Zero-Sum Differential Games Involving Hybrid Controls [1,2]

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Abstract We study a zero-sum differential game with hybrid controls in which both players are allowed to use continuous as well as discrete controls. Discrete controls act on the system at a given set interface. The state of the system is changed discontinuously when the trajectory hits predefined sets, an autonomous jump set A or a controlled jump set C where one controller can choose to jump or not. At each jump, trajectory can move to a different Euclidean space. One player uses all the three types of controls namely, continuous controls, autonomous jumps and controlled jumps and the other player uses continuous controls and autonomous jumps. We prove the continuity of the associated lower and upper value functions V^- and V^+. Using the dynamic programming principle satisfied by V^- and V^+, we derive lower and upper quasivariational inequalities satisfied by them in the viscosity sense. We characterize the lower and upper value functions as the unique viscosity solution of the corresponding quasivariational inequalities. Lastly we state an Isaacs' like condition for the game to have a value.

Key words Dynamic programming principle, viscosity solutions, quasivariational inequalities, Hybrid control, differential games

1 Introduction

We study here, a zero sum differential game involving hybrid controls. The main motivation comes from the hybrid control systems arising in many engineering problems like constrained robotic systems, automated highway systems, and flight control systems. (See Ref. 1). Branicky, Borkar and Mitter have presented a model for the hybrid control system where continuous controls and discrete controls act on the system at a given set interface. The state of the system is changed discontinuously when the trajectory hits predefined sets, namely, an autonomous jump set A or a controlled jump set C where controller can choose to jump or not. At each jump, trajectory can move to a different Euclidean space. They prove right continuity of the value function. By using dynamic programming principle they have arrived at the quasivariational inequality satisfied by the value function in the viscosity sense. In Ref. 2, Bensoussan and Menaldi study a similar system and prove that the value function is continuous. They prove the uniqueness for a special case when the autonomous jump set is empty. We have shown for this hybrid control problem in Ref. 3 that the value function is Hölder continuous under a transversality condition. We have proved the uniqueness of the value function by a very different method than Ref. 2 and in a more general case namely autonomous jump set being nonempty.

In this work we extend the model in Ref. 1 to the game theoretic set up. We allow player P1 to use all the three types of controls namely, continuous controls, autonomous jumps and controlled jumps and the other player P2 to use continuous controls and autonomous jumps. As in Ref. 4 we then use Elliot Kalton approach to define the strategies and lower and upper value function. The viscosity solution techniques applied to differential game problems using Elliot Kalton strategies and dynamic programming principle give good existence and uniqueness results for the Hamilton-Jacobi-Isaacs' equations satisfied by the value functions of the game problem.(See chapter 6, Ref. 4 and references therein). The games where both players use continuous controls Refs.4-5, when both players use switching strategies Ref. 6 and a problem where one player uses impulse controls and the other one continuous control and/or switching controls Ref. 7 are to name a few.

In this paper our aim is to to prove the local Hölder continuity of lower and upper value functions, to derive corresponding lower and upper quasivariational inequalities satisfied by them in the viscosity sense and to characterize them as unique viscosity solutions of these QVI's. Finally we give Isaacs' like condition for upper and lower values to coincide and thus game to have a value.

2 Notations and Assumptions

The trajectory $X(t)$ of the hybrid system evolves by the following equation:

$$\dot{X}(t) = f_i(X(t), u_1(t), u_2(t)); \quad X(0) = x \tag{1}$$

where $x \in \Omega_i$; Ω_i is the closure of a connected open subset of $I\!R^{d_i}$, $d_i \in Z_+$. The trajectory on hitting the autonomous jump set A_i, jumps to the destination set D_j, ($D_j \subseteq \Omega_j$ possibly), according to the given transition map g depending on discrete controls from the discrete control sets V_1 for P1 and V_2 for P2 where, V_1, V_2 are compact metric spaces. The trajectory then, will continue its evolution under f_j till it hits again A_j or C_j. On hitting C_j the player P1 can choose either to jump or not to jump. If it chooses to jump, then the trajectory is moved to a new point in say D_k. The state space Ω is actually, $\Omega := \cup \ \Omega_i; \Omega_i \in I\!R^{d_i}, d_i, i \in Z_+$. Predefined sets, A, C, D are unions of sets in $I\!R^{d_i}$:

$$A = \bigcup_{i=1}^{\infty} A_i; \ C = \bigcup_{i=1}^{\infty} C_i; \ D = \bigcup_{i=1}^{\infty} D_i; \text{ and } A_i, C_i, D_i \subseteq \Omega_i \subseteq I\!R^{d_i}.$$

Also, $f : \Omega \times U_1 \times U_2 \to \Omega$. with the understanding, $f = f_i(X(t), u_1(t), u_2(t))$ whenever $X(t) \in \Omega_i$ and u_j in the continuous control space, $\mathcal{U}_j = \{u_j : [\,0, \infty) \to U_j \mid u_j \text{ measurable.}\}$ Where U_j are compact metric spaces ; j = 1,2. Similarly the transition map $g : A \times V_1 \times V_2 \to D$ is actually $g(x, v_1, v_2) = g_i(x, v_1, v_2)$ whenever $x \in A_i$.

Thus trajectory of hybrid problem gives rise to a sequence of hitting times of A, which we denote by σ_i and sequence of hitting times of C, where P1 chooses to jump which is denoted by ξ_i. We denote $X(\sigma_i, u_1(\cdot), u_2(\cdot))$ by x_i, and $g(x_i, v_1, v_2)$ by x_i' and destination of $X(\xi_i, u_1(\cdot), u_2(\cdot))$ by $X(\xi_i)'$. When it is clear from the context, we use $X_x(t)$ instead of $(X_x(t, u_1(\cdot), u_2(\cdot)))$.

We give the inductive limit topology on Ω namely, $x_n \in \Omega$ converges to $x \in \Omega$ if for some N large and $\forall\, n \geq N;\ x, x_n \in \Omega_i$, for some i, and $\|x_n - x\|_{I\!R^{d_i}} < \varepsilon$.

Following are our basic assumptions:

(A1): A_i, C_i, D_i are closed, with non empty interior. ∂A_i, ∂C_i are C^1 ; D_i's are uniformly bounded and $\partial A_i \supseteq \partial\Omega_i \quad \forall\, i$.

(A2): The transition map g is bounded, uniformly Lipschitz continuous, with Lipschitz constant G.

(A3): Vector field f is Lipschitz continuous with Lipschitz constant L in the state variable x and uniformly continuous in variables u_1 and u_2. Moreover,

$$|f(x, u_1, u_2)| \leq F \quad \forall\, x \in \Omega \text{ and } \forall\, (u_1, u_2) \in U_1 \times U_2. \tag{2}$$

Further ∂A_i is compact for all i, and for some $\xi_0 > 0$, transversality condition holds:

$$f(x_0, u_1, u_2) \cdot \eta(x_0) \leq -\xi_0 \;\; \forall \;\; x_0 \in \partial A_i \;\; \forall\, (u_1, u_2) \in U_1 \times U_2 \tag{3}$$

where $\eta(x_0)$ is the unit outward normal to ∂A_i at x_0.

(A4): Let d be the distance between two sets A_i and C_i defined by $d(A_i, C_i) = \inf_{x \in A_i, y \in C_i} d(x, y)$.

We assume, $d(A_i, C_i) > \beta > 0, \quad \forall\, i$ and $\inf_i \; d(A_i, D_i) = \beta > 0$. Note that, this rules out infinitely many autonomous jumps in finite time.

(A5): The sets U_1, U_2 and V_1, V_2 are compact.

The total discounted cost functional is given by:

$$J(x, u_1(\cdot), u_2(\cdot), v_1, v_2, \xi_i, X(\xi_i)') = \int_0^\infty k(X_x(t), u_1(t), u_2(t)) \exp(-\lambda t) dt$$

$$+ \sum_{i=1}^{\infty} c_a(X(\sigma_i), v_1, v_2) \exp(-\lambda \sigma_i) + \sum_{i=1}^{\infty} c_c(X(\xi_i), X(\xi_i)') \; exp(-\lambda \xi_i) \qquad (4)$$

where λ is the discount factor, $k : \Omega \times U_1 \times U_2 \rightarrow I\!R_+$ is the running cost, $c_a : A \times V_1 \times V_2 \rightarrow I\!R_+$ is the autonomous jump cost and $c_c : C \times D \rightarrow I\!R_+$ is the controlled jump cost.

