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Uniqueness of unbounded viscosity solutions for impulse control problem ☆

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Abstract

We study here the impulse control problem in infinite as well as finite horizon. We allow the cost functionals and dynamics to be unbounded and hence the value function can possibly be unbounded. We prove that the value function is the unique viscosity solution in a suitable subclass of continuous functions, of the associated quasivariational inequality. Our uniqueness proof for the infinite horizon problem uses stopping time problem and for the finite horizon problem, comparison method. However, we assume proper growth conditions on the cost functionals and the dynamics.

Keywords: Dynamic programming principle; Viscosity solution; Quasivariational inequality; Impulse control

1. Introduction

The study of optimal control problems with continuous controls, gives rise to Hamilton–Jacobi–Bellman equations which are satisfied by the value function corresponding to the

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problem, if it is smooth. Different kinds of control problems like optimal stopping time problem and problems involving switching control and impulse control have been studied in the literature. It is known that the value function of these problems, whenever smooth, satisfy different variational and quasivariational inequalities. But most of the time these value functions are only continuous and not sufficiently smooth. The notion of viscosity solutions, a kind of generalized solutions, introduced by Crandall and Lions [6] is extremely well suited for these problems. The value function satisfies the corresponding equations or inequalities in the viscosity sense. These control problems are studied in the viscosity solution set up, for example, in [2,5]. See also [4], and the references therein. Yong in [9], also studies differential games where one person uses impulse controls and other uses continuous controls. In all these works the uniqueness results are obtained assuming that the dynamics and cost functionals are bounded and uniformly continuous and hence the value functions are in the bounded uniformly continuous function class, $BUC(\mathbb{R}^d)$.

Our aim in this work is to relax the boundedness assumption on cost functionals and the dynamics for impulse control problem and to characterize the value function as the only solution of the associated quasivariational inequality for the infinite horizon problem in a suitable subclass of $C_{bb}(\mathbb{R}^d)$, bounded below continuous functions on $\mathbb{R}^d$, with suitable growth. Uniqueness in the class of lower bounded uniformly continuous functions, $UC_{bb}(\mathbb{R}^d)$ is obtained via a sequence of stopping time problems. In the process we also show uniqueness in the same class for the stopping time problem. By allowing the dynamics also to be unbounded, and cost functionals to be continuous with a certain growth, uniqueness in the class of continuous functions with suitable growth is then proved using the same method. Similar parallel results for the finite horizon problem in uniformly continuous function class, $UC_{bb}(\mathbb{R}^d \times [0,T])$ as well as continuous function class $C_{bb}(\mathbb{R}^d \times [0,T])$, are proved using comparison principle method. The same method has been used in [3,8] but our idea is different from theirs. While they modify the auxiliary function a finite number of times to arrive at the comparison result, we choose the parameters in the auxiliary function suitably for the same purpose.

Earlier Ishii, in [7] proved the uniqueness of unbounded viscosity solutions of HJB equations in different function classes under different assumptions on Hamiltonian. Our work extends some of these results to quasivariational inequalities arising from impulse control problem with unbounded cost functionals and dynamics and also the earlier results of [2,3], for impulse control problems with bounded costs and dynamics in infinite horizon. Yong in [8], proves uniqueness of unbounded value functions for infinite horizon optimal control problems with continuous, impulse and switching controls, in the class of uniformly continuous functions and in the subclass of uniformly continuous functions with sublinear growth under assumptions very similar to ours. Our method for value functions in uniformly continuous class is different. Further, we extend the results of [8] to the case of continuous functions with a certain growth m, depending on the dynamics and the discount factor and this m can be larger than one. We also deal with the finite horizon case in the same spirit as in [7], first proving a local comparison theorem in a cone and then a global comparison theorem, for quasivariational inequalities. Similarly for the switching problem Ball et al. [1] have extended the result of [5] in to the case of unbounded solutions.

Now we describe the control problem. The trajectory of the impulse control problem in infinite horizon is given by

$$\begin{aligned} &\dot{X}(t) = f\big(X(t), u(t)\big) \quad \text{if } t \in (\theta_i, \theta_{i+1}), \\ &X_x(0) = x, \\ &X_x(\theta_i^+) = X_x(\theta_i^-) + \xi_i, \end{aligned} \tag{1.1}$$

where $f : \mathbb{R}^d \times U \to \mathbb{R}^d$, $\theta = (\theta_i)_{i \in \mathbb{N}}$ is a nondecreasing sequence of positive reals which satisfies $\theta_n \to \infty$ when $n \to \infty$ and $\xi = (\xi_i)_{i \in \mathbb{N}}$ is a sequence of elements of $(\mathbb{R}^+)^d$. Finally, $u(t) : [0, \infty) \to U$ is any measurable function, where U is a compact metric space and $\beta = (\theta, \xi, u(\cdot))$ is the control variable. The total discounted cost $J(x, \beta)$ is given by

$$J(x, \beta) = \int_0^\infty k\big(X_x(t), u(t)\big) e^{-\lambda t}\, dt + \sum_{i \in \mathbb{N}} c\big(X_x({\theta_i}^-), \xi_i\big) e^{-\lambda \theta_i}, \tag{1.2}$$

where $X_x(t)$ is the solution of (1.1), k is the running cost, $c(x, \xi)$ is the impulse cost and $\lambda > 0$ is the discount factor. The optimal cost functional is then defined to be $V(x)$:

$$V(x) = \inf_\beta \left\{ \int_0^\infty k\big(X_x(t), u(t)\big) e^{-\lambda t}\, dt + \sum_{i \in \mathbb{N}} c\big(X_x({\theta_i}^-), \xi_i\big) e^{-\lambda \theta_i} \right\}. \tag{1.3}$$

We make the following assumptions:

(A1) $f : \mathbb{R}^d \times U \to \mathbb{R}^d$ is bounded by F and Lipschitz continuous in the first variable:

$$\big|f(x, u) - f(y, u)\big| \leqslant L|x - y| \quad \forall u \in U. \tag{1.4}$$

(A2) $k : \mathbb{R}^d \times U \to \mathbb{R}^+$ is nonnegative and uniformly continuous in the first variable with modulus of continuity ω_k. That is:

$$\big|k(x, u) - k(y, u)\big| \leqslant \omega_k\big(|x - y|\big) \quad \forall x, y \in \mathbb{R}^d \text{ and } u \in U. \tag{1.5}$$

(A3) U is a compact metric space.

(A4) $c(x, \xi)$ is nonnegative and uniformly continuous in the first variable with modulus of continuity ω_c. That is

$$\big|c(x, \xi) - c(y, \xi)\big| \leqslant \omega_c\big(|x - y|\big) \quad \forall x, y \in \mathbb{R}^d \text{ and } \xi \in (\mathbb{R}^+)^d.$$

Moreover, there exist constants C_1 and C_2 such that, for all ξ,

$$C_1\big(1 + |x|\big) \leqslant c(x, \xi) \leqslant C_2\big(1 + |x|\big). \tag{1.6}$$

(A5) The discount factor $\lambda > L$ where L is as in (A1).

It can be shown by using similar arguments as in the case of bounded cost functions as in [2,4] that, under the assumptions, (A1)–(A5) V defined by (1.3) is nonnegative uniformly continuous on $\mathbb{R}^d$. Hence it belongs to $UC_{bb}(\mathbb{R}^d)$. Moreover, it satisfies optimality principle called dynamic programming principle. We can further show that it satisfies the following quasivariational inequality in the viscosity sense:

$$\max\big\{V + H(x, DV), V - MV\big\} = 0 \quad \forall x \in \mathbb{R}^d, \tag{1.7}$$

where M is defined by

$$M\phi(x) = \inf_{\xi \in (\mathbb{R}^+)^d} \big\{\phi(x+\xi) + c(x,\xi)\big\}, \quad \phi \in UC_{bb}(\mathbb{R}^d), \tag{1.8}$$

and

$$H(x,p) = \sup_{u \in U} \left\{ \frac{-f(x,u)\cdot p - k(x,u)}{\lambda} \right\}. \tag{1.9}$$

Observe that under the assumptions (A1)–(A4), the above Hamiltonian satisfies

$$\big|H(x,p) - H(y,q)\big| \leqslant F|p-q| + \omega\big(|x-y|\big), \tag{1.10}$$

where ω is a modulus of continuity depending on those of f and k.

In [4, Chapter 3] and [2] cost functionals are assumed to be bounded, hence value function belongs to the class $BUC(\mathbb{R}^d)$. The uniqueness of the value function as the solution of quasivariational inequality in $BUC(\mathbb{R}^d)$ is established in [2] by using an iteration of stopping time problems. As our cost functionals are no longer bounded above we get the value function to be in the bounded below uniformly continuous function class, $UC_{bb}(\mathbb{R}^d)$. Now we state the uniqueness theorem in $UC_{bb}(\mathbb{R}^d)$ for (1.7) which will be proved in Section 2.

Theorem 1.1. *Assume* (A1)–(A5)*. Then, V defined by* (1.3) *is the unique viscosity solution in the class $UC_{bb}(\mathbb{R}^d)$ of the quasivariational inequality given by* (1.7)*.*

Now let us relax boundedness assumption on the dynamics f and uniform continuity assumption on the cost functionals k and c. We allow f to have linear growth and k, c to have power like growth. We list below the assumptions in this particular case:

(A6) f is Lipschitz and has a linear growth: for $x, y \in \mathbb{R}^d$

$$\big|f(x,u) - f(y,u)\big| \leqslant L|x-y| \quad \text{and} \quad \big|f(x,u)\big| \leqslant F\big(1+|x|\big). \tag{1.11}$$

(A7) $k : \mathbb{R}^d \times U \to \mathbb{R}$ is continuous with growth m such that $m < \frac{1}{L}$, where L is as in (A6).

$$\big|k(x,u)\big| \leqslant C\big(1+|x|^m\big) \quad \forall x \in \mathbb{R}^d \text{ and } u \in U. \tag{1.12}$$

(A8) $c(x,\xi)$ is positive and continuous in both the variables:

$$C\big(1+|\xi|^m\big) \geqslant c(x,\xi) \geqslant C_0 > 0. \tag{1.13}$$

(A9) The discount factor $\lambda > mL$.

It can be shown by using arguments similar to those in the case of bounded cost functionals as in [2,4] that under the assumptions (A6)–(A8), V defined by (1.3) is continuous and it satisfies dynamic programming principle. Using this, we can show that it solves the quasivariational inequality in the viscosity sense given by (1.7).

