left( \frac{\omega - \lambda_n}{b_n} \right) (U_0, \xi_n^*)_{H'}. \]

Using (7.24), we now answer in the negative the question of stabilizability.

**Theorem 7.11.** The system (5.4) is not boundary stabilizable in \( H' \) with an exponential decay rate \( \omega \), for \( \omega > \omega_0 \) and \( \omega \) in the resolvent set of \( A \), by a boundary control \( q \in L^2(0, \infty) \) at \( x = \pi \) for the velocity component \( u \).

**Proof.** Let us assume that the system is stabilizable using boundary control \( f \). Then, by Theorem 7.6, there exists a positive constant \( K_1 \) such that for any initial condition \( U_0 \in H' \), we have
\[ (7.25) \quad \|f\|_{L^2(0, \infty)} \leq K_1 \|U_0\|_{H'}. \]

Then, for the initial condition \( X_n = \frac{\xi_n}{\|\xi_n\|} \in H' \) with \( n \geq n_0 \), there exists a control \( f_n \) satisfying (7.25), which stabilizes the system (7.9). Since \( Z \) is continuously embedded in \( H' \), we have
\[ \|f_n\|_{L^2(0, \infty)} \leq K_2, \quad \forall n \geq n_0. \]
On the other hand, if we consider the projected system (7.21) on $E_n$ with the initial condition

$$\langle X_n, \xi_n^* \rangle_{H'} = \frac{1}{\|\xi_n\|_Z};$$

then, from the above calculations, the minimal norm control is

$$q_n(t) = -2e^{-t(\omega - \lambda_n)} \frac{\omega - \lambda_n}{b_n\|\xi_n\|_Z}.$$ 

Thus the norm of $q_n$ is

$$\|q_n\|_{L^2(0, \infty)} = \sqrt{\frac{2(\omega - \lambda_n)}{|b_n\|\xi_n\|_Z}}.$$ 

As $f_n$ also stabilizes the same equation, we have

$$(7.26) \quad \|f_n\|_{L^2(0, \infty)} \geq \|q_n\|_{L^2(0, \infty)}.$$ 

With (2.8) and (7.19), we obtain

$$bQ_0 - \nu_0\lambda_n = -\frac{\omega_0^2}{n^2} + o\left(\frac{1}{n^2}\right),$$

$$\theta_n = \frac{2}{b\pi}(\lambda_n - \omega)\left((\omega_0 - \omega) + \left(\frac{\omega_0\omega + 2\lambda_n\omega - 2\lambda_n\omega_0}{\nu_0n^2}\right)\right) + o\left(\frac{1}{n^2}\right),$$

$$\|\xi_n\|_Z = \sqrt{\frac{\pi}{2}}\left(b + \frac{\lambda_n^2}{n^2Q_0}\right).$$

Using the expression for $b_n$ from (7.22) and the above estimates, it follows that

$$|b_n| = \frac{2\omega_0^2(\omega - \omega_0)(\omega - \lambda_n)}{b\pi n^2} + o\left(\frac{1}{n^2}\right).$$

Thus

$$\|b_n\|\xi_n\|_Z = O\left(\frac{1}{n^2}\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$ 

Hence, the sequence $\{\|q_n\|_{L^2(0, \infty)}\}_n$ is unbounded but $\{\|f_n\|_{L^2(0, \infty)}\}_n$ is bounded. This contradicts (7.26). Thus the system is not boundary stabilizable for arbitrary $U_0 \in H'$. 

Let us end the paper by completing a result stated in Theorem 4.1 in [2]. It is shown there that system (5.4) is stabilizable in $H^1_m(I_\pi) \times H^1_{0, \pi}(I_\pi)$ with an exponential decay rate $-\omega$ with $0 < \omega < \omega_0$, by boundary controls in $H^2(0, \infty)$ satisfying

$$(7.27) \quad \|qU_0\|_{H^2(0, \infty)} \leq K_3\|U_0\|_{H^1_m \times H^1},$$

where $qU_0$ is the control stabilizing the initial condition $U_0 \in H^1_m(I_\pi) \times H^1_{0, \pi}(I_\pi)$. Here, using the same arguments as in the proof of Theorem 7.11, we prove that the system is not stabilizable $H^1_m(I_\pi) \times H^1_{0, \pi}(I_\pi)$ with an exponential decay rate $-\omega$ when $\omega > \omega_0$. 

Corollary 7.12. The system (5.4) with initial condition in \( U_0 \in H^1_0(I_x) \times H^1_0(I_x) \) is not boundary stabilizable in \( H^1_0(I_x) \times H^1_0(I_x) \) with the exponential decay rate \( -\omega \), for \( \omega > \omega_0 \) and \( \omega \) belonging to the resolvent set of \( A \), by a boundary control \( q_{u_0} \in H^2(0, \infty) \) satisfying (7.27) and the compatibility condition \( q_{u_0}(0) = u_0(\pi) \).

Proof. Let us argue by contradiction. We choose the initial condition \( X_n = \xi_n \notin H^1_{\binfty}(I_x) \times H^1(I_x) \) for \( n \in \mathbb{N}^* \), where

\[
\| \xi_n \|_{H^1_{\binfty} \times H^1} = \| \xi_n \|_{Z} + \| \nabla \xi_n \|_{Z}.
\]

Notice that

\[
\nabla \xi_n(x) = \left( \frac{-n \sin(nx)}{Q_0} \cos(nx) \right), \quad \| \nabla \xi_n \|_{Z} = \pi \sqrt{\frac{n}{2} \left( b + \frac{\lambda_2^2}{n^2 Q_0} \right)}.
\]

If (7.27) holds true, we have

\[
\| q_n \|_{L^2(0, \infty)} \leq \| q_X \|_{L^2(0, \infty)} \leq \| q_X \|_{H^1(0, \infty)} \leq K_3 \| X_n \|_{H^1_{\binfty} \times H^1} = K_3,
\]

where \( q_n \) is the \( L^2 \)-control of minimal norm stabilizing \( X_n \). Now we follow the same arguments as before. Since \( |b_n| \| \xi_n \|_{H^1_{\binfty} \times H^1} \) is now of order \( O(\frac{1}{n}) \) and is still going to zero, the control \( q_n \) of minimum norm stabilizing \( X_n \) is such that \( \| q_n \|_{L^2(0, \infty)} \) is unbounded. Thus, we obtain a contradiction and the proof is complete. \( \square \)

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