Following Elliot-Kalton approach as in Refs. 4-5, we define the set of all non anticipating or (Elliot-Kalton) strategies for P1 and denote it by Γ:

$\Gamma = \{\alpha\colon\; U_2 \times V_2 \rightarrow U_1 \times V_1 \times [0, \infty) \times D\;\}$ where α's are such that,

$$(u_2(s), v_2) = (u_2{}'(s), v_2{}') \;\text{ for } s \leq t_0 \Rightarrow \; \alpha(u_2(s), v_2) = \alpha(u_2{}'(s), v_2{}') \;\text{ for } s \leq t_0$$

Similarly we can define the set of non anticipating strategies β for P2 denoted by Δ.

The lower value function V^- and upper value function V^+ are defined as follows.

$$V^-(x) = \inf_{\alpha\in\Gamma} \sup_{u_2(\cdot),v_2} J(x,\ \alpha(u_2,v_2),\ u_2(\cdot),\ v_2\ )$$

$$V^+(x) = \sup_{\beta\in\Delta} \inf_{(u_1(\cdot),v_1,\xi_i,X(\xi_i)')} J(x,u_1(\cdot),v_1,\ \xi_i,\ X(\xi_i)',\ \beta(u_1,v_1,\xi_i,X(\xi_i)')(\cdot)\ )$$

We assume the following conditions on the cost functionals:

(C1): k is uniformly continuous in x and it is bounded by k_0 . Also k is Lipschitz continuous in x variable with Lipschitz constant k_1.

(C2): c_a and c_c are uniformly continuous and bounded below by $C' > 0$. Moreover c_a is bounded above by C_0 and Lipschitz continuous with Lipschitz constant C_1.

In section 3 we show that the lower and upper value functions are bounded and locally Hölder continuous. In section 4, we use the dynamic programming principle to derive a PDE; Hamilton-Jacobi-Isaacs' quasivariational inequalities (QVI) satisfied by V^- and V^+ in the viscosity sense. In Section 5 we characterize the value functions as the unique viscosity solution of the respective QVI and state a condition for value to exist.

3 Continuity of the Value Function

Let the trajectory given by solution of (1) and starting from the point x be denoted by $X_x(t,u_1(\cdot),u_2)(\cdot)$. Since $x\in\Omega$, in particular it belongs to some $\Omega_1 \subseteq I\!\!R^{d_1}$. Then we have from assumption A3 and theory of ordinary differential equations:

$$|X_x(t, u_1(\cdot), u_2(\cdot)) - X_z(t, u_1(\cdot), u_2(\cdot))| \leq \exp(Lt)|x - z|, \tag{5}$$

$$|X_x(t, u_1(\cdot), u_2(\cdot)) - X_x(\bar{t}, u_1(\cdot), u_2(\cdot)) \leq F|t - \bar{t}|. \tag{6}$$

Define the first hitting time of trajectory starting from x and evolving with the fixed controls $u_1(\cdot), u_2(\cdot)$ by: $t(x, u_1(\cdot), u_2(\cdot)) = \inf \ \{ t > 0 \mid X_x(t, u_1(\cdot), u_2(\cdot)) \in A\}$. Next proposition deals with the continuity of the first hitting time in the topology of $\mathbb{R}^{d_i}$ if $x \in \Omega_i$. Note that this is equivalent to proving its continuity in Ω with respect to inductive limit topology.

Proposition 3.1: Assume (A1)- (A5). Then the first hitting time t is locally Lipschitz continuous with respect to the initial point. i.e., there exists a $\delta_1 > 0$ depending on f, distance function from A_i and ξ_0 such that $\forall\ y$, $\bar{y}$ in $B(x, \delta_1)$, a δ_1 neighborhood of x in Ω

$$|t(y, u_1(\cdot), u_2(\cdot)) - t(\bar{y}, u_1(\cdot), u_2(\cdot))| < \tilde{C}|y - \bar{y}|, \quad \text{where} \quad \tilde{C} \quad \text{depends on} \quad \xi_0.$$

Proof: Step 1: Estimates for points near ∂A_i. For simplicity of notations we drop the suffix i now onwards remembering that the distances are in $\mathbb{R}^{d_i}$. First we show that there exists $\delta > 0$ and $\tilde{C} > 0$ such that,

$$t(x, u_1(\cdot), u_2(\cdot)) < \tilde{C}\ d(x) \quad \forall\ x \ \in B(A, \delta) \setminus \mathring{A}$$

where $B(A, \delta)$ is δ neighborhood of A and $d(x)$ is a signed distance of x from ∂A given by,

$$d(x) = \begin{cases} -d(x, \partial A) & \text{if } x \in \overset{\circ}{A} \\ 0 & \text{if } x \in \partial A \\ d(x, \partial A) & \text{if } x \in \bar{A}^c \end{cases}$$

It is possible to choose $R > 0$ such that in a small neighborhood of ∂A, say $B(\partial A, R)$, the above signed distance function d is C^1, thanks to our assumption (A1). We denote derivative of d by ∇d. For all $(\hat{u}_1, \hat{u}_2)$ in $U_1 \times U_2$ we can choose $r_0 < R$ such that,

$$f(x, \hat{u}_1, \hat{u}_2) \cdot \nabla d(x) < -\xi_0 \qquad \forall\, x \in B(x_0, r_0) \tag{7}$$

Observe that we can choose r_0 independent of x_0 by using compactness of ∂A. We fix the controls $u_1(\cdot)$ and $u_2(\cdot)$ and now on denote them by u_1, u_2 respectively. For the trajectory starting from $x \in B(x_0, r_0)$, evolving with the fixed controls u_1, u_2,

$$\begin{aligned} d(X(s)) - d(x) &= \int_0^s \nabla d(x) \cdot f(x, u_1, u_2) d\tau + \int_0^s (\nabla d(X(\tau)) - \nabla d(x)) \cdot f(X(\tau), u_1, u_2)\, d\tau \\ &+ \int_0^s \nabla d(x) \cdot (f(X(\tau), u_1, u_2) - f(x, u_1, u_2)) d\tau \end{aligned}$$

Let c be the bound on ∇d on $B(\partial A, r_0)$. If s is small so that $X(\tau)$ is in the r_0 neighborhood of ∂A, then ∇d is continuous. So is f. Also using (3) and (2), we have,

$$d(X(s)) - d(x) \leq -\xi_0 s + o(Fs) + o(cLs) < -\xi_0 s/2$$

for $0 < s < \bar{s}$, where $\bar{s}$ is dependent only on modulus of continuity of f and ∇d and independent of x. Choose $\delta = \min\{r_0, \bar{s}\xi_0/2\}$. If x is in δ ball around x_0 then, $d(x) < \bar{s}/\xi_0 2$

and, choosing $s_x = 2d(x)/\xi_0$, will imply $s_x < \bar{s}$ and hence $d(X(s_x)) < 0$. Thus by our definition of d, $X(s_x) \in \overset{\circ}{A}$ which implies $t(x, u_1, u_2) < s_x = 2d(x)/\xi_0$. Then for $\tilde{C} = 2/\xi_0$ we have, $t(x, u_1, u_2) < \tilde{C} d(x) \quad \forall\, x \in B(x_0, \delta) \setminus \overset{\circ}{A}$.

Step 2: Estimate for any two points in Ω. Using estimate (6),

$$|X_{\bar{x}}(t, u_1, u_2) - X_x(t, u_1, u_2)| \leq |\bar{x} - x| \exp(Lt) \tag{8}$$

Let us denote $t(x, u_1, u_2)$ by τ. Then, $X_x(\tau) \in \partial A$. Define $\delta_1 = \delta \exp(-L\tau)$ where δ is as in Step 1. Let us choose x such that $|x - \bar{x}| < \delta_1$. Then,

$$|X_{\bar{x}}(\tau, u_1, u_2) - X_x(\tau, u_1, u_2)| < \delta.$$

Hence, $X_{\bar{x}}(\tau, u_1, u_2) \in B(X_x(\tau), \delta) \setminus \overset{\circ}{A}$. Therefore by Step 1,

$$t(X_{\bar{x}}(\tau, u_1, u_2), u_1, u_2) < \tilde{C} d(X_{\bar{x}}(\tau, u_1, u_2), u_1, u_2); \tag{9}$$

$$\begin{aligned} t(\bar{x}, u_1, u_2) &\leq \tau + \inf\{\, t \mid X_{X_{\bar{x}}(\tau, u_1, u_2)}(t, u_1, u_2) \in \partial A\} \\ &\leq \tau + t(X_{\bar{x}}(\tau, u_1, u_2), u_1, u_2). \end{aligned}$$

Using (9) we have,

$$t(\bar{x}, u_1, u_2) - \tau \leq t(X_{\bar{x}}(\tau, u_1, u_2), u_1, u_2) \leq \tilde{C}|x - \bar{x}| \exp(L\tau). \tag{10}$$

Interchanging the roles of x and $\bar{x}$, $\tau - t(\bar{x}, u_1, u_2) \leq \tilde{C}|x - \bar{x}| \exp(Lt(\bar{x}, u_1, u_2))$.