With the assumptions (1.11) and (1.12) our Hamiltonian defined by (1.9) satisfies the following structural condition:

$$\big|H(x,p) - H(x,q)\big| \leqslant F\big(1+|x|\big)|p-q| \tag{1.14}$$

for all $x, p, q \in \mathbb{R}^d$ and

$$\bigl|H(x,p) - H(y,p)\bigr| \leqslant \omega_R\bigl(|x-y|\bigl(1+|p|\bigr)\bigr) \tag{1.15}$$

for all $p \in \mathbb{R}^d$, $x, y \in B(0,R)$, $R > 0$, where ω_R is a local modulus of continuity depending on k.

Unlike the case of bounded dynamics, the value function, in this case is not necessarily uniformly continuous but lies in the continuous function class $C(\mathbb{R}^d)$. Let us now estimate growth of the value function. For this we recall the ODE estimate [4, Chapter 3, Section 5, Remark 5.6]

$$\bigl|X_x(t)\bigr| \leqslant |x|e^{Lt} + \frac{M_0}{L}\bigl(e^{Lt}-1\bigr),$$

where $M_0 = \sup_u |f(0,u)|$. Observe that if there are no impulses, then

$$\begin{aligned}\bigl|J(x,u)\bigr| &\leqslant \int_0^\infty C\bigl(1+\bigl|X_x(t)\bigr|^m\bigr)e^{-\lambda t}\,dt \leqslant \int_0^\infty C\bigl(1+e^{mLt}\bigl(|x|^m + C\bigr)\bigr)e^{-\lambda t}\,dt\\ &\leqslant C\bigl(1+|x|^m\bigr),\end{aligned}$$

thanks to the assumption (A9). Let us define

$$Q_m\bigl(\mathbb{R}^d\bigr) = \left\{u \in C_{bb}\bigl(\mathbb{R}^d\bigr) \;\middle|\; \sup \frac{|u(x)|}{1+|x|^m} < \infty\right\}$$

for $m > 0$. The right class to look for a unique solution to the quasivariational inequality is

$$Q := \bigcup Q_m(\mathbb{R}^d), \quad m < \frac{1}{L}.$$

The condition $m < \frac{1}{L}$ cannot be relaxed. Otherwise it is known that (1.7) may not admit a unique solution. For example, $u(x) = |x|^m$ solves the equation

$$u - \frac{1}{m}\langle x, Du\rangle = 0 \quad \text{in } \mathbb{R}^d$$

in the viscosity sense. Our result in this case is

Theorem 1.2. *Assume* (A3), (A6), (A7), (A8) *and* (A9). *Then the value function* V *defined by* (1.3) *is the unique solution of the quasivariational inequality* (1.7) *in the class* $Q = \bigcup Q_m(\mathbb{R}^d)$, $m < \frac{1}{L}$ *where* L *is as in* (1.11).

Next for the finite horizon problem, for fixed $T > 0$, the evolution of the trajectory occurs according to

$$\dot{X}(t) = f\bigl(t, X(t), u(t)\bigr) \quad \text{for } t \in (\theta_i, \theta_{i+1}),\ \theta_i \in (s,T), \tag{1.16}$$

$$X(s) = x; \qquad X_x(\theta_i^+) = X_x(\theta_i^-) + \xi_i, \tag{1.17}$$

where $\theta_i \leqslant T$. As before $\xi = (\xi_i)_{i\in\mathbb{N}}$ is a sequence of elements of $(\mathbb{R}^+)^d$ and $u(t) : [s,T] \to U$ is any measurable function, $\beta = (\theta, \xi, u(\cdot))$ is the control variable. The total cost is then given by

$$J(s,x,\beta) = \int_s^T k_1\bigl(t, X(t), u(t)\bigr)\,dt + \sum_{s\leqslant\theta_i\leqslant T} c_1\bigl(\theta_i, X_x(\theta_i^-), \xi_i\bigr) + \tilde{g}\bigl(X(T)\bigr), \tag{1.18}$$

where $X(t)$ is actually $X_{(s,x)}(t)$, the solution of (1.16), (1.17) and if $\theta_i = T$ for some i then we take $\tilde{g}(X(T)) = \tilde{g}(X(T^+))$. Here k_1 is the running cost, c_1 is the impulse cost and $\tilde{g}$ is the terminal cost. For notational convenience we drop the superscript in $c_1(t, X_x(\theta_i^-), \xi_i)$ now onwards.

The optimal cost functional $V(s,x)$ is defined by

$$V(s,x) = \inf_\beta \left\{ \int_s^T k_1\bigl(t, X(t), u(t)\bigr)\,dt + \sum_{s\leqslant\theta_i\leqslant T} c_1\bigl(\theta_i, X_x(\theta_i), \xi_i\bigr) + \tilde{g}\bigl(X(T)\bigr) \right\}. \tag{1.19}$$

We make the following assumptions on f and the cost functional k_1, c_1 and $\tilde{g}$ of finite horizon problem.

(A10) 1. $f(t, X_x(t), u(t)$ is continuous in t variable and Lipschitz in the second variable uniformly for all u with Lipschitz constant L and bounded by F.
2. $k_1(t,x,u)$ is nonnegative continuous and uniformly continuous in first two variable for all $u \in U$.
3. $c_1(t, X_x(t), \xi)$ is nonnegative and uniformly continuous in first two variables and is bounded below by a positive constant C_0.
4. c_1 is monotonically increasing in t variable, namely, for $s \leqslant t$,

$$c_1\bigl(s, X_x(s), \xi\bigr) \leqslant c_1\bigl(t, X_x(t), \xi\bigr).$$

Moreover,

$$\frac{c_1(t, X_x(t), \xi)}{|\xi|} \to \infty \quad \text{as } |\xi| \to \infty, \text{ uniformly for all } t \in [0,T]. \tag{1.20}$$

5. Terminal cost $\tilde{g}$ is nonnegative and uniformly continuous.

Under the assumptions (A3) and (A10) V defined by (1.19) is uniformly continuous in both the variables. Moreover, V satisfies the optimality principle called dynamic programming principle. We can further show that it satisfies the following quasivariational inequality in the viscosity sense:

$$\max\bigl\{-V_t + H(x, DV), V - MV\bigr\} = 0 \quad \text{in } (0,T) \times \mathbb{R}^d, \tag{1.21}$$

$$V(T,x) = g(x), \tag{1.22}$$

where

$$g(x) = \min\Bigl\{\tilde{g}(x), \inf_{\xi}\bigl\{\tilde{g}(x) + c_1\bigl(T, X_x(T), \xi\bigr)\bigr\}\Bigr\}.$$

Here V_t is the derivative of V with respect to t variable and DV is the derivative with respect to x variable. M is defined by

$$M\phi(t,x) = \inf_{\xi\in(\mathbb{R}^+)^d}\bigl\{\phi(t, x+\xi) + c_1\bigl(t, X_x(t), \xi\bigr)\bigr\} \tag{1.23}$$

for any $\phi \in UC_{bb}(\mathbb{R}^d)$ and

$$H(t,x,p) = \sup_{u\in U}\bigl\{-f(t,x,u).p - k_1(t,x,u)\bigr\}. \tag{1.24}$$

Now we state our uniqueness theorem for the finite horizon problem which will be proved in Section 3.

Theorem 1.3. *Assume* (A3) *and* (A10). *Let* v_1 *and* $v_2 \in UC_{bb}([0,T]\times\mathbb{R}^d)$ *be two solutions of the quasivariational inequality given by* (1.21), (1.22). *Then* $v_1 = v_2$.

Next our aim is to relax the boundedness assumption on the dynamics. Here we allow f to be unbounded with linear growth, that is:

(A10′) 1. f is locally Lipschitz with Lipschitz constant L_R in the ball of radius R around x in $\mathbb{R}^d$ and has a linear growth:

$$\bigl|f(t,x,u) - f(t,y,u)\bigr| \leqslant L_R|x-y| \quad \forall |x|, |y| \leqslant R,\ \forall t \in [0,T], \tag{1.25}$$

$$\bigl|f(t,x,u)\bigr| \leqslant F\bigl(1+|x|\bigr). \tag{1.26}$$

2. k_1, c_1 are nonnegative and continuous and c_1 is bounded below by a positive constant C_0; c_1 is monotonically increasing in t variable and

$$c_1(t,x,\xi) \to \infty \quad \text{as } |\xi| \to \infty. \tag{1.27}$$

3. Terminal cost $\tilde{g}$ is nonnegative and continuous.

Value function, in this case is not necessarily uniformly continuous and lies in the continuous function class $C_{bb}(\mathbb{R}^d \times [0,T])$. Uniqueness theorem which we will prove uses ideas from [7, Theorem 2.5] for HJB equations. Here we extend it to quasivariational inequalities arising from impulse control problem. With the assumptions (1.25), (1.26) our Hamiltonian defined by (1.24) satisfies the following structural condition:

$$\bigl|H(t,x,p) - H(t,x,q)\bigr| \leqslant F\bigl(1+|x|\bigr)|p-q| \tag{1.28}$$

for all $x, p, q \in \mathbb{R}^d$ and

$$\bigl|H(t,x,p) - H(t,y,p)\bigr| \leqslant \omega_R\bigl(|x-y|\bigl(|p|+1\bigr)\bigr) \tag{1.29}$$

for all $p \in \mathbb{R}^d, x, y \in B(0,R), R > 0$, where ω_R is a local modulus of continuity. Now we state our uniqueness theorem in this case.

Theorem 1.4. *Assume* (A3) *and* (A10′). *Let* $v_1, v_2 \in C_{bb}(\mathbb{R}^d \times [0,T])$ *be two viscosity solutions of the QVI given by* (1.21), (1.22). *Then* $v_1 = v_2$ *in* $C_{bb}(\mathbb{R}^d \times [0,T])$.

2. Infinite horizon problem

Recall that the uniqueness result for the viscosity solution of the quasivariational inequality satisfied by the value function of the impulse control problem in the class $BUC(\mathbb{R}^d)$ can be proved using the uniqueness of solution of the stopping time problem (see [2]). So to extend this result to the class $UC_{bb}(\mathbb{R}^d)$, we first need to establish the uniqueness for the stopping time problem in $UC_{bb}(\mathbb{R}^d)$.

Consider the optimal stopping time problem for which evolution of the trajectory is given by

$$\dot{X}(t) = f\big(X(t), u(t)\big), \quad t \in (0,\infty);$$
$$X_x(0) = x.$$

The value function of the stopping time problem with stopping cost ψ is given by

$$V_\psi(x) = \inf_{\nu \geqslant 0, u(\cdot)} \int_0^\nu k\big(X_x(s), u(s)\big)e^{-\lambda s}\,ds + \psi\big(X(\nu)\big)e^{-\lambda\nu}. \tag{2.1}$$

If $k, \psi \in BUC(\mathbb{R}^d)$, and (A1)–(A3) hold then $V_\psi \in BUC(\mathbb{R}^d)$ and the variational inequality satisfied by V_ψ in the viscosity sense is

$$\max\big\{V_\psi(x) + H\big(x, DV_\psi(x)\big); V_\psi(x) - \psi(x)\big\} = 0, \tag{2.2}$$

where H is given by (1.9), see [4, Chapter 3, Theorem 4.10]. The uniqueness proof follows as a corollary to the uniqueness of HJB equation satisfied by the value function of infinite horizon optimal control problem with continuous controls, see [4, Chapter 3, Theorem 4.11]. If k and ψ are nonnegative uniformly continuous and unbounded, the corresponding value function for the stopping time problem will be nonnegative and will lie in the class $UC_{bb}(\mathbb{R}^d)$. It is easy to show that it will also satisfy the same variational inequality as given by (2.2). Thus arguing as in [4], to establish uniqueness of solution of stopping time problem we need uniqueness of solution of the infinite horizon problem, i.e., of the HJB equation in the class $UC_{bb}(\mathbb{R}^d)$, given by

$$V(x) + H\big(x, DV(x)\big) = 0.$$

Ishii has proved the above result in the class $UC(\mathbb{R}^d)$, in [7, Theorem 1.1 and Remark 1.1], under the assumptions

$$H \in UC(\mathbb{R}^d \times B_R) \quad \text{for each } R > 0, \tag{2.3}$$

$$\big|H(x,p) - H(y,p)\big| \leqslant \omega\big(|x-y|\big(1+|p|\big)\big) \quad \forall x, y, p \in \mathbb{R}^d. \tag{2.4}$$

This result also holds for the subclass $UC_{bb}(\mathbb{R}^d)$. The required conditions on H can be verified for our Hamiltonian given by (1.9), using assumptions (A1)–(A3). (In fact, with our assumption we can derive stronger condition on H given by (1.10).) Hence the HJB equation has a unique solution in $UC_{bb}(\mathbb{R}^d)$, and hence (2.2) also has a unique viscosity solution in $UC_{bb}(\mathbb{R}^d)$. We summarize this result in the next proposition.

Proposition 2.1. *Assume* (A1)–(A3). *Then the value function of the stopping time problem given by* (2.1), *with nonnegative unbounded running and stopping cost, is the unique viscosity solution of the variational inequality, given by* (2.2) *in the class* $UC_{bb}(\mathbb{R}^d)$.

For the proof of Theorem 1.1, we need two lemmas which we prove first. Recall that for $w \in UC_{bb}(\mathbb{R}^d)$, Mw is defined by

$$Mw(x) = \inf_{\xi}\bigl\{w(x+\xi) + c(x,\xi)\bigr\}.$$

We define another nonlinear operator denoted by $\mathcal{T} : UC_{bb}(\mathbb{R}^d) \to UC_{bb}(\mathbb{R}^d)$,

$$\mathcal{T}w(x) = \inf_{\nu \geqslant 0, u}\left\{\int_0^{\nu} k\bigl(X_x(s), u(s)\bigr)e^{-\lambda s}\,ds + Mw\bigl(X_x(\nu,u)\bigr)e^{-\lambda\nu}\right\}. \tag{2.5}$$

Following the approach in [2], we analyze first the properties of M and $\mathcal{T}$, in Lemmas 2.1 and 2.2. While the proof of the first lemma is similar to that in [2], the major difference occurs in the proof of Lemma 2.2(d), because of the unbounded cost functionals. In the first lemma we list the properties of M.

Lemma 2.1. *For all* $w, w_1 \in UC_{bb}(\mathbb{R}^d)$ *M has following properties*:

1. $Mw \in UC_{bb}(\mathbb{R}^d)$.
2. $Mw \geqslant Mw_1$ *if* $w \geqslant w_1$.
3. *M has concavity property; i.e. for all* $\mu \in [0,1]$

$$M\bigl(\mu w + (1-\mu)w_1\bigr) \geqslant \mu Mw + (1-\mu)Mw_1.$$

Proof. 1. Given $\varepsilon > 0$, there exists ξ_ε such that

$$Mw(x) + \varepsilon \geqslant w(x+\xi_\varepsilon) + c(x,\xi_\varepsilon).$$

Then,

$$\begin{aligned} Mw(z) - Mw(x) &\leqslant w(z+\xi_\varepsilon) + c(z,\xi_\varepsilon) - w(x+\xi_\varepsilon) - c(x,\xi_\varepsilon) + \varepsilon \\ &\leqslant \omega_w\bigl(|x-z|\bigr) + c(z,\xi_\varepsilon) - c(x,\xi_\varepsilon) + \varepsilon \\ &\leqslant \omega_w\bigl(|x-z|\bigr) + \omega_c\bigl(|x-z|\bigr) + \varepsilon. \end{aligned}$$

As ε is arbitrary and w and c are uniformly continuous and w and c are bounded below it follows that $Mw \in UC_{bb}(\mathbb{R}^d)$.

2. Since $w(x+\xi) \geqslant w_1(x+\xi)$ $\forall\xi$, it follows that $Mw(x) \geqslant Mw_1(x)$ $\forall x \in \mathbb{R}^d$.

3. Set $v = (\mu w + (1-\mu)w_1)$. By the definition of M, given $\varepsilon > 0$ there exists ξ_ε such that

$$Mv(x) + \varepsilon \geqslant v(x+\xi_\varepsilon) + c(x,\xi_\varepsilon).$$

Observe that, by the definition of infimum,

$$\mu\big(Mw(x)\big) \leqslant \mu w(x+\xi_\varepsilon) + \mu c(x,\xi_\varepsilon) \quad \text{and}$$
$$(1-\mu)\big(Mw_1(x)\big) \leqslant (1-\mu)w_1(x+\xi_\varepsilon) + (1-\mu)c(x,\xi_\varepsilon).$$

Adding the above two inequalities we get

$$\begin{aligned}\mu\big(Mw(x)\big) + (1-\mu)\big(Mw_1(x)\big) &\leqslant \mu w(x+\xi_\varepsilon) + (1-\mu)w_1(x+\xi_\varepsilon) + c(x,\xi_\varepsilon)\\ &= v(x+\xi_\varepsilon) + c_x(\xi_\varepsilon)\\ &\leqslant Mv(x) + \varepsilon\\ &= M\big(\mu w + (1-\mu)w_1\big)(x) + \varepsilon.\end{aligned}$$

Since ε is arbitrary, we have

$$\mu\big(Mw(x)\big) + (1-\mu)\big(Mw_1(x)\big) \leqslant M\big(\mu w + (1-\mu)w_1\big)(x).$$

This proves the lemma. □

Now we list the properties of $\mathcal{T}$ in the next lemma.

Lemma 2.2 (Properties of $\mathcal{T}$). *For all $w, w_1 \in UC_{bb}(\mathbb{R}^d)$, $\mathcal{T}$ has following properties*:

(a) $\mathcal{T}w \in UC_{bb}(\mathbb{R}^d)$.
(b) $\mathcal{T}w \geqslant \mathcal{T}w_1$ *if* $w \geqslant w_1$.
(c) $\mathcal{T}$ *has concavity property, i.e., for all* $\mu \in [0,1]$

$$\mathcal{T}\big(\mu w + (1-\mu)w_1\big) \geqslant \mu\mathcal{T}w + (1-\mu)\mathcal{T}w_1.$$

(d) *If* $w_i \in UC_{bb}(\mathbb{R}^d)$ *and* $w_1 - w_2 \leqslant \gamma w_1$ *for any* $\gamma \in [0,1]$ *then* $\exists \mu_0 \in (0,1)$ *such that*

$$\mathcal{T}w_1 - \mathcal{T}w_2 \leqslant \gamma(1-\mu)\mathcal{T}w_1 \quad \textit{for all } \mu \in (0,\mu_0].$$

Proof. The first three properties of $\mathcal{T}$ are similar to those of M and can be proved with similar techniques. So we take up the proof of property (d). For $w_1 - w_2 \leqslant \gamma w_1$ implies $(1-\gamma)w_1 \leqslant w_2$. Hence by property (b),

$$\mathcal{T}\big((1-\gamma)w_1\big) \leqslant \mathcal{T}w_2. \tag{2.6}$$

Now using property (c), we get

$$\mathcal{T}\big((1-\gamma)w_1\big) \geqslant (1-\gamma)\mathcal{T}w_1 + \gamma\mathcal{T}0,$$
$$\mathcal{T}w_2 \geqslant (1-\gamma)\mathcal{T}w_1 + \gamma\mathcal{T}0 \quad \text{by (2.6)}.$$

Hence

$$\mathcal{T}w_1 - \mathcal{T}w_2 \leqslant \gamma\mathcal{T}w_1 - \gamma\mathcal{T}0.$$

Let us consider the value function of the control problem without any impulses, namely:

$$\bar{w}(x) = \inf_u\left\{\int_0^\infty k\big(X(s),u(s)\big)e^{-\lambda s}\,ds\right\}.$$

We will show that for some fixed $\mu_0 \in (0,1)$,

$$\mathcal{T}0 \geqslant \mu_0 \bar{w} \quad \text{and} \tag{2.7}$$

$$\mathcal{T}w \leqslant \bar{w} \quad \forall w \in UC_{bb}(\mathbb{R}^d). \tag{2.8}$$

(2.7) and (2.8) together when substituted back in the estimate for $\mathcal{T}w_1 - \mathcal{T}w_2$ give

$$\begin{aligned} \mathcal{T}w_1 - \mathcal{T}w_2 &\leqslant \gamma \mathcal{T}w_1 - \gamma \mathcal{T}0 \\ &\leqslant \gamma \mathcal{T}w_1 - \gamma \mu_0 \bar{w} \\ &\leqslant \gamma \mathcal{T}w_1 - \gamma \mu_0 \mathcal{T}w_1 \quad (\text{using (2.8) with } w = w_1) \\ &= \gamma(1-\mu_0)\mathcal{T}w_1. \end{aligned}$$

This proves the property (d) for all $\mu \in (0, \mu_0]$. Thus it is enough to prove now (2.7) and (2.8). To prove (2.7), observe that by the definition of $\bar{w}$,

$$\bar{w}(x) \leqslant \int_0^\infty k\big(X(s), u(s)\big)e^{-\lambda s}\,ds \quad \text{for all } u. \tag{2.9}$$