Thus we have,

$$|t(x, u_1, u_2) - t(\bar{x}, u_1, u_2)| \leq \tilde{C}\ |x - \bar{x}|\ \exp(L\{t(\bar{x}, u_1, u_2) \vee t(x, u_1, u_2)\}) \tag{11}$$

□

Theorem 3.1 Continuity of the Lower and Upper Value Functions: Assume (A1)-(A5) and (C1), (C2). Then lower and upper value functions V^-, V^+ of hybrid game problem are bounded and locally Hölder continuous.

Proof: We prove the theorem only for the lower value function as the proof for the upper value function is completely analogous.

First we show that lower value function is bounded. Consider the J given by (4). Assuming that P2 chooses not to do any controlled jumps, by our assumptions (C1) and (C2),

$$\begin{aligned} |J(x, u_1(\cdot), u_2(\cdot), v_1, v_2, \xi_i, X(\xi_i)')| &\leq k_0 \int_0^{\infty} \exp(-\lambda t)\, dt + \sum_{i=1}^{+\infty} C_0 \exp(-\lambda\sigma_i) \\ &\leq k_0/\lambda + C_0 \sum_{i=1}^{\infty} \exp(-\lambda\sigma_i) \end{aligned}$$

From (A4), recalling that, $\beta > d(A_i, D_i)$, $\sigma_{i+1} \geq \sigma_i + \beta/\sup|f| \geq \sigma_i + \beta/F$. Hence,

$$\sum_{i=1}^{\infty} \exp(-\lambda\sigma_i) \leq \exp(-\lambda\sigma_1)(1/1 - \exp(-\lambda\beta/F) \tag{12}$$

$$\Rightarrow |J(x, u_1(\cdot), u_2(\cdot), v_1, v_2, \xi_i, X(\xi_i)')| \leq k_0/\lambda + C_0 \exp(-\lambda\sigma_1)(1/1 - \exp(-\lambda\beta/F)).$$

Since this is true for all the controls, we can conclude that $V^-(x)$ is bounded.

We now show that V^- is locally Hölder continuous. Let $x, z \in \Omega$. We denote by Π_i the projection on the i th variable of the any strategy $\alpha \in \Gamma$. Given $\varepsilon > 0$, there exists $\alpha_0 \in \Gamma$, depending on ε such that

$$\begin{aligned} V^-(z) &\geq \sup_{u_2, v_2} J(z, \alpha_0(u_2, v_2)(\cdot), u_2(\cdot), v_2) - \varepsilon \\ &\geq \int_0^\infty k(X_z(t), (\Pi_1 \circ \alpha_0)(u_2, v_2)(t), u_2(t)) \exp(-\lambda t) dt \;\; + \\ & \sum_{i=1}^\infty c_a \left(X_z(\Sigma_i), (\Pi_2 \circ \alpha_0)(u_2, v_2), v_2\right) \exp(-\lambda \Sigma_i) - \varepsilon. \end{aligned}$$

By the definition of $V^-(x)$, for every α, and hence in particular for α_0,

$$V^-(x) \leq \sup_{u_2, v_2} J(x, \alpha_0(u_2, v_2)(\cdot), u_2(\cdot), v_2)$$

and there exists $\bar{u}_2, \bar{v}_2$ such that, $V^-(x) \leq J(x, \alpha_0(\bar{u}_2, \bar{v}_2)(\cdot), \bar{u}_2(\cdot), \bar{v}_2(\cdot)) + \varepsilon$. Thus,

$$\begin{aligned} V^-(x) - V^-(z) &\leq J(x, \alpha_0(\bar{u}_2, \bar{v}_2)(\cdot), \bar{u}_2(\cdot), \bar{v}_2) - J(z, \alpha_0(\bar{u}_2, \bar{v}_2)(\cdot), \bar{u}_2(\cdot), \bar{v}_2) + 2\varepsilon \\ &\leq \int_0^\infty I dt + \sum_{i=1}^\infty S \; + \; 2\varepsilon \end{aligned}$$

where $I = |k(X_x(t), (\Pi_1 \circ \alpha_0)(\bar{u}_2, \bar{v}_2)(t), \bar{u}_2(t)) - k(X_z(t), (\Pi_1 \circ \alpha_0)(\bar{u}_2, \bar{v}_2)(t), \bar{u}_2(t))| \exp(-\lambda t)$ and $S = |c_a(X_x(\sigma_i), (\Pi_2 \circ \alpha_0)(\sigma_i), \bar{v}_2) - c_a(X_z(\Sigma_i), (\Pi_2 \circ \alpha_0)(\Sigma_i), \bar{v}_2)| \exp(-\lambda(\sigma_i \vee \Sigma_i))$.

Here we have denoted by $\{\Sigma_i\}$, the hitting times of A, for trajectory starting from z and by $\{\sigma_i\}$ hitting times of A, for trajectory starting from x evolving with the controls,

$\alpha_0(\bar{u}_2, \bar{v}_2), \bar{u}_2(\cdot), \bar{v}_2$. We split the integral and summation as follows:

$$V^-(x) - V^-(z) \leq \int_0^T I dt + \sum_{i=1}^{n} S + \int_T^\infty I dt + \sum_{i=n+1}^{\infty} S + 2\varepsilon \tag{13}$$

where n and T both are to be chosen suitably large so that the tail end of the integral and summation become small and $\Sigma_n < T < \sigma_{n+1}$. By using the bound k_0 on k given by (C1) we get,

$$\int_T^\infty I dt \leq 2k_0/\lambda \exp(-\lambda T) \tag{14}$$

and by using bound C_0 on c_a given by (C2) and doing calculations on similar lines of (12) we get the estimate,

$$\sum_{i=n+1}^{\infty} S \leq 2C_0 \, (\exp(-\lambda\beta/F))^n (1/\{1 - \exp(-\lambda\beta/F)\}) \tag{15}$$

Without loss of generality let $\sigma_1 < \Sigma_1$. We will show that for a T large, chosen suitably, there exists $\bar{\delta} > 0$ such that if $|x - z| < \bar{\delta}$ then the sequence of σ_i and Σ_i is such that,

$$0 < \sigma_1 \leq \Sigma_1 < \sigma_2 \leq \Sigma_2 \leq \ldots \leq \sigma_n \leq T \leq \Sigma_n. \tag{16}$$

where n is the number of hitting times of A in the interval $[0, T]$. Assuming the above claim we further split the integral as follows:

$$\int_0^T I dt \leq \int_0^{\sigma_1} I dt + \int_{\sigma_1}^{\Sigma_1} I dt + \ldots + \int_{\sigma_n}^{\Sigma_n} I \, dt. \tag{17}$$

In order to evaluate these integrals, we need to find some estimates on successive hitting times of trajectories starting from x and z. We state these estimates in following lemmas which will be proved after the proof of the theorem is complete. For simplicity here onwards, we suppress the fixed control variables, $\bar{u}_2(\cdot), \bar{v}_2$ and $\alpha_0(\bar{u}_2, \bar{v}_2)(\cdot)$ for X and k.

Lemma 3.1: Let σ_1 and Σ_1 be as in the theorem above, and let,

$$x_1 = X_x(\sigma_1) \qquad z_1 = X_z(\Sigma_1); \qquad x_1, z_1 \in \partial A$$

If $|x-z| < \delta_1$ where δ_1 is as in proposition (3.1), then there exists $\tilde{C}$, a constant depending on ξ_0 such that:

$$|\sigma_1 - \Sigma_1| \leq \tilde{C}\ \exp(L(\Sigma_1 \vee \sigma_1))|x-z|; \quad |x_1 \ - z_1| \leq (1 + F\tilde{C})\ \exp(L(\Sigma_1 \vee \sigma_1))|x-z|.$$

Let the jump destination points of x_1 and z_1 be $x_1{}' = g(x_1, (\Pi_2 \circ \alpha_0)(\sigma_1), \bar{v}_2)$ and $z_1{}' = g(z_1, (\Pi_2 \circ \alpha_0)(\Sigma_1), \bar{v}_2)$ in $\Omega_2 \subseteq I\!\!R^{d_2}$, and let the evolution of trajectories take place in Ω_2 till the next hitting time. Let σ_i and Σ_i be the *ith* hitting times of trajectories starting from x and z respectively. We denote by x_i, z_i and x_i', z_i' the following:

$$\begin{aligned} x_i &= X_{x_{i-1}}(\sigma_i - \sigma_{i-1}); \qquad x_i' = g(x_i, (\Pi_2 \circ \alpha_0)(\sigma_i), \bar{v}_2) \\ z_i &= X_{z_{i-1}}(\Sigma_i - \Sigma_{i-1}); \qquad z_i' = g(z_i, (\Pi_2 \circ \alpha_0)(\Sigma_i), \bar{v}_2) \end{aligned}$$

Lemma 3.2: Assume (16) and let us denote $F\tilde{C} + G(1 + F\tilde{C})$ by μ. Then

$$|\sigma_i - \Sigma_i| \le \tilde{C}\exp(L\Sigma_i)\mu^{i-1}|x - z|; \quad |x_i - z_i| \le \exp(L\Sigma_i)(F\tilde{C} + 1)\mu^{i-1}|x - z|$$

whenever $|x - z| < \delta_i$ where $\delta_i := \min\{\delta_1, \delta_2, ..., \delta_1\exp(-L\Sigma_i)/\mu^{i-1}\}$.