We remark here that any $v \in UC(\mathbb{R}^d)$, uniformly continuous function on $\mathbb{R}^d$, has linear growth at infinity. That is,

$$\big|v(x)\big| \leqslant C\big(1+|x|\big) \quad \text{for some } C>0 \text{ a constant.} \tag{2.10}$$

(See, for example, [7, Remark 1.2].) Hence, by (A2), and C corresponding to the running cost k,

$$\bar{w}(x) \leqslant C\int_0^\infty \big(1+\big|X_x\big(s,u(s)\big)\big|\big)e^{-\lambda s}\,ds.$$

Now by using standard estimates from ODE and Gronwall's lemma (see, for example, [4, Chapter 3, Section 5]) we can show that

$$\big|X_x\big(t,u(t)\big)\big| \leqslant |x|e^{Lt} + \frac{M_0}{L}\big(e^{Lt}-1\big), \tag{2.11}$$

where L is as in assumption (A1), and $M_0 = \sup_u |f(0,u)|$. Hence using this estimate we get

$$\bar{w}(x) \leqslant C\int_0^\infty \left(1+|x|e^{Lt} + \frac{M_0}{L}\big(e^{Lt}-1\big)\right)e^{-\lambda s}\,ds.$$

By assumption (A5)

$$\begin{aligned} \bar{w}(x) &\leqslant C\left\{\frac{1}{\lambda} + \frac{|x|}{\lambda-L} + \frac{M_0}{L(\lambda-L)} - \frac{M_0}{\lambda L}\right\} = CC'\left\{1 + \frac{|x|}{(\lambda-L)C'}\right\} \\ &\leqslant C''\big(1+|x|\big). \end{aligned}$$

Hence by (A4),

$$\bar{w}(x) \leqslant \frac{C''}{C_1} C_1\big(1+|x|\big) \leqslant \frac{C''}{C_1} c(x,\xi) \quad \forall \xi.$$

Choose the best $\mu_0 \in (0,1)$ such that

$$\frac{C''}{C_1} \leqslant \frac{1}{\mu_0}. \tag{2.12}$$

Hence,

$$\bar{w}(x) \leqslant \frac{1}{\mu_0} c(x,\xi) \quad \forall x \text{ and } \xi. \tag{2.13}$$

Now by the definition of $\mathcal{T}$,

$$\mathcal{T}0(x) = \inf_{\nu \geqslant 0, u} \left\{ \int_0^\nu k\big(X_x(s), u(s)\big) e^{-\lambda s}\, ds + M0\big(X_x(\nu)\big) e^{-\lambda \nu} \right\}.$$

Observing that $M0(x) = \inf_\xi c(x,\xi) = c(x,\xi_x)$, for some ξ_x,

$$\mathcal{T}0(x) \geqslant \inf_{\nu \geqslant 0, u} \left\{ \int_0^\nu k\big(X_x(s), u(s)\big) e^{-\lambda s}\, ds + c\big(X_x(\nu), \xi_x\big) e^{-\lambda \nu} \right\}.$$

Substituting from (2.13) we get

$$\begin{aligned} \mathcal{T}0(x) &\geqslant \inf_{\nu \geqslant 0, u} \left\{ \int_0^\nu k\big(X_x(s), u(s)\big) e^{-\lambda s}\, ds + \mu_0 \bar{w}\big(X_x(\nu)\big) e^{-\lambda \nu} \right\} \\ &\geqslant \mu_0 \inf_{\nu \geqslant 0, u} \left\{ \int_0^\nu k\big(X_x(s), u(s)\big) e^{-\lambda s}\, ds + \bar{w}\big(X_x(\nu)\big) e^{-\lambda \nu} \right\}, \end{aligned}$$

where the last inequality follows as $k \geqslant \mu_0 k$. By the dynamic programming principle for $\bar{w}(x)$ (see [4, Chapter 3, Proposition 2.5]) we get

$$\mathcal{T}0(x) \geqslant \mu_0 \bar{w}(x) \quad \forall x \text{ where } \mu_0 \text{ is chosen as in (2.12).}$$

This proves (2.7). Now to prove the estimate in (2.8), by the definition of $\mathcal{T}$, for all $w \in UC_{bb}(\mathbb{R}^d)$,

$$\mathcal{T}w(x) \leqslant \left\{ \int_0^\nu k\big(X_x(s), u(s)\big) e^{-\lambda s}\, ds + Mw\big(X_x(\nu)\big) e^{-\lambda \nu} \right\} \quad \forall u.$$

Below we show that $Mw(X_x(\nu))e^{-\lambda\nu}$ tends to 0 as ν tends to ∞. Then sending ν to ∞ in the above estimate we will get, for all x,

$$\mathcal{T}w(x) \leqslant \int_0^\infty k\big(X_x(s), u(s)\big) e^{-\lambda s}\, ds \quad \forall u.$$

By taking infimum in u, will imply $\mathcal{T}w \leqslant \bar{w}\ \forall w \in UC_{bb}(\mathbb{R}^d)$. Hence we have to show now that for each x and u,

$$Mw\bigl(X_x(\nu)\bigr)e^{-\lambda\nu} \to 0 \quad \text{as } \nu \to \infty. \tag{2.14}$$

By using assumption (A4), for all x and ξ,

$$Mw(x) \leqslant w(x+\xi) + c(x,\xi) \leqslant w(x+\xi) + C_2\bigl(1+|x|\bigr).$$

Hence fixing ξ, and using the growth condition for w coming from uniform continuity, we get

$$\begin{aligned} Mw\bigl(X_x(\nu)\bigr)e^{-\lambda\nu} &\leqslant w\bigl(X_x(\nu)+\xi\bigr)e^{-\lambda\nu} + C_2\bigl(1+\bigl|X_x(\nu)\bigr|\bigr)e^{-\lambda\nu} \\ &\leqslant C\bigl(1+\bigl|X_x(\nu)\bigr|+|\xi|\bigr)e^{-\lambda\nu} + C_2\bigl(1+\bigl|X_x(\nu)\bigr|\bigr)e^{-\lambda\nu}. \end{aligned}$$

Hence we can majorise the right-hand side of the above expression, using (2.11) by

$$Ce^{(L-\lambda)\nu} + |x|e^{(L-\lambda)\nu} + \bigl(|\xi| + C\bigr)e^{-\lambda\nu} \quad \text{for some constant } C.$$

By assumption (A5), $L - \lambda < 0$, hence the above expression will tend to 0 as ν tends to ∞. Thus $Mw(X_x(\nu,u))e^{-\lambda\nu} \to 0$ as $\nu \to \infty$. This proves (2.14). So, (2.8) follows:

$$\mathcal{T}w(x) \leqslant \bar{w}(x) \quad \forall w.$$

This finishes the proof of the property (d) and hence of the Lemma 2.2. □

Remark 2.1. In [2], property (d) of above lemma follows for all $\mu \in (0,1)$ using $\|\bar{w}\|_\infty$. Here $\bar{w}$ may not be bounded. So we avoid that estimate to get property (d) for some $\mu_0 \in (0,1)$. But this is enough to prove the uniqueness.

Proof of Theorem 1.1. Suppose by contradiction there are two viscosity solutions in $UC_{bb}(\mathbb{R}^d)$ of QVI, say v_1 and v_2. Hence both v_1 and v_2 satisfy

$$\begin{aligned} &\max\bigl\{v_1(x) + H\bigl(x, Dv_1(x)\bigr);\ v_1(x) - Mv_1(x)\bigr\} = 0, \\ &\max\bigl\{v_2(x) + H\bigl(x, Dv_2(x)\bigr);\ v_2(x) - Mv_2(x)\bigr\} = 0. \end{aligned}$$

Now we will obtain the uniqueness of the value function of the impulse control problem by using Lemmas 2.1 and 2.2, and arguments similar to [2]. Without loss of generality we assume that v_1 and v_2 are nonnegative. For else if $-M_0$ is a lower bound on v_1 and v_2 we can define $\tilde{v}_1 = v_1 + M_0$ and $\tilde{v}_2 = v_2 + M_0$. Then $\tilde{v}_1$ and $\tilde{v}_2$ will be nonnegative and will satisfy QVI with H replaced by $\tilde{H} = H - M_0$. Observe that $\tilde{H}$ satisfies the conditions of the Proposition 2.1, namely, (2.3), (2.4).

By definition, $\mathcal{T}v_1$ is the value function for the optimal stopping time problem with stopping cost Mv_1. Hence it satisfies the variational inequality:

$$\max\bigl\{(\mathcal{T}v_1)(x) + H\bigl(x, D(\mathcal{T}v_1)(x)\bigr);\ (\mathcal{T}v_1)(x) - Mv_1(x)\bigr\} = 0$$

in the viscosity sense. By Proposition 2.1, it follows that the above variational inequality has a unique solution. Also by assumption, v_1 is the solution of the QVI given by

$$\max\bigl\{v_1(x) + H\bigl(x, Dv_1(x)\bigr);\ v_1(x) - Mv_1(x)\bigr\} = 0.$$

Hence by uniqueness of solution of stopping time problem, we get $\mathcal{T}v_1 = v_1$. Thus by the same argument for two solutions v_1 and v_2 of QVI, we have

$$\begin{aligned} &\mathcal{T}v_1 = v_1 \quad \text{and} \quad \mathcal{T}v_2 = v_2, \\ &v_1, v_2 \geqslant 0, \quad \Rightarrow \quad v_1 - v_2 \leqslant v_1. \end{aligned} \tag{2.15}$$

Hence by property (d) of Lemma 2.2, with $\gamma = 1$ we get

$$\mathcal{T}v_1 - \mathcal{T}v_2 \leqslant (1-\mu_0)\mathcal{T}v_1$$

and using (2.15), $v_1 - v_2 \leqslant (1-\mu_0)v_1$. Using property (d) with $\gamma = (1-\mu_0)$ we get

$$\mathcal{T}v_1 - \mathcal{T}v_2 \leqslant (1-\mu_0)^2\mathcal{T}v_1;$$

again using (2.15) we get

$$v_1 - v_2 \leqslant (1-\mu_0)^2 v_1.$$

Proceeding in this way after nth step we will get

$$v_1 - v_2 \leqslant (1-\mu_0)^n v_1.$$

Since $\mu_0 \in (0,1)$ then $(1-\mu_0)^n \to 0$, as $n \to \infty$. Thus $v_1 - v_2 \leqslant 0$ as $n \to \infty$. Hence

$$v_1 \leqslant v_2.$$

Interchanging the roles of v_1 and v_2 we get other way inequality, i.e., $v_2 \leqslant v_1$. Hence $v_1 = v_2$ and the uniqueness follows. □

We now give the proof of Theorem 1.2. We outline the proof below as it is very similar to the proof of the Theorem 1.1.

Proof of Theorem 1.2. As in the proof of the Theorem 1.1, we will need the uniqueness of stopping time problem in the class Q. The next proposition states this result. Proof of the proposition follows from the proof of the uniqueness theorem in class Q for unbounded viscosity solutions of HJB equations, namely [7, Theorem 1.5].

Proposition 2.2. *Assume* (A3), (A6)–(A9). *Then the value function of the stopping time problem given by* (2.1)*, with nonnegative unbounded running and stopping cost is the unique viscosity solution of the variational inequality, given by* (2.2) *in the class* Q.

Using Proposition 2.2, proceeding on similar lines as in the proof of Theorem 1.1, we can prove Theorem 1.2. Note that in the proof of Theorem 1.1, it was important that the property (d) of Lemma 2.2 holds for appropriate μ_0. We can choose such a μ_0 in the proof of Theorem 1.2 as well, by exploiting the assumption that both running cost and impulse cost have same growth as $|x|^m$. □

Remark 2.2. Note that Theorem 1.2 can be proved by comparison method also, as is done in [7, Theorem 1.5]. In [8] author proves the result for value functions in the class uniformly continuous functions and in the subclass with sublinear growth having unbounded cost functionals. Theorem 1.2 in this paper proves the uniqueness in the class Q, for cost functions and value functions with a certain growth m, which may be larger than 1.

3. Finite horizon problem

In this section we prove Theorems 1.3 and 1.4 by comparison method. This method has been used in [3] for impulse control problem and in [8] for an optimal control problem with continuous, switching and impulse controls. Both these results are for infinite horizon. The idea used by them is to modify the auxiliary function a finite number of times to avoid the case $v_2 = Mv_2$. Our idea here is different. Instead of changing the auxiliary function many times we choose the parameters in the auxiliary function suitably so that the case $v_2 = Mv_2$ does not arise. Once $v_2 = Mv_2$ is avoided for the supersolution, we use the usual comparison principle for HJB equation.

Proof of Theorem 1.3. We will show that for any two viscosity solutions v_1 and v_2 in $UC_{bb}([0,T] \times \mathbb{R}^d)$ of QVI given by (1.21), $v_1 \leqslant v_2$. Interchanging the roles of v_1 and v_2 we can then get the other way inequality and the uniqueness will follow. We want to show that

$$v_1(t,x) \leqslant v_2(t,x) \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d.$$

Suppose the contrary, i.e., there exists $(\bar{t},\bar{x}) \in [0,T] \times \mathbb{R}^d$ such that

$$\sup(v_1 - v_2) \geqslant v_1(\bar{t},\bar{x}) - v_2(\bar{t},\bar{x}) = 2\delta > 0.$$

Let $\gamma \in (0,1)$ such that

$$\gamma v_1(\bar{t},\bar{x}) - v_2(\bar{t},\bar{x}) > \delta. \tag{3.1}$$

We are assured that such a γ exists because if $v_1(\bar{t},\bar{x}) < 0$ then we can choose any $\gamma \in (0,1)$ and if $v_1(\bar{t},\bar{x}) > 0$ then we can choose

$$1 - \gamma < \frac{\delta}{v_1(\bar{t},\bar{x})}. \tag{3.2}$$

We define the auxiliary function Φ by doubling the x and t variables.

$$\begin{aligned}\Phi(t,x,s,y) = \gamma v_1(t,x) - v_2(s,y) - \frac{|x-y|^2 + |t-s|^2}{\varepsilon} - \kappa\bigl(\langle x\rangle^m + \langle y\rangle^m\bigr) \\ - \eta(t+s),\end{aligned} \tag{3.3}$$

where $m \in (0,1)$ is fixed and $\langle x\rangle^m = (1+|x|^2)^{m/2}$ and $\eta = \delta/8T$. $\gamma, \varepsilon, \kappa$ are positive parameters to be chosen suitably later on. Observe that by (A10-5), $g \geqslant 0$. Since $v_1(T,x) = v_2(T,x) = g(x)$,

$$\begin{aligned}\gamma v_1(t,x) - v_2(s,y) &\leqslant \gamma v_1(t,x) - \gamma v_1(T,x) + v_2(T,x) - v_2(s,y) \\ &\leqslant C\bigl(1 + |t-T|\bigr) + C\bigl(1 + |s-T| + |x-y|\bigr),\end{aligned}$$

where the last inequality follows because v_1 and v_2 are uniformly continuous. Hence as $|x|, |y| \to \infty$, $\Phi \to -\infty$ and Φ attains its supremum at some finite point in $[0,T] \times \mathbb{R}^d \times [0,T] \times \mathbb{R}^d$, say, (t_0,x_0,s_0,y_0). Using the fact that $\sup \Phi = \Phi(t_0,x_0,s_0,y_0)$ we get the following estimate in Lemma 3.1 which will be proved after finishing the proof of the theorem.

$$|t_0 - s_0| + |x_0 - y_0| \leqslant \sqrt{\varepsilon}C \quad \text{for some constant } C \text{ independent of } \kappa,\ \varepsilon,\ \text{and } \gamma. \tag{3.4}$$

We first claim that t_0 or s_0 cannot be equal to T for ε sufficiently small, that is:

$$t_0 < T \quad \text{and} \quad s_0 < T. \tag{3.5}$$

If not, let $t_0 = T$. Then,

$$\begin{aligned}
\Phi(t_0, x_0, s_0, y_0) &= \Phi(T, x_0, s_0, y_0) \\
&= \gamma v_1(T, x_0) - v_2(s_0, y_0) - \frac{|x_0 - y_0|^2 + |T - s_0|^2}{\varepsilon} \\
&\quad - \kappa\big(\langle x_0\rangle^m + \langle y_0\rangle^m\big) - \eta(T + s_0) \\
&\geqslant \Phi(\bar t, \bar x, \bar t, \bar x) \\
&= \gamma v_1(\bar t, \bar x) - v_2(\bar t, \bar x) - 2\kappa\langle \bar x\rangle^m - 2\eta\bar t \\
&\geqslant \delta - 2\kappa\langle \bar x\rangle^m - 2\eta\bar t \quad \text{by (3.1)}.
\end{aligned}$$

Using $v_1(T, y_0) = v_2(T, y_0) = g(y_0) \geqslant 0$, above inequalities will imply

$$\begin{aligned}
&v_1(T, x_0) - v_1(T, y_0) + v_2(T, y_0) - v_2(s_0, y_0) \\
&\quad = v_1(T, x_0) - v_2(s_0, y_0) \geqslant \gamma v_1(T, x_0) - v_2(s_0, y_0) \geqslant \delta - 2\kappa\langle \bar x\rangle^m - 2\eta\bar t \quad \Rightarrow \\
&g(x_0) - g(y_0) + v_2(T, y_0) - v_2(s_0, y_0) \geqslant \delta - 2\kappa\langle \bar x\rangle^m - 2\eta\bar t \quad \Rightarrow \\
&\omega_g\big(|x_0 - y_0|\big) + \omega_{v_2}\big(|T - s_0|\big) \geqslant \delta - 2\kappa\langle \bar x\rangle^m - 2\eta\bar t.
\end{aligned}$$

Choose κ_0 such that $2\kappa_0\langle \bar x\rangle^m \leqslant \delta/4$ and observe that $\eta\bar t \leqslant \eta T = \delta/8$. Then,

$$\omega_g\big(|x_0 - y_0|\big) + \omega_{v_2}\big(|T - s_0|\big) \geqslant \delta/2.$$

Now as $\varepsilon \to 0$, $|x_0 - y_0|$, $|T - s_0|$ become small and left-hand side of above expression tends to 0 whereas right-hand side is $\delta/2$. Thus we arrive at a contradiction. Hence our assumption is wrong and $t_0 < T$. Similarly we can show that $s_0 < T$. Thus (3.5) is proved. Note here that the above inequalities hold true for all $\kappa \leqslant \kappa_0$. Hereafter we choose κ such that $\kappa \leqslant \kappa_0$. Since v_1 is a subsolution, by the definition of viscosity subsolution we have, for a test function ϕ_1 such that $v_1 - \phi_1$ has maximum at (t_0, x_0)

$$\begin{aligned}
&(v_1)_t(t_0, x_0) + H\big(t_0, x_0, D\phi_1(t_0, x_0)\big) \leqslant 0 \quad \text{and} \\
&v_1(t_0, x_0) - M(v_1)(t_0, x_0) \leqslant 0,
\end{aligned} \tag{3.6}$$

and by the definition of supersolution for v_2 we have

$$\begin{aligned}
&(v_2)_s(s_0, y_0) + H\big(y_0, D\phi_2(s_0, y_0)\big) \geqslant 0 \quad \text{or} \\
&v_2(s_0, y_0) - M(v_2)(s_0, y_0) \geqslant 0
\end{aligned} \tag{3.7}$$

for ϕ_2 such that $v_2 - \phi_2$ has local minimum at (s_0, y_0). If both v_1, v_2 satisfy HJB we can proceed by the usual comparison principle, for example, as in [7]. Otherwise we consider the case when $v_2 = Mv_2$ at (s_0, y_0). Our aim is to show that this case does not arise if we restrict γ further suitably. For if, $v_2(s_0, y_0) = Mv_2(s_0, y_0)$, then by (1.20), we know that the infimum will be attained in Mv_2 at some ξ_0, that is

$$v_2(s_0, y_0) = Mv_2(s_0, y_0) = v_2(s_0, y_0 + \xi_0) + c_1\bigl(s_0, X_{y_0}(s_0), \xi_0\bigr).$$

Also by the definition of viscosity subsolution we have that

$$v_1(t_0, x_0) \leqslant Mv_1(t_0, x_0) \leqslant v_1(t_0, x_0 + \xi_0) + c_1\bigl(t_0, X_{x_0}(t_0), \xi_0\bigr).$$