Using these lemmas now we estimate the integral in (17). This involves 2 types of integrals:

$$1. \int_{\sigma_i}^{\Sigma_i} I\ dt\ ; \quad 2. \int_{\Sigma_i}^{\sigma_{i+1}} I dt.$$

If $|x - z| < \delta_n$ where $\delta_n = \min\{\delta_1, \delta_2, ..., \delta_1 \exp(-L\Sigma_n)/\mu^{n-1}\}$, we can estimate the above integrals using lemma (3.1) and lemma (3.2). We use the bound on k to evaluate the first integral.

$$\int_{\sigma_i}^{\Sigma_i} I\ dt \le (2k_0/\lambda)(\ \exp(-\lambda\sigma_i) -\ \exp(-\lambda\Sigma_i)) \le 2k_0\ |\sigma_i - \Sigma_i|$$

Using lemma (3.2),

$$\int_{\sigma_i}^{\Sigma_i} I\ dt \le 2k_0\tilde{C}\mu^{i-1}\ \exp(L\Sigma_i)|x - z| \tag{18}$$

To evaluate the second integral we use Lipschitz continuity of k. Hence,

$$\int_{\Sigma_i}^{\sigma_{i+1}} I\ dt \le k_1 \int_{\Sigma_i}^{\sigma_{i+1}} |X_{x_i'}(t - \sigma_i) - X_{z_i'}(t - \Sigma_i)|\ \exp(-\lambda t)dt \tag{19}$$

By semi group property, and (6),

$$\begin{aligned} |X_{x_i'}(t - \sigma_i) - X_{z_i'}(t - \Sigma_i)| &= |X_{X_{x_i'}(\Sigma_i - \sigma_i)}(t - \Sigma_i) - X_{z_i'}(t - \Sigma_i)| \\ &\le \exp(L(t - \Sigma_i))|X_{x_i'}(\Sigma_i - \sigma_i) - z_i'| \end{aligned}$$

By similar calculations, we get,

$$|X_{x_i'}(\Sigma_i - \sigma_i) - z_i'| \leq \mu^i \exp(L\Sigma_i)|x - z| \tag{20}$$

Substituting above estimates in (19) we get,

$$\int_{\Sigma_i}^{\sigma_{i+1}} I dt \leq k_1 \exp(-L\Sigma_i)\mu^i \exp(L\Sigma_i)|x-z| \int_{\Sigma_i}^{\sigma_{i+1}} \exp((L-\lambda)t)dt$$

For $L \neq \lambda$,

$$\int_{\Sigma_i}^{\sigma_{i+1}} I dt \quad \leq \quad k_1\mu^i|x-z| \quad \{ \exp((L-\lambda)T-1)\}/L-\lambda \tag{21}$$

and for $L = \lambda$

$$\int_{\Sigma_i}^{\sigma_{i+1}} I \ dt \quad \leq \quad k_1\mu^i|x-z| \ |\sigma_{i+1} - \Sigma_i| \leq k_1\mu^i \ |x-z| \ \ 2T. \tag{22}$$

For $L \neq \lambda$, by using (18), (21), $\int_0^T I \ dt$ becomes,

$$\int_0^T I dt \leq \sum_{i=1}^{n} 2k_0\tilde{C}\mu^{i-1} \exp(LT)|x-z| + \sum_{i=1}^{n}(k_1\mu^i/L-\lambda)\{ \exp((L-\lambda)T-1)\}|x-z|$$

hence, for $L \neq \lambda$

$$\int_0^T I \ dt \quad \leq \quad \left[\mu^{n-1}/\mu - 1\right] |x-z| \left[2k_0\tilde{C} + k_1 \exp((L-\lambda)T-1)/L-\lambda\right] \tag{23}$$

and for $L = \lambda$, using (18) and (22),

$$\int_0^T I \ dt \quad \leq \quad \sum_{i=1}^{n} 2k_0\tilde{C}\mu^{i-1}|x-z| + \sum_{i=1}^{n} k_1 T\mu^i|x-z|$$

thus, for $L = \lambda$

$$\int_0^T I\ dt \quad \leq \quad [\mu^n - 1/\mu - 1]\,|x - z|\,\left[2k_0\tilde{C} + 2k_1 T\right] \tag{24}$$

Furthermore we need to estimate the summation term arising from discrete cost, in the estimate (13). By using (C2) and lemma (3.2) we get,

$$\sum_{i=1}^{n} S \leq \sum_{i=1}^{n} 2C_1|x_i - z_i|\ \exp(-\lambda(\sigma_i \vee \Sigma_i)) \quad \leq 2C_1 \sum_{i=1}^{n} (F\tilde{C} + 1)\ \exp(LT)\mu^{i-1}|x - z|$$

Hence,

$$\sum_{i=1}^{n} S \leq 2C_1(F\tilde{C} + 1)\ \exp(LT)|x - z|\ \left(\mu^{n-1} - 1/\mu - 1\right) \tag{25}$$

Since μ is a constant, without loss of generality we can assume, $\mu^n/\mu - 1 < 2\mu^n$. Also observe that, $\sigma_i - \sigma_{i+1} \geq \beta/F$ implies that $T \geq \sigma_{n+1} - \sigma_1 \geq n\beta/F$ and hence

$$n < TF/\beta. \tag{26}$$

Using (23), (25), (26), (14), and (15) in (13), for $L \neq \lambda$ we have,

$$\begin{aligned} V^-(x) - V^-(z) \quad \leq \quad & 4k_0\tilde{C}e^{LT}\mu^{TF/\beta}\ |x - z|\ +\ 2k_1\mu^{TF/\beta}\ \exp((L-\lambda)T - 1)/L - \lambda|x - z| \\ & +\quad 2k_0/\lambda\ \exp{-\lambda T} \quad + \quad 2C_1\ \exp(LT)\mu^{TF/\beta}\ |x - z| \\ & +\quad 2C_0(\ \exp(-\lambda\beta/F))^{TF/\beta}\ 1/1 -\ \exp(-\lambda\beta/F) + 2\varepsilon \end{aligned} \tag{27}$$

Now we further restrict $|x - z| < (\delta_1)^{1/1-\theta}$ for some θ such that $0 < \theta < 1$, where δ_1 is as in proposition (3.1). Then choosing T such that $\mu^{TF/\beta}\exp(LT) = |x-z|^{-\theta}$, or

$$T = -\theta \log |x-z|/\lambda + F \log \mu/\beta. \tag{28}$$

This together with the choice of $|x-z|$ implies:

$$\delta_n = \delta_1/\exp(L\Sigma_n)\mu^{n-1} > \delta_1/\exp(LT)\mu^{TF/\beta} \ = \ \delta_1|x-z|^\theta \ > \ |x-z|$$

Thus $|x-z| < \delta_n$ and hence estimate (27) holds true for our choice of T. Then substituting the value of T in this estimate, for $L \neq \lambda$ we get,

$$\begin{aligned} V^-(x) - V^-(z) \ &\leq \ 4k_0\tilde{C}|x-z|^{1-\theta} + (k_1/L-\lambda)|x-z|^{1-\theta} \ + \ C_1|x-z|^{1-\theta} \\ &+ \ (2k_0/\lambda)|x-z|^{\lambda\theta/(F\log\mu/\beta)+L} \ + \ 2C_0|x-z|^{\lambda\theta/(F\log\mu/\beta)+L} \ + \ 2\varepsilon \end{aligned}$$

Thus V^- is Hölder continuous in a $\delta_1^{1/1-\theta}$ ball around x with Hölder constant θ_1,

$$\theta_1 = \min\{1-\theta,\ \lambda\theta/\{(F\log\mu/\beta)+L\}\} \quad \text{for } 0<\theta<1.$$

Similar calculations for $L=\lambda$, yield V^- is locally Hölder continuous in the $\delta_1^{1/1-\theta}$ ball around x with Hölder constant θ_1 for all θ_1 such that,

$$\theta_1 \ < \ \min\{1-\theta,\ L\theta/\{(F\log\mu/\beta)+L\}\} \quad \text{for } \ 0<\theta<1$$

This proves the local Hölder continuity of V^-.

Now we want to justify our claim in (16). ie. if $\sigma_1 < \Sigma_1$ we can choose $|x-z|$ small enough such that (16) holds. If we restrict $|x-z|$ such that $|x-z| \le \min\{\beta/(4F\tilde{C}), (\beta/(4\tilde{C}F))^{1/1-\theta}\}$, then by Lemma 3.2,

$$|\Sigma_i - \sigma_i| \le \tilde{C}\exp(LT)\mu^{TF/\beta}|x-z|.$$

By our choice of T, $|\Sigma_i - \sigma_i| \le \tilde{C}|x-z|^{1-\theta} \le \beta/(4F) < 1/2|\sigma_i - \sigma_{i+1}|$. This together with the assumption $\sigma_1 < \Sigma_1$, implies $\sigma_i < \Sigma_i < \sigma_{i+1} \quad \forall\, i$. So our claim is justified. This completes the proof of continuity of V^-. □

Proof of lemma 3.1: Note here that by proposition (3.1) we have the estimate on $|\sigma_1 - \Sigma_1|$ given by (11),

$$|\sigma_1 - \Sigma_1| < \tilde{C}\ \exp(L(\Sigma_1 \vee \sigma_1))\ |x-z| \tag{29}$$

Using this, we estimate $|x_1 - z_1|$. By assumption, $\sigma_1 < \Sigma_1$. Thus,

$$|x_1 - z_1| = |X_x(\sigma_1) - X_z(\Sigma_1)| \le |X_x(\sigma_1) - X_z(\sigma_1)| + |X_z(\sigma_1) - X_z(\Sigma_1)|$$

Using (6) we get, $|X_x(\sigma_1) - X_z(\sigma_1)| < \exp(L\sigma_1)|x-z|$ while (6) and (29) lead to, $|X_z(\sigma_1) - X_z(\Sigma_1)| \le F|\sigma_1 - \Sigma_1| \le F\tilde{C}\ \exp(L\Sigma_1)|x-z|$. Combining these estimates, $|x_1 - z_1| \le \exp(L\Sigma_1)|x-z|(1+F\tilde{C}) \quad \text{for} \quad z \in B(x, \delta_1)$. □

Proof of lemma 3.2: We assume without loss of generality that, $x_i', z_i' \in \Omega_{i+1} \subseteq I\!R^{d_{i+1}}$.

We apply Proposition (3.1) and the above lemma recursively to find estimates on successive hitting times and points to conclude the estimate in the lemma. □

4 Quasivariational Inequality

Following theorem deals with the dynamic programming principle satisfied by V^-. DPP for V^+ is analogous.

Theorem 4.1: Dynamic Programming Principle: DPP^- for lower value function V^-: for any $T > 0$,

$$V^-(x) = \inf_{\alpha\in\Gamma} \sup_{u_2(\cdot),v_2} \left\{ \int_0^T k(X_x(t), \Pi_1 \circ \alpha(u_2, v_2)(t), u_2(t)) \exp(-\lambda t) dt \right.$$

$$\left. + \sum_{\sigma_i < T} S_a(\alpha) + \sum_{\Pi_3\circ\alpha(u_2,v_2)<T} S_c(\alpha) + \exp(-\lambda T) V^-(X_x(T, \Pi_1 \circ \alpha(u_2, v_2), u_2)) \right\}$$

where for fixed controls $u_2(\cdot), v_2$, $S_a(\alpha) = \exp(-\lambda\sigma_i) c_a(X(\sigma_i), \Pi_2 \circ \alpha(u_2, v_2)(\sigma_i), v_2)$ and $S_c(\alpha) = \exp(-\lambda \Pi_3\circ\alpha(u_2, v_2)) c_c(X(\Pi_3\circ\alpha(u_2, v_2)), \Pi_4\circ\alpha(u_2, v_2))$.

If σ_1 is the first hitting time of A then $DPPA^-$ for V^- is given by:,

$$V^-(x) = \inf_{\alpha\in\Gamma} \sup_{u_2(\cdot),v_2} \left\{ \int_0^{\sigma_1} k(X_x(t, \Pi_1 \circ \alpha(u_2, v_2)(t), u_2(t)) \exp(-\lambda t) dt \right.$$

$$\left. + \exp(-\lambda\sigma_1) M^- V^-(X_x(\sigma_1, \Pi_1 \circ \alpha_0(u_2, v_2), u_2) \right\}$$

where $M^-\phi(x) = \inf_{v_1} \sup_{v_2} \{\phi(g(x, v_1, v_2)) + c_a(x, v_1, v_2)\}$.

If ξ_1 is the first hitting time of C then $DPPC^-$ is given by:,

$$V^-(x) = \inf_{\alpha\in\Gamma} \sup_{u_2(\cdot),v_2} \left\{ \int_0^{t_1} k(X_x(t, \Pi_1 \circ \alpha(u_2, v_2)(t), u_2(t)) \exp(-\lambda t) dt \right.$$

$$\left. + \exp(-\lambda t_1) N V^-(X_x(\xi_1, \Pi_1 \circ \alpha(u_2, v_2), u_2)) \right\}$$

where $N\phi(x) = \inf_{x' \in D} \{\phi(x') + c_c(x, x')\}$.

Proof: We prove the general dynamic programming principle for any $T > 0$ given by (DPP$^-$). The proofs of (DPPA$^-$) and (DPPC$^-$) then follow on similar lines. Let us denote RHS of (DPP$^-$) by $w(x)$. Fix $\varepsilon > 0$. Note that by definition of V^-, for any z, there exists $\alpha_z \in \Gamma$ such that, for all $u_2(\cdot), v_2$,

$$V^-(z) \geq J(z, \alpha_z(u_2, v_2)(\cdot), u_2(\cdot), v_2) - \varepsilon \tag{30}$$

We first prove that $V^-(x) \leq w(x)$. Choose $\bar{\alpha} \in \Gamma$ such that,

$$w(x) \geq \sup_{u_2(\cdot),v_2} \left\{ \int_0^T k(X_x(t), \Pi_1 \circ \bar{\alpha}(u_2, v_2)(t), u_2(t)) \exp(-\lambda t) dt + \sum_{\sigma_i < T} S_a(\bar{\alpha}) \right.$$

$$\left. + \sum_{\xi_i < T} S_c(\bar{\alpha}) + \exp(-\lambda T) V^-(X_x(T), \bar{\alpha}(u_2, v_2), u_2, v_2)) \right\} - \varepsilon \tag{31}$$

Now define $\delta \in \Gamma$ by,

$$\delta(u_2, v_2)(s) = \bar{\alpha}(u_2, v_2)(s) \text{ if } s \leq T; \qquad \delta(u_2, v_2)(s) = \alpha_z(u_2, v_2)(s - T) \text{ if } s > T$$

with $z = X_x(T, \bar{\alpha}(u_2, v_2), u_2, v_2)$, and α_z as chosen in (30),

$$X_x(s + T, \delta(u_2, v_2), u_2, v_2) = X_z(s, \alpha_z(u_2, v_2)(s + T), u_2(s + T), v_2).$$

So by change of variables, $\tau = s + T$ we have,

$$J(z, \alpha_z(u_2, v_2)(\cdot + T), u_2(\cdot + T), v_2) =$$

$$\{\int_T^\infty k(X_x(\tau), \Pi_1 \circ \delta(u_2, v_2)(\tau), u_2(\tau)) \exp(-\lambda(T-\tau))d\tau + \sum_{\sigma_i > T} S_a(\delta) + \sum_{\xi_i > T} S_c(\delta)\}$$

Substituting above in (30), we get,

$$V^-(z) \geq \left\{\int_T^\infty k(X_x(\tau), \Pi_1 \circ \delta(u_2, v_2)(\tau), u_2(\tau)) \exp(-\lambda(T-\tau))d\tau \right.$$
$$\left. + \sum_{\sigma_i > T} S_a(\delta) + \sum_{\xi_i > T} S_c(\delta)\right\} - \varepsilon$$

Substituting back in (31),

$$\begin{aligned} w(x) \quad &\geq \quad \sup_{u_2(\cdot), v_2} \left\{\int_0^\infty k(X_x(t), \Pi_1 \circ \delta(u_2, v_2)(t), u_2(t)) \exp(-\lambda t)dt \right. \\ &\quad \left. + \sum_{i=1}^\infty S_a(\delta) + \sum_{i=1}^\infty S_c(\delta)\right\} - 2\varepsilon \\ &\geq \quad J(x, \delta(u_2, v_2)(\cdot), u_2(\cdot), v_2) - 2\varepsilon \end{aligned}$$

Hence, $w(x) \geq V^-(x) - 2\varepsilon$. Since ε is arbitrary, $w(x) \geq V^-(x)$.