Substituting these inequalities in $\Phi(t_0, x_0, s_0, y_0)$, we get

$$\begin{aligned}
\Phi(t_0, x_0, s_0, y_0) &\leqslant \gamma v_1(t_0, x_0 + \xi_0) + \gamma c_1\bigl(t_0, X_{x_0}(t_0), \xi_0\bigr) - v_2(s_0, y_0 + \xi_0) \\
&\quad - c_1\bigl(s_0, X_{y_0}(s_0), \xi_0\bigr) - \frac{|x_0 - y_0|^2 + |t_0 - s_0|^2}{\varepsilon} \\
&\quad - \kappa\bigl(\langle x_0\rangle^m + \langle y_0\rangle^m\bigr) - \eta(t_0 + s_0) \\
&\leqslant \gamma v_1(t_0, x_0 + \xi_0) + \gamma c_1\bigl(t_0, X_{x_0}(t_0), \xi_0\bigr) - v_2(s_0, y_0 + \xi_0) \\
&\quad - \gamma c_1\bigl(t_0, X_{y_0}(t_0), \xi_0\bigr) + \gamma c_1\bigl(t_0, X_{y_0}(t_0), \xi_0\bigr) \\
&\quad - c_1\bigl(s_0, X_{y_0}(s_0), \xi_0\bigr) - \gamma c_1\bigl(s_0, X_{y_0}(s_0), \xi_0\bigr) \\
&\quad + \gamma c_1\bigl(s_0, X_{y_0}(s_0), \xi_0\bigr) - \frac{|x_0 - y_0|^2 + |t_0 - s_0|^2}{\varepsilon} \\
&\quad - \kappa\bigl(\langle x_0\rangle^m + \langle y_0\rangle^m\bigr) - \eta(t_0 + s_0) \\
&\leqslant \Phi(t_0, x_0 + \xi_0, s_0, y_0 + \xi_0) + \gamma\omega_{c_1}\bigl(e^{Lt_0}|x_0 - y_0|\bigr) \\
&\quad + \gamma\omega_{c_1}\bigl((F+1)|t_0 - s_0|\bigr) - (1-\gamma)c_1\bigl(s_0, X_{y_0}(s_0), \xi_0\bigr) \\
&\quad + \kappa\bigl(\langle x_0 + \xi_0\rangle^m + \langle y_0 + \xi_0\rangle^m\bigr) - \kappa\bigl(\langle x_0\rangle^m + \langle y_0\rangle^m\bigr) \\
&\leqslant \Phi(t_0, x_0 + \xi_0, s_0, y_0 + \xi_0) - (1-\gamma)c_1\bigl(s_0, X_{y_0}(s_0), \xi_0\bigr) + 2\kappa|\xi_0| \\
&\quad + \gamma\omega_{c_1}\bigl(e^{Lt_0}|x_0 - y_0|\bigr) + \gamma\omega_{c_1}\bigl((F+1)|t_0 - s_0|\bigr).
\end{aligned}$$

Here in the last inequality we have used the fact that $\langle x\rangle^m$ is Lipschitz continuous with Lipschitz constant less than 1 and ω_{c_1} is the modulus of continuity of c_1. Now we claim that $|\xi_0|$ is bounded. Using (1.20), we have that for a given G there exists R_1 such that

$$c_1\bigl(t, X_{y_0}(t), \xi_0\bigr) > G|\xi_0| \quad \text{for } |\xi_0| > R_1.$$

But $c_1(s_0, X_{y_0}(s_0), \xi_0) = v_2(s_0, y_0) - v_2(s_0, y_0 + \xi_0)$ and by uniform continuity of v_2

$$\begin{aligned}
&C\bigl(1 + |\xi_0|\bigr) \geqslant v_2(s_0, y_0) - v_2(s_0, y_0 + \xi_0) = c_1\bigl(s_0, X_{y_0}(s_0), \xi_0\bigr) \geqslant G|\xi_0| \\
&\quad \text{if } |\xi_0| > R_1.
\end{aligned}$$

Also, for the choice of $G > C$ this would give a bound on $|\xi_0|$. Hence $|\xi_0|$ is bounded by some R_1, for all ξ_0 such that $v_2(s_0, y_0) = v_2(s_0, y_0 + \xi_0) + c_1(s_0, X_{y_0}(s_0), \xi_0)$. Recalling from (A10-3) that $c_1(t, X_x(t), \xi)$ is bounded below by C_0, the above inequality can be written as

$$\begin{aligned}
\Phi(t_0, x_0, s_0, y_0) &\leqslant \Phi(t_0, x_0 + \xi_0, s_0, y_0 + \xi_0) - (1-\gamma)C_0 + 2\kappa R_1 \\
&\quad + \gamma\omega_{c_1}\bigl(e^{Lt_0}|x_0 - y_0|\bigr) + \gamma\omega_{c_1}\bigl((F+1)|t_0 - s_0|\bigr).
\end{aligned}$$

Now we choose ε such that

$$\gamma\omega_{c_1}\big(Ce^{LT}\sqrt{\varepsilon}\big)+\gamma\omega_{c_1}\big((F+1)\sqrt{\varepsilon}\big)<\kappa. \tag{3.8}$$

Hence for $R=2R_1+1$ we can write

$$\Phi(t_0,x_0,s_0,y_0)\leqslant\Phi(t_0,x_0+\xi_0,s_0,y_0+\xi_0)-(1-\gamma)C_0+\kappa R. \tag{3.9}$$

If we restrict γ further to satisfy

$$(1-\gamma)>\frac{\kappa R}{C_0}, \tag{3.10}$$

we get

$$\Phi(t_0,x_0,s_0,y_0)<\Phi(t_0,x_0+\xi_0,s_0,y_0+\xi_0).$$

This is a contradiction to the fact that (t_0,x_0,s_0,y_0) is the supremum point of Φ. The above choice of γ in (3.10) is consistent with the earlier choice namely (3.2), if we fix $\kappa<\kappa_0$ such that

$$\kappa<\min\left\{\frac{C_0\delta}{Rv_1(\bar t,\bar x)},\frac{C_0}{R}\right\}. \tag{3.11}$$

Thus the case $v_2=Mv_2$ cannot occur at the maximum point of the auxiliary function Φ for such a choice of ε,γ and κ. Hence v_2 satisfies HJB at (s_0,y_0).

Now we proceed by the usual method of comparison principle. For, we define test functions ϕ_1 and ϕ_2 by

$$\phi_1(t,x)=v_2(s_0,y_0)+\frac{|x-y_0|^2+|t-s_0|^2}{\varepsilon}+\kappa\big(\langle x\rangle^m+\langle y_0\rangle^m\big)+\eta(t+s_0),$$
$$\phi_2(s,y)=\gamma v_1(t_0,x_0)-\frac{|x_0-y|^2+|t_0-s|^2}{\varepsilon}-\kappa\big(\langle x_0\rangle^m+\langle y\rangle^m\big)-\eta(s+t_0).$$

Note here that

$$D_x\phi_1(t_0,x_0)=2\frac{x_0-y_0}{\varepsilon}+\kappa m\langle x_0\rangle^{m-2}x_0,$$
$$D_y\phi_2(s_0,y_0)=2\frac{x_0-y_0}{\varepsilon}-\kappa m\langle y_0\rangle^{m-2}y_0,$$
$$\phi_{1_t}(t_0,x_0)=2\frac{t_0-s_0}{\varepsilon}+\eta;\qquad \phi_{2_s}(s_0,y_0)=2\frac{t_0-s_0}{\varepsilon}-\eta.$$

Then $\gamma v_1-\phi_1$ attains its maximum at (t_0,x_0), that is $v_1-\frac{\phi_1}{\gamma}$ attains its maximum at (t_0,x_0) and $v_2-\phi_2$ attains its minimum at (t_0,y_0). Hence by the definition of viscosity sub- and super-solution,

$$\frac{(\phi_1)_t}{\gamma}+H\left(t_0,x_0,\frac{D\phi_1(t_0,x_0)}{\gamma}\right)\leqslant 0\quad\Rightarrow$$
$$(\phi_1)_t+\gamma H\left(t_0,x_0,\frac{D\phi_1(t_0,x_0)}{\gamma}\right)\leqslant 0,$$
$$(\phi_2)_s+H\big(s_0,y_0,D\phi_2(s_0,y_0)\big)\geqslant 0.$$

Now subtracting one from the other we will have

$$\begin{aligned}
&(\phi_1)_t - (\phi_2)_s \\
&\quad \leqslant H\big(s_0, y_0, D\phi_2(s_0, y_0)\big) - \gamma H\left(t_0, x_0, \frac{D\phi_1(t_0, x_0)}{\gamma}\right) \\
&\quad \leqslant \inf_u \left(-k_1(s_0, y_0, u) - f(s_0, y_0, u) \cdot \left(2\frac{x_0 - y_0}{\varepsilon} - \kappa m \langle y_0\rangle^{m-2} y_0\right)\right) \\
&\qquad - \inf_u \left(-\gamma k_1(t_0, x_0, u) - f(t_0, x_0, u) \cdot \left(2\frac{x_0 - y_0}{\varepsilon} + \kappa m \langle x_0\rangle^{m-2} x_0\right)\right).
\end{aligned}$$

Using the definition of infimum in the second term, for any $\bar{\varepsilon}$ there exists $\bar{u}$ such that

$$\begin{aligned}
&(\phi_1)_t - (\phi_2)_s \\
&\quad \leqslant -k_1(s_0, y_0, \bar{u}) - \left(f(s_0, y_0, \bar{u}) \cdot \left(2\frac{x_0 - y_0}{\varepsilon} - \kappa m \langle y_0\rangle^{m-2} y_0\right)\right) \\
&\qquad + \gamma k_1(t_0, x_0, \bar{u}) + \left(f(t_0, x_0, \bar{u}) \cdot \left(2\frac{x_0 - y_0}{\varepsilon} + \kappa m \langle x_0\rangle^{m-2} x_0\right)\right) + \bar{\varepsilon}.
\end{aligned}$$

Now adding and subtracting

$$k_1(t_0, x_0, \bar{u}) \quad \text{and} \quad f(s_0, y_0, \bar{u}) \cdot \left(2\frac{x_0 - y_0}{\varepsilon} + \kappa m \langle x_0\rangle^{m-2} x_0\right),$$

using (A1), (A2) and fact that $\bar{\varepsilon}$ is arbitrary, we get

$$\begin{aligned}
2\eta &\leqslant \omega_{k_1}\big(|t_0 - s_0| + |x_0 - y_0|\big) + (1 - \gamma)\big|k_1(t_0, x_0, \bar{u})\big| \\
&\quad + F\kappa m \big|\langle x_0\rangle^{m-2} x_0 + \langle y_0\rangle^{m-2} y_0\big| + L\big(|t_0 - s_0| + |x_0 - y_0|\big).
\end{aligned}$$

Note that $m|\langle x_0\rangle^{m-2} x_0|$ and $m|\langle y_0\rangle^{m-2} y_0|$ remain bounded for all $0 < m < 1$. Now using the fact that $t_0 < T$ and $\sqrt{\kappa}|x_0|$ is bounded by some constant $\hat{C}$ we get that $k_1(t_0, x_0, \bar{u})$ remains bounded in some $r = r(\kappa)$ ball. Then, using (3.4) and sending first ε to 0 and then κ and $(1 - \gamma)$ to 0, respecting the choices (3.8), (3.2), (3.10) and (3.11) we get

$$2\eta \leqslant 0.$$

But $\eta = \frac{\delta}{8T} > 0$, hence we arrive at a contradiction.