To prove other way inequality, ie. $w(x) \leq V^-(x)$, we choose $\bar{u}_2(\cdot), \bar{v}_2$ such that,

$$\begin{aligned} w(x) \leq \quad & \int_0^T k(X_x(t), \Pi_1 \circ \alpha_x(\bar{u}_2, \bar{v}_2)(t), \bar{u}_2(t)) \exp(-\lambda t)dt \qquad (32) \\ & + \sum_{\sigma_i < T} S_a(\alpha_x) + \sum_{\xi_i < T} S_c(\alpha_x) \; + \; \exp(-\lambda T) V^-(X_x(T, \alpha_x(\bar{u}_2, \bar{v}_2), \bar{u}_2, \bar{v}_2)) + \varepsilon \end{aligned}$$

where α_x is as in (30) and S_a and S_c are now with respect to fixed controls $\bar{u}_2, \bar{v}_2$. Now for each $u_2(\cdot), v_2$, define $\tilde{u}_2, \tilde{v}_2$ by,

$$(\tilde{u}_2(s), \tilde{v}_2) = (\bar{u}_2(s), \bar{v}_2) \text{ if } s \leq T; \qquad (\tilde{u}_2(s), \tilde{v}_2) = (u_2(s-T), v_2) \text{ if } s > T \tag{33}$$

then, define $\hat{\alpha} \in \Gamma$ and z by,

$$\hat{\alpha}(u_2(s), v_2) = \alpha_x(\tilde{u}_2, \tilde{v}_2) \tag{34}$$

$$z = X_x(T, \alpha_x(\bar{u}_2, \bar{v}_2), \bar{u}_2, \bar{v}_2 \tag{35}$$

and choose $u_2(\cdot), v_2$ such that,

$$V^-(z) \leq J(z, \hat{\alpha}(u_2, v_2)(\cdot), u_2(\cdot), v_2) + \varepsilon \tag{36}$$

So by using (32), (35) and (36) we get,

$$\begin{aligned} w(x) \ &\leq \ 2\varepsilon + \int_0^T k(X_x(t), \Pi_1 \circ \alpha_x(\bar{u}_2, \bar{v}_2)(t), \bar{u}_2(t)) \exp(-\lambda t) dt \\ &+ \ \sum_{\sigma_i < T} S_a(\alpha_x) + \sum_{\xi_i < T} S_c(\alpha_x) \end{aligned} \tag{37}$$

Observe that,(33), (34), (35) implies $X_x(\tau, \alpha_x(\tilde{u}_2, \tilde{v}_2), \tilde{u}_2, \tilde{v}_2) = X_z(\tau - T, \hat{\alpha}(u_2, v_2), u_2, v_2)$ if $\tau > T$. So doing the change of variables $\tau = s + T$,

$$\begin{aligned} J(z, \hat{\alpha}(u_2, v_2)(\cdot), u_2(\cdot), v_2) \ &= \ \int_T^\infty k(X_x(\tau, \alpha_x(\tilde{u}_2, \tilde{v}_2)(\tau), \tilde{u}_2(\tau), \tilde{v}_2) \exp(T - \tau) d\tau \\ &+ \ \sum_{\sigma_i > T} S_a(\alpha_x) + \sum_{\xi_i > T} S_c(\alpha_x) \end{aligned} \tag{38}$$

Using (33), (34) and (35) in (38) implies $w(x) \leq J(x, \alpha_x(\tilde{u}_2, \tilde{v}_2)(\cdot), \tilde{u}_2(\cdot), \tilde{v}_2) + 2\varepsilon$. Combining this and (30) we have, $w(x) \leq V^-(x) + 3\varepsilon$ which proves the other way inequality. □

Theorem 4.2: Quasivariational Inequality: Under the assumptions $(A1-A5)$ and $(C1),(C2)$, the value function V^- satisfies the following lower quasivariational inequality in the viscosity sense:

$$V^-(x) = \begin{cases} M^-V^-(x) & \forall\, x \in A \\ \max\{NV^-(x), -H^-(x, \nabla V^-(x))\} & \forall\, x \in C \\ -H^-(x, \nabla V^-(x)) & \forall\, x \in \Omega \setminus (A \cup C) \end{cases} \qquad (QVI^-)$$

where $H^-(x,p) = \inf_{u_1} \sup_{u_2} \{-k(x,u_1,u_2) - f(x,u_1,u_2)\cdot p\}/\lambda$

Proof: Let $x \in A$. Then the first hitting time of trajectory is $\sigma_1 = 0$. Hence by DPPA$^-$ we get $V^-(x) = M^-V^-(x)$.

Now we consider the case $x \in \Omega \setminus A \cup C$. Let, $r = \min\{d(x,\partial A), d(x,\partial C)\}$. Choose $R < r$. Then in the ball $B(x,R)$ no impulses are applied. Now V^- is continuous at x, and let $\phi \in C^1(\Omega)$ be such that $V^- - \phi$ has local maximum at x and $V^-(x) = \phi(x)$. Then, to show that V^- is a subsolution we must prove, $\phi(x) + H^-(x, \nabla\phi(x)) \leq 0$. Should this fail, there would exist some $\nu > 0$ such that,

$$\phi(x) + H^-(x, \nabla\phi(x)) \geq \nu > 0 \qquad (39)$$

Then the lemma similar to the one in Ref. 4 (Chapter 8, Lemma 1.11) will imply that for sufficiently small $t > 0$, there exists $\alpha^* \in \Gamma$ such that for all $u_2 \in U_2$

$$\int_0^t \{k(X_x(s)) + f(X_x(s)) \cdot \nabla\phi(X_x(s)) - \lambda\phi(x)\}\ \exp(-\lambda s) ds \leq -\nu t/4 \qquad (40)$$

where, $X_x(s) = (X_x(s, \alpha^*(u_2, v_2)(s), u_2(s)), \Pi_1 \circ \alpha^*(u_2, v_2)(s), u_2(s))$. Choose τ small enough such that $\tau < t$ and $X_x(\tau) \in B(x, R)$. By our choice of R and τ, τ is less than first hitting time that is $\tau < \sigma_1$ and $\tau < \xi_1$. Since $V^- - \phi$ has maximum at x and $V^-(x) = \phi(x)$,

$$\exp(-\lambda\tau)\phi(X_x(\tau)) - \phi(x) \geq \exp(-\lambda\tau)V^-(X_x(\tau)) - V^-(x) \tag{41}$$

Substituting (41) in (40), we get,

$$\inf_{\alpha\in\Gamma} \sup_{u_2(\cdot), v_2} \left\{ \int_0^\tau k(X_x(s)) + \exp(-\alpha\tau)V^-(X_x(\tau, \alpha^*(u_2, v_2), u_2) \right\} - V^-(x) \leq -\nu\tau/4 < 0$$

This contradicts DPP$^-$. Hence V^- is a viscosity subsolution of HJB equation.

To show that V^- is a viscosity supersolution, let $V^- - \phi$ has local minimum at x and $V^-(x) = \phi(x)$. Assume by contradiction, $\phi(x) + H^-(x, \nabla\phi(x)) = -\nu < 0$. By definition of H^-, there exists, $u_2^* \in U_2$ such that

$$\phi(x) - f(x, u_1, u_2^*).\nabla\phi(x) - k(x, u_1, u_2^*) \leq -\nu$$

for all $u_1 \in U_1$. So for τ small enough that is $\tau < t$ and any $\alpha \in \Gamma$,

$$\phi(X_x(s)) - f(X_x(s)) \cdot \nabla\phi(X_x(s)) - k(X_x(s)) \leq -\nu/2$$

where, $X_x(s) = (X_x(s), \Pi_1 \circ \alpha(u_2^*, v_2)(s), u_2^*(s))$. For $0 \leq s \leq \tau$, multiplying by $\exp(-\lambda s)$ and integrating from 0 to τ, for τ small,

$$\left\{ \phi(x) - \exp(-\lambda\tau)\phi(X_x(\tau)) - \int_0^\tau k(X_x(s))\exp(-\lambda s)ds \right\} \leq -\nu t/4 \tag{42}$$

From, $\phi(x) - \exp(-\lambda\tau)\phi(X_x(\tau)) \geq V^-(x) - \exp(-\lambda\tau)V^-(X_x(\tau))$ we obtain,

$$\exp(-\lambda\tau)V^-(X_x(\tau)) + \int_0^\tau k(X_x(s)) \exp(-\lambda s)ds \ \geq \ \nu\tau/2 + V^-(x).$$

$$\text{Thus,} \ \inf_{\alpha\in\Gamma} \sup_{u_2(\cdot),v_2} \left\{ \int_0^\tau k(X_x(s)) \exp(-\lambda s)ds + \exp(-\lambda\tau)V^-(X_x(\tau)) \right\} > V^-(x)$$

which is a contradiction to DPP$^-$ and this completes the proof that V^- is a viscosity supersolution of HJI equation.

Now consider the case $x \in C$. If $V^-(x) = NV^-(x)$, there is nothing to prove. Suppose $V^-(x) < NV^-(x)$. Then we show that whenever there exists a $r > 0$ and a ball $B(x,r)$ around x such that it is not optimal to apply any impulses on $B(x,r)$. Then we can do the analysis in this ball to conclude as in the case of $x \in \Omega \setminus A \cup C$. We claim that there exists $\varepsilon > 0$ such that,

$$\begin{aligned} V^-(x) &= \inf_{\alpha} \sup_{u_2(\cdot),v_2} \left\{ \int_0^{t_1} k(X_x(t), \Pi_1 \circ \alpha(u_2,v_2)(t), u_2(t)) \exp(-\lambda t)dt \right. \\ &\qquad \left. + NV^-(X_x(t_1)) \mid t_1 > \varepsilon \right\} \end{aligned}$$

where t_1 is first time when controller chooses to jump. If not, then $\varepsilon = 0$. which implies $\xi_1 = 0$ which by DPPC implies $V^-(x) = NV^-(x)$; which is a contradiction to $V^-(x) < NV^-(x)$. Hence $\varepsilon > 0$. Choose $r < \min\{d(x, X_x(\varepsilon)), d(A,C)\}$. Then in the ball $B(x,r)$, no impulses are applied. So we can do the analysis in this ball around x and can conclude as in the earlier case. This proves the quasivariational inequality for the case $x \in C$. □

For upper value function define $M^+\phi(x)$ and $H^+(x,p)$ by:

$$
\begin{aligned}
M^+\phi(x) &= \sup_{v_2} \inf_{v_1} \{\phi(g(x, v_1, v_2)) + c_a(x, v_1, v_2)\} \\
H^+(x,p) &= \sup_{u_2} \inf_{u_1} \{-k(x, u_1, u_2) - f(x, u_1, u_2) \cdot p\} / \lambda
\end{aligned}
$$

The upper value function V^+ satisfies following QVI^+:

$$
V^+(x) = \begin{cases} M^+V^+(x) & \forall\, x \in A \\ \max\{NV^+(x), -H^+(x, \nabla V^+(x))\} & \forall\, x \in C \\ -H^+(x, \nabla V^+(x)) & \forall\, x \in \Omega \setminus (A \cup C) \end{cases} \qquad (QVI^+)
$$

5 Existence of Value

Our aim in this section is to characterize the lower and upper value functions V^- and V^+ as the unique viscosity solutions of corresponding quasivariational inequalities. Further we give a condition for equality to hold between V^- and V^+ and thus the game to have the value.

Theorem 5.1: Uniqueness: Assume $(A1 - A5)$ and $(C1), (C2)$. Let $w_1, w_2 \in BC(\Omega)$ be 2 viscosity solutions of quasivariational inequality given by (QVI^-). Then $w_1 = w_2$.

Proof: The idea of the proof is to show that $w_1(x) \leq w_2(x)\ \forall\, x \in \Omega$. For this we define the following auxiliary function Φ defined on $\bigcup_{i=1}^{\infty}\ (\Omega_i \times \Omega_i)$ by:

$$\Phi(x,y) = \gamma w_1(x) - w_2(y) - \frac{1}{2\epsilon}|x-y|^2 - \kappa(< x >^m + < y >^m) \tag{43}$$

where $m \in (0,1)$, $< x >^m = (1+|x|^2)^{m/2}$ and $\gamma \in (0,1)$, ε and κ are positive parameters to be chosen suitably later on. Let, $\sup_i \sup_{\Omega_i} w_1 - w_2 > 0$. If the above supremum is not attained at some finite point (x_0, y_o) in some fixed Ω_i, we can work with an approximate supremum of Φ over $\Omega \times \Omega$, that is for given $n > 0$, we can find $(x_n, y_n) \in \Omega_n$ such that, $\sup_{\Omega\times\Omega} \Phi(x,y) < \Phi(x_n, y_n) + (1/n)$. The proof in this case is very similar to the one given below, however for simplicity, we give the proof in the case when the supremum of Φ is attained at (x_0, y_0) in some fixed Ω_i.

Since x_0 and y_0 can lie in different sets in Ω_i $w_1(x_0)$ and $w_2(y_0)$ satisfy different equations from QVI^-. We list below the different cases which arise:

Case 1. $(x_0, y_0) \in A \times C$ or $C \times A$

Case 2. $(x_0, y_0) \in \Omega \setminus (A \cup C) \times \Omega \setminus (A \cup C)$

Case 3. $x_0, y_0 \notin A$ and one of x_0 or $y_0 \in C$.

Case 4. $x_0, y_0 \notin C$ and one of the x_0 or $y_0 \in A$

We claim that under suitable choice of ε, γ and κ only Case 2 occurs and no other case can occur. In Case 2, both w_1 and w_2 satisfy HJI and we can complete the proof by slightly modifying usual comparison method. First we list some estimates in the following lemma which can be proved by standard techniques using the facts that w_1, w_2 are bounded, continuous and (x_0, y_0) is maximum point of Φ.

Lemma 5.1: Let Φ be as in (43) and (x_0, y_0) be such that $\Phi(x_0, y_0) = \sup \Phi$. Let $\hat{C}$ be generic constant. Then,

1. $(|x_0 - y_0|^2/\epsilon) \leq \hat{C}$ $\hat{C}$ independent of κ and ε.
2. $\sqrt{\kappa}|x_0|, \sqrt{\kappa}|y_0| \leq \hat{C}$ for some $\hat{C}$ independent of κ and ε.
3. $(|x_0 - y_0|^2/\epsilon) \leq \omega(\sqrt{\hat{C}\epsilon})$ where ω is the local modulus of continuity of w_1 and w_2 in the ball of radius $\tilde{C}/\sqrt{\kappa}$.

Now we consider the different cases listed earlier.

Case 1: $(x_0, y_0) \in A \times C$ or $C \times A$. Without loss of generality let, $(x_0, y_0) \in A \times C$. Since, $d(A, C) > \beta, \Rightarrow$ $|x_0 - y_0| > \beta$. On the other hand, by lemma (5.1)(i), $|x_0 - y_0| < \sqrt{\hat{C}\epsilon}$. So choosing ϵ such that $\sqrt{\hat{C}\epsilon} < \beta/2$, $|x_0 - y_0| < \beta/2$, which is a contradiction. Hence case 1 does not occur, for small ε.

Case 2: Let $(x_0, y_0) \in \Omega \setminus (A \cup C) \times \Omega \setminus (A \cup C)$. Then, using the definition of the viscosity sub and supersolution for w_1 and w_2 respectively, we have,

$$w_1(x_0) - w_2(y_0) \leq H^-(y_0,\ (x_0-y_0)/\varepsilon - \kappa m < y_0 >^{m-2} y_0)$$

$$-H^-(x_0,\ (1/\gamma)((x_0-y_0)/\varepsilon + \kappa m < x_0 >^{m-2} x_0))$$

$$\leq (L/\gamma)(|x_0-y_0|^2/\varepsilon) + (L/\gamma)\kappa m| < x_0 >^{m-2} x_0||x_0-y_0| + k_1|x_0-y_0|$$

$$+F(1-1/\gamma)(|x_0-y_0|/\varepsilon) + F\kappa m|x_0 < x_0 >^{m-2} + y_0 < y_0 >^{m-2}|$$

Here in the second inequality we have used the fact that under our assumptions (A1)-(A5) Hamiltonian H^- satisfies the structural condition:

$$|H^-(x,p) - H^-(y,q) \leq F|p-q| + L|q||x-y| + k_1(|x-y|) \quad \forall\ x,p$$

Now by using the estimates in the lemma 5.1 and the fact that $| < x_0 >^m x_0|$ remains bounded for $m \in (0,1)$ we have that,

$$w_1(x_0) - w_2(y_0) \leq O(\sqrt{\varepsilon}) + \omega(\sqrt{\hat{C}\varepsilon}) + (1-1/\gamma)|x_0-y_0|/\varepsilon + O(\kappa).$$

Choosing γ such that $1-1/\gamma < \varepsilon$, we will get, $(1-1/\gamma)(x_0-y_0)/\varepsilon < O(\sqrt{\varepsilon})$.