Hence our assumption that $\sup(v_1 - v_2) > 0$ is wrong and we have

$$v_1 \leqslant v_2 \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d.$$

Then by interchanging the roles of v_1 and v_2 we will get the uniqueness. □

Now we prove the estimate (3.4) used in the proof as the following lemma.

Lemma 3.1. *Let Φ be as in* (3.3) *and let* $\sup \Phi = \Phi(t_0, x_0, s_0, y_0)$. *Then,*

1. $|x_0 - y_0| + |t_0 - s_0| \leqslant \sqrt{\varepsilon} C$ *for some constant C independent of κ and ε.*
2. $\sqrt{\kappa}|x_0| \leqslant C$, $\sqrt{\kappa}|y_0| \leqslant C$ *for some C independent of ε and κ.*

Proof. Since $\sup \Phi(t,x,s,y) = \Phi(t_0,x_0,s_0,y_0)$

$$2\Phi(t_0,x_0,s_0,y_0) \geqslant \Phi(t_0,x_0,t_0,x_0) + \Phi(s_0,y_0,s_0,y_0),$$

$$2\frac{|x_0-y_0|^2+|t_0-s_0|^2}{\varepsilon} \leqslant \gamma v_1(t_0,x_0) + v_2(t_0,x_0) - \gamma v_1(s_0,y_0) - v_2(s_0,y_0).$$

Now by uniform continuity of v_1 and v_2 and Cauchy–Schwarz inequality we get

$$\begin{aligned}
&2\frac{|x_0-y_0|^2+|t_0-s_0|^2}{\varepsilon}\\
&\quad\leqslant C\big(1+|x_0-y_0|+|t_0-s_0|\big) + \gamma C\big(1+|x_0-y_0|+|t_0-s_0|\big) \quad \Rightarrow\\
&|x_0-y_0|^2+|t_0-s_0|^2 \leqslant \varepsilon C\big(1+|x_0-y_0|+|t_0-s_0|\big) \quad \text{since } \gamma < 1\\
&\quad\leqslant \varepsilon C + \frac{\varepsilon^2 C^2}{2} + \frac{1}{2}\big(|x_0-y_0|^2+|t_0-s_0|^2\big) \quad \Rightarrow\\
&\frac{1}{2}\big(|x_0-y_0|^2+|t_0-s_0|^2\big) \leqslant \varepsilon C + \frac{\varepsilon^2 C^2}{2}
\end{aligned}$$

and hence, $|x_0-y_0|+|t_0-s_0| \leqslant \sqrt{\varepsilon}C$ for some C independent of κ and ε. □

Now we take up the proof of Theorem 1.4. We first prove a local comparison theorem in a cone and then using that prove the global comparison, as done for HJB equations in [7, Theorems 2.4 and 2.5]. Now we state the local uniqueness theorem in a cone.

Theorem 3.1. *Let*

$$\mathcal{C} = \big\{(x,t)\colon\ 0 < t < T \text{ and } |x| < Ct\big\}.$$

Let v_1 and v_2 in $C(\bar{\mathcal{C}})$ be two viscosity solutions of QVI given by (1.21), (1.22). *Then $v_1 = v_2$ in $\mathcal{C}$.*

Proof. We prove that for any two solutions v_1 and v_2 of (1.21), (1.22) in $\mathcal{C}$, $v_1 \leqslant v_2$ in $\mathcal{C}$. Then by interchanging the roles of v_1 and v_2 we will get the required result. The idea of the proof is similar to that of Theorem 1.3. We will modify the auxiliary function in Theorem 1.3 slightly so as to avoid the supremum point of Φ escaping to the lateral boundary of $\mathcal{C}$. Let there exist $\delta < T$ and $(\bar{t},\bar{x}) \in \mathcal{C}$ such that

$$\sup_{\mathcal{C}}(v_1 - v_2) \geqslant v_1(\bar{t},\bar{x}) - v_2(\bar{t},\bar{x}) = 2\delta > 0 \quad \text{and} \quad |\bar{x}| \leqslant C\bar{t} - 4\delta. \tag{3.12}$$

If $v_1(\bar{t},\bar{x}) < 0$ we can choose any $\gamma \in (0,1)$, else we fix $\gamma \in (0,1)$ such that

$$(1-\gamma) > \frac{\delta}{v_1(\bar{t},\bar{x})}. \tag{3.13}$$

Then we are assured that

$$\gamma v_1(\bar{t},\bar{x}) - v_2(\bar{t},\bar{x}) > \delta.$$

Let $\tilde{\mathcal{C}} = \mathcal{C} \times \mathcal{C}$, and

$$M_0 > \sup_{\tilde{\mathcal{C}}}\big\{\gamma v_1(t,x) - v_2(s,y)\big\}.$$

Then $M_0 \geqslant 2\delta$. Let $h \in C^1(\mathbb{R})$ be such that $h' \leqslant 0$, $h(r) = 0$ for $r \leqslant -\delta$, $h(r) = -3M_0$ for $r \geqslant 0$. Let us denote $\langle x\rangle_\beta = (|x|^2 + \beta^2)^{1/2}$.

We now define the auxiliary function Φ on $\tilde{\mathcal{C}}$ by

$$\begin{aligned}\Phi(t,x,s,y) = \gamma v_1(t,x) - v_2(s,y) - \frac{|x-y|^2 + |t-s|^2}{\varepsilon} - \eta(t+s)\\ + h\big(\langle x\rangle_\beta - Ct\big) + h\big(\langle y\rangle_\beta - Cs\big),\end{aligned}$$

where ε, η, β are positive parameters to be chosen suitably later on. Φ attains its supremum at some point in $\tilde{\mathcal{C}}$ say, (t_0, x_0, s_0, y_0). Using the fact that $\sup \Phi = \Phi(t_0, x_0, s_0, y_0)$ we get the following estimate similar to the one in Lemma 2.3.1:

$$|x_0 - y_0|^2 + |t_0 - s_0|^2 \leqslant 2M_0\varepsilon \tag{3.14}$$

and

$$\frac{|x_0 - y_0|^2 + |t_0 - s_0|^2}{\varepsilon} \leqslant \omega_1(\varepsilon). \tag{3.15}$$

We now claim that $t_0 < T$ and $s_0 < T$. Moreover, $|x_0| < Ct_0$ and $|y_0| < Cs_0$ for η and β small. If $|x_0| = Ct_0$ then,

$$\begin{aligned}\sup\Phi &= \Phi(t_0,x_0,s_0,y_0)\\ &\leqslant \gamma v_1(t_0,x_0) - v_2(s_0,y_0) + h\big(\langle x_0\rangle_\beta - Ct_0\big) + h\big(\langle y_0\rangle_\beta - Cs_0\big)\\ &\leqslant M_0 - 3M_0 = -2M_0.\end{aligned}$$

But on the other hand, for any $\beta < \delta$ and $\eta < \delta/4\bar{t}$,

$$\Phi(\bar{t},\bar{x},\bar{t},\bar{x}) \geqslant \delta - 2\eta\bar{t} + 2h\big(\langle\bar{x}\rangle_\beta - C\bar{t}\big) \geqslant \delta/2, \tag{3.16}$$

which is a contradiction. Hence $|x_0| < Ct_0$. In a similar fashion we can show that $|y_0| < Ct_0$ for $\beta < \delta$ and $\eta < \delta/4\bar{t}$.

Now if $t_0 = T$,

$$\begin{aligned}\Phi(t_0,x_0,s_0,y_0) &= \Phi(T,x_0,s_0,y_0)\\ &\leqslant \gamma v_1(T,x_0) - v_2(s_0,y_0)\\ &\leqslant \gamma\big(g(x_0) - g(y_0)\big) + (\gamma-1)\omega_{v_2}\big(|T-s_0|\big)\\ &\leqslant \gamma\omega_g\big(|x_0-y_0|\big) \quad \text{as } \gamma < 1.\end{aligned}$$

This implies by the estimate (3.14), that $\Phi(t_0, x_0, s_0, y_0)$ can be made arbitrarily small by choosing ε appropriately. This contradicts (3.16) and our claim is proved. Similarly we can show that $s_0 < T$.

If both v_1, v_2 satisfy HJB at (t_0, x_0) and (s_0, y_0) we can proceed by the usual comparison principle, for example, as in [7, Theorem 2.5]. Otherwise, we consider the case when $v_2 = Mv_2$ at (s_0, y_0). Our aim is to show that this case does not arise for $\gamma < 1$. For if, $v_2(s_0, y_0) = Mv_2(s_0, y_0)$, then by (1.20), we know that the infimum will be attained in Mv_2 at some ξ_0, that is

$$v_2(s_0,y_0) = Mv_2(s_0,y_0) = v_2(s_0, y_0+\xi_0) + c_1\big(s_0, X_{y_0}(s_0), \xi_0\big).$$

Also by the definition of viscosity subsolution we have that

$$v_1(t_0,x_0) \leqslant Mv_1(t_0,x_0) \leqslant v_1(t_0,x_0+\xi_0) + c_1\big(t_0, X_{x_0}(t_0), \xi_0\big).$$

Substituting these inequalities in $\Phi(t_0,x_0,s_0,y_0)$, we get

$$\begin{aligned}
&\Phi(t_0,x_0,s_0,y_0)\\
&\quad \leqslant \gamma v_1(t_0,x_0+\xi_0) + \gamma c_1\big(t_0,X_{x_0}(t_0),\xi_0\big) - v_2(s_0,y_0+\xi_0) - c_1\big(s_0,X_{y_0}(s_0),\xi_0\big)\\
&\qquad - \frac{|x_0-y_0|^2+|t_0-s_0|^2}{\varepsilon} - \eta(t_0+s_0) + h\big(\langle x_0\rangle_\beta - Ct_0\big) + h\big(\langle y_0\rangle_\beta - Cs_0\big)\\
&\quad \leqslant \Phi(t_0,x_0+\xi_0,s_0,y_0+\xi_0) + \gamma c_1\big(t_0,X_{x_0}(t_0),\xi_0\big) - c_1\big(s_0,X_{y_0}(s_0),\xi_0\big)\\
&\qquad - h\big(\langle x_0+\xi_0\rangle_\beta - Ct_0\big) + h\big(\langle x_0\rangle_\beta - Ct_0\big) - h\big(\langle y_0+\xi_0\rangle_\beta - Cs_0\big)\\
&\qquad + h\big(\langle y_0\rangle_\beta - Cs_0\big)\\
&\quad \leqslant \Phi(t_0,x_0+\xi_0,s_0,y_0+\xi_0) - (1-\gamma)c_1\big(t_0,X_{x_0}(t_0),\xi_0\big) + c_1\big(t_0,X_{x_0}(t_0),\xi_0\big)\\
&\qquad - c_1\big(s_0,X_{y_0}(s_0),\xi_0\big).
\end{aligned}$$