Also for given $x \in \Omega$, $\Phi(x,x) \leq \Phi(x_0,y_0)$, so $\gamma w_1(x) - w_2(x) \leq \gamma w_1(x_0) - w_2(y_0) + 2\kappa < x >^m$. Hence for x fixed, $\gamma w_1(x) - w_2(x) \leq O(\sqrt{\varepsilon}) + O(\kappa)$. Now first sending ε and then κ to 0 and observing that this implies γ goes to 1 by our choice of γ, we get that $w_1(x) \leq w_2(x) \quad \forall\ x \in \Omega$.

Case 3: $x_0, y_0 \notin A$, and one of $x_0, y_0 \in C$. Without loss of generality let $y_0 \in C$. Since $x_0 \notin A$ and w_1 is a subsolution of QVI^- implies $w_1(x_0) + H^-(x_0, \nabla w_1(x_0)) \leq 0$.

$$y_0 \in C \Rightarrow \quad \max\{w_2(y_0) + H^-(y_0, \nabla w_2(y_0)), w_2(y_0) - Nw_2(y_0)\} = 0.$$

Also, w_2 is a solution of QVI^-, in particular it is a supersolution. Hence either $w_2 + H^- \geq 0$ or $w_2 - Nw_2 \geq 0$ at y_0. If $w_2(y_0) + H^-(y_0, \nabla w_2(y_0)) \geq 0$ we can proceed as in Case 2. Else assume $w_2(y_0) - Nw_2(y_0) \geq 0$. Since w_2 is also a subsolution, $w_2(x) \leq Nw_2(x) \;\; \forall \;\; x \in C$ and therefore, $w_2(y_0) = Nw_2(y_0) = \inf_{y' \in D} w_2(y') + c_c(y_0, y')$. If infimum in $Nw_2(y_0)$ is not attained, we deal with an approximate minimum at the expense of an extra term $1/n$. For, given $n > 0$ we can choose $y_0' \in D_j$ such that, $Nw_2(y_0) > w_2(y_0') + c_c(y_0, y_0') - (1/n)$. If $y_0' \notin D_i$ and it belongs to some other D_j, we will have to work with an approximate supremum of Φ as indicated earlier. In that case by similar calculations as below, for a suitable choice of γ we get a contradiction. But for simplicity of calculations we work with a $y_0' \in D_i \subseteq \Omega_i$, where infimum is attained in $Nw_2(y_0)$.

Let R be the bound on D, fix κ such that $\kappa < \min\{1, C'/2R\}$ where C' is as in (C2).

Then using the fact that $w_1(y_0) \leq Nw_1(y_0) \leq w_1(y_0{}') + c_c(y_0, y_0{}')$,

$$\begin{aligned}
\Phi(x_0, y_0) &= \gamma w_1(x_0) - w_2(y_0{}') - c_c(y_0, y_0{}') - 1/\varepsilon(|x_0 - y_0|^2) - \kappa(< x_0 >^m + < y_0 >^m) \\
&\leq \gamma w_1(x_0) + \gamma w_1(y_0{}') - \gamma w_1(y_0) - w_2(y_0{}') - (1-\gamma)c_c(y_0, y_0{}') \\
&\quad -1/\varepsilon(|x_0 - y_0|^2) - \kappa(< x_0 >^m + < y_0 >^m) \\
&\leq \Phi(y_0', y_0') + 2\kappa < y_0' >^m + \gamma w_1(x_0) - \gamma w_1(y_0) \\
&\quad -1/\varepsilon(|x_0 - y_0|^2) - \kappa(< x_0 >^m + < y_0 >^m) - (1-\gamma)C' \\
&\leq \Phi(y_0', y_0') + 2\kappa R + \gamma\omega(\hat{C}\sqrt{\varepsilon}) - (1-\gamma)C'
\end{aligned}$$

By choosing γ such that, $\gamma < (C' - 2\kappa R)/(C' + \omega(\hat{C}\sqrt{\varepsilon}))$ we can make $\Phi(x_0, y_0) < \Phi(y_0', y_0')$. This is a contradiction as $\sup \Phi = \Phi(x_0, y_0)$. Hence the case $w_2(y_0) - Nw_2(y_0) \geq 0$ can not occur for this choice of γ, depending on ε and κ and we can proceed as in Case 2 and conclude that $w_1 \leq w_2$.

Case 4: $x_0, y_0 \notin C$, and one of $x_0, y_0 \in A$. Without loss of generality let $y_0 \in A$. $x_0 \notin C$ and w_1 is a subsolution of QVI^- implies $w_1(x_0) + H^-(x_0, \nabla w_1(x_0)) \leq 0$. Since $y_0 \in A \Rightarrow$ $w_2(y_0) - M^- w_2(y_0) = 0$ therefore,

$$w_2(y_0) = M^- w_2(y_0) = \inf_{v_1} \sup_{v_2} \left\{ w_2(g(y_0, v_1, v_2)) + c_a(y_0, v_1, v_2) \right\}.$$

By compactness of V_1 and V_2 above inf sup will be attained at some v_1 and v_2. Observe that the point $g(y_0, v_1, v_2) \in D$ and let R be the bound on D. We fix κ such that $\kappa < \min\{1, C'/2R\}$. Now doing the calculations similar to the one in the Case 3, we can get,

$$\Phi(x_0, y_0) \leq \Phi(g(y_0, v_1, v_2), g(y_0, v_1, v_2)) + 2\kappa R + \gamma w_1(x_0) - \gamma w_1(y_0) - (1-\gamma)C'.$$

By choosing $\gamma < (C' - 2\kappa R)/(2C' + \omega(\hat{C}\sqrt{\varepsilon}))$ we can make $\Phi(x_0, y_0) < \Phi(g(y_0, v_1, v_2), g(y_0, v_1, v_2))$. This is a contradiction. Hence the case $w_2(y_0) - M^- w_2(y_0) \geq 0$ can not occur and we can proceed as in Case 2 and conclude that $w_1 \leq w_2$ for this case also.

Thus $w_1 \leq w_2 \quad \forall \quad x \in \Omega$. By interchanging the roles of w_1 and w_2 we get other way inequality and hence $w_1 = w_2$ and hence the uniqueness. □

Similarly one can show that V^+ is the unique viscosity solution of upper quasivariational inequality QVI^+.

Corollary 5.1: Existence of Value Assume (A1- A5) and (C1), (C2). If H^-, H^+, M^- and M^+ satisfy:

$$H^-(x,p) = H^+(x,p) \quad \forall \ x,\ p \tag{44}$$

$$\text{and} \quad M^- V^-(x) = M^+ V^-(x) \quad \forall \ x \in A \tag{45}$$

then $V^- = V^+$ and hence game has a value.

Proof: By the assumption (44), (45), V^- and V^+ are solutions of the QVI^-. By the above uniqueness theorem V^- is the unique viscosity solution of QVI^- and hence,

$V^-(x) = V^+(x) \quad \forall \;\; x \in \Omega$. Thus game has the value under the condition (44, 45). $\Box$

Remark 5.1: We remark that when, f and k are linear in all its variables , g and c_a are linear in v_1, v_2, the total discounted cost functional J turns out to be linear in u_1, u_2, v_1, v_2. For example we can take $\Omega \subseteq I\!R^d$, U_i compact in $I\!R^d$, $f(x, u_1, u_2) = [{a_1}^T u_1]x + [{a_2}^T u_2]x$, where $a_i \in I\!R^d$, constants, and ${a_i}^T$ denotes its transpose. $k(x, u_1, u_2) = k^1 < u_1, x > + k^2 < u_2, x >$, $k^i \in I\!R$, $g = v_1 g^1(x) + v_2 g^2(x)$ and $c_a = \psi^1 < b_1, v_1 > + \psi^2 < b_2, v_2 >$ where b_1, b_2 are constants and g^1, g^2, ψ^1 and ψ^2 are real valued functions. Then by Von Neumann's minimax theorem, (See Ref. 8, section 2.13, theorem, 2G.) $V^- = V^+$, and $H^-(x, p) = H^+(x, p) \;\; \forall \; x, p$ we can also conclude that, $M^- V^- = M^+ V^-$ and thus the condition (44),(45) holds.

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