Here in the last inequality we have used the mean value theorem for h and the fact that $h' < 0$ and $|\xi_0|$ is bounded by some large constant R. For, if $v_2(s_0,y_0) = Mv_2(s_0,y_0)$, then by the assumption (1.27), there exists ξ_0 such that

$$\begin{aligned}
&Mv_2(s_0,y_0) = v_2(s_0,y_0) = v_2(s_0,y_0+\xi_0) + c_1\big(s_0,X_{y_0}(s_0),\xi_0\big) \quad \Rightarrow\\
&c_1\big(s_0,X_{y_0}(s_0),\xi_0\big) = v_2(s_0,y_0) - v_2(s_0,y_0+\xi_0).
\end{aligned}$$

Now right-hand side of the above equation is bounded because v_2 is bounded on $\mathcal{C}$ and the second term $v_2(s_0,y_0+\xi_0)$ is bounded below. This together with (1.27) implies that ξ_0 cannot go to infinity. This will give a bound on $|\xi_0|$, say R. Recalling that c_1 is bounded below by C_0, the above inequality can be written as

$$\begin{aligned}
\Phi(t_0,x_0,s_0,y_0) \leqslant{}& \Phi(t_0,x_0+\xi_0,s_0,y_0+\xi_0) - (1-\gamma)C_0 + \omega_{c_1}\big(|t_0-s_0|\big)\\
&+ \omega_{c_1}\big(e^{Lt_0}|x_0-y_0|\big).
\end{aligned} \tag{3.17}$$

Now by choosing ε and γ appropriately, that is,

$$(1-\gamma) > \frac{\omega_{c_1}(\hat{C}\sqrt{\varepsilon})}{C_0}, \tag{3.18}$$

we can make

$$\Phi(t_0,x_0,s_0,y_0) < \Phi(t_0,x_0+\xi_0,s_0,y_0+\xi_0).$$

This is a contradiction to the fact that (t_0,x_0,s_0,y_0) is the supremum point of Φ. This choice (3.18) of γ can be made consistent with the earlier choice (3.13) if we choose ε such that

$$\omega_{c_1}(\hat{C}\sqrt{\varepsilon}) < \frac{\delta}{v_1(\bar{t},\bar{x})}. \tag{3.19}$$

Thus the case $v_2 = Mv_2$ cannot occur at the maximum point of the auxiliary function Φ for such a choice of parameters. Hence v_2 satisfies HJB at (s_0,y_0). Now we proceed by the usual method of comparison principle. We define test functions ϕ_1 and ϕ_2 by

$$\begin{aligned}\phi_1(t,x) &= v_2(s_0,y_0) + \frac{|x-y_0|^2+|t-s_0|^2}{\varepsilon} + \eta(t+s_0)\\ &\quad - h\bigl(\langle x\rangle_\beta - Ct\bigr) - h\bigl(\langle y_0\rangle_\beta - Cs_0\bigr),\\ \phi_2(s,y) &= \gamma v_1(t_0,x_0) - \frac{|x_0-y|^2+|t_0-s|^2}{\varepsilon} - \eta(t_0+s)\\ &\quad + h\bigl(\langle x_0\rangle_\beta - Ct_0\bigr) + h\bigl(\langle y\rangle_\beta - Cs\bigr).\end{aligned}$$

Let $X = (\langle x_0\rangle_\beta - Ct_0)$ and $Y = (\langle y_0\rangle_\beta - Cs_0)$. Then,

$$\begin{aligned}D_x\phi_1(t_0,x_0) &= 2\frac{x_0-y_0}{\varepsilon} - h'(X)\frac{x_0}{\langle x_0\rangle_\beta},\\ D_y\phi_2(s_0,y_0) &= 2\frac{x_0-y_0}{\varepsilon} + h'(Y)\frac{y_0}{\langle y_0\rangle_\beta};\\ \phi_{1_t}(t_0,x_0) &= 2\frac{t_0-s_0}{\varepsilon} - Ch'(X) + \eta,\\ \phi_{2_s}(s_0,y_0) &= 2\frac{t_0-s_0}{\varepsilon} + Ch'(Y) - \eta,\end{aligned}$$

$\gamma v_1 - \phi_1$ attains its maximum at (t_0,x_0), that is $v_1 - \frac{\phi_1}{\gamma}$ attains its maximum at (t_0,x_0) and $v_2 - \phi_2$ attains its minimum at (t_0,y_0). Hence by the definition of viscosity sub- and super-solution,

$$\begin{aligned}&\frac{(\phi_1)_t}{\gamma} + H\left(t_0,x_0,\frac{D\phi_1(t_0,x_0)}{\gamma}\right) \leqslant 0,\\ &(\phi_2)_s + H\bigl(s_0,y_0,D\phi_2(s_0,y_0)\bigr) \geqslant 0.\end{aligned}$$

Multiplying the first inequality by γ, and subtracting one from other,

$$(\phi_1)_t - (\phi_2)_s \leqslant H\bigl(s_0,y_0,D\phi_2(s_0,y_0)\bigr) - \gamma H\left(t_0,x_0,\frac{D\phi_1(t_0,x_0)}{\gamma}\right).$$

Under the assumptions (A10′) and (A3), H satisfies the following structural condition in the cone $\mathcal{C}$:

$$\begin{aligned}&\bigl|H(t,x,p) - \gamma H(t,x,q/\gamma)\bigr| \leqslant C|p-q| + (1-\gamma)k_1(t,x,\bar u),\\ &\quad\text{for some } \bar u, \qquad (3.20)\\ &\bigl|H(t,x,q) - H(s,y,q)\bigr| \leqslant \omega\bigl(|x-y|+|t-s|\bigr)\\ &\qquad\qquad + \omega\bigl(\bigl(|x-y|+|t-s|\bigr)|q|\bigr). \qquad (3.21)\end{aligned}$$

Hence we get

$$\begin{aligned}&2\eta - C\bigl\{h'(Y) + h'(X)\bigr\}\\ &\quad \leqslant (1-\gamma)\bigl|k_1(t_0,x_0,\bar u)\bigr| + C\left|h'(Y)\frac{y_0}{\langle y_0\rangle_\beta} + h'(X)\frac{x_0}{\langle x_0\rangle_\beta}\right|\end{aligned}$$

$$+\,\omega\big(|x_0-y_0|+|t_0-s_0|\big)+\omega\left(\frac{|x_0-y_0|^2}{\varepsilon}+|x_0-y_0|\big|h'(Y)\big|\right)$$
$$+\,\omega\left(\frac{|x_0-y_0||t_0-s_0|}{\varepsilon}+|t_0-s_0|\big|h'(Y)\big|\right).$$

Observe that k_1 is continuous on the cone $\mathcal{C}$ hence there exists $\tilde{C}$ such that

$$\big|k_1(t_0,x_0,\bar{u})\big|\leqslant\tilde{C}.$$

Since $h'<0$, by sending ε to 0 and γ to 1, respecting (3.13), (3.18) and (3.19) we get

$$2\eta\leqslant o(1).$$

This is a contradiction and hence $v_1\leqslant v_2$. □

Corollary 3.1. *The above theorem holds true in the cone*

$$\big\{(x,t)\colon\ 0<t<T,\,|x-x_0|<Ct\big\}$$

for any $x_0\in\mathbb{R}^d$, if structural assumptions (3.20) *and* (3.21) *on Hamiltonian hold in this cone.*

Proof of Theorem 1.4. Fix $T_0<T$ such that $T-T_0<1/F$, where F is as in (1.26). We fix $x_0\in\mathbb{R}^d$. Define $r=FT(1+|x_0|)/(1-FT)>0$, so that

$$r=FT\big(1+|x_0|+r\big):=CT.$$

Define the cone

$$\mathcal{C}_{x_0}=\big\{(x,t)\colon\ T-T_0<t<T \text{ and } |x-x_0|<F\big(1+|x_0|+r\big)\big(t-(T-T_0)\big)\big\}.$$

We next claim that in this cone, $v_1=v_2$. For we will apply the theorem and corollary stated and proved below with $C=F(1+|x_0|+r)$ for x_0 fixed. Then we can write

$$\mathbb{R}^d\times(T-T_0,T)=\bigcup_{x_0\in\mathbb{R}^d}\mathcal{C}_{x_0}$$

and hence, $v_1\leqslant v_2$ on $[T-T_0,T]\times\mathbb{R}^d$. The proof for $T-T_0>1/F$ can be obtained by iterating the above argument on time intervals of fixed length smaller than $1/F$. This will complete the proof. □

References

[1] A.J. Ball, C. Jerawan, Comparison theorems for viscosity solutions of quasivariational inequalities with application to optimal control with switching costs, J. Math. Anal. Appl. 251 (2000) 40–64.

[2] G. Barles, Deterministic impulse control problems, SIAM J. Control Optim. 23 (1985) 419–432.

[3] G. Barles, Quasi-variational inequalities and first order Hamilton–Jacobi equations, Nonlinear Anal. 9 (1985) 131–148.

[4] M. Bardi, I. Capuzzo-Dolcetta, Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations, Birkhäuser, Boston, MA, 1997.

[5] I. Capuzzo-Dolcetta, L.C. Evans, Optimal switching for ordinary differential equations, SIAM J. Control Optim. 22 (1984) 143–161.

[6] M.G. Crandall, P.L. Lions, Viscosity solutions of Hamilton–Jacobi equations, Trans. Amer. Math. Soc. 277 (1984) 1–42.

[7] H. Ishii, Uniqueness of unbounded viscosity solutions of Hamilton–Jacobi equations, Indiana Univ. Math. J. 33 (1984) 721–748.

[8] J. Yong, Systems governed by ordinary differential equations with continuous, switching and impulse controls, Appl. Math. Optim. 20 (1989) 223–235.

[9] J. Yong, Zero sum differential games involving impulse controls, Appl. Math. Optim. 29 (1994) 243–261.