Eigenvalue problems with weights in Lorentz spaces

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Abstract Given $V$, $w$ locally integrable functions on a general domain $\Omega$ with $V \geq 0$ but $w$ allowed to change sign, we study the existence of ground states for the nonlinear eigenvalue problem:

$$-\Delta u + Vu = \lambda |u|^{p-2}u, \quad u|_{\partial \Omega} = 0,$$

with $p$ subcritical. These are minimizers of the associated Rayleigh quotient whose existence is ensured under suitable assumptions on the weight $w$. In the present paper we show that an admissible space of weight functions is provided by the closure of smooth functions with compact support in the Lorentz space $L(\tilde{p}, \infty)$ with $\frac{1}{\tilde{p}} + \frac{p}{2p} = 1$. This generalizes previous results and gives new sufficient conditions ensuring existence of extremals for generalized Hardy–Sobolev inequalities. The existence in such a generality of a principal eigenfunction in the linear case $p = 2$ is applied to study the bifurcation for semilinear problems of the type

$$-\Delta u = \lambda (a(x)u + b(x)r(u)).$$

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where $a, b$ are indefinite weights belonging to some Lorentz spaces, and the function $r$ has subcritical growth at infinity.

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1 Introduction

Given $p \geq 2$, $\Omega$ a non-empty open connected set of $\mathbb{R}^N$ and $V, w \in L^1_{\text{loc}}(\Omega)$, we are interested in the existence of “ground states” to the problem

$$-\Delta u + V(x)u = \lambda w(x)|u|^{p-2}u, \quad u|_{\partial \Omega} = 0,$$

for some appropriate value of the parameter $\lambda$. Note that if $p > 2$, the pair $(\lambda, u)$ solves (1.1) for some $\lambda > 0$ if and only if $(1, \lambda^{\frac{1}{p-2}} u)$ is a solution of (1.1).

More precisely consider the “Beppo Levi space” ([10]) defined as

$$D^{1,2}_0(\Omega) := C^\infty_0(\Omega) \|u\| := \left(\int_{\Omega} |\nabla u|^2\right)^{1/2},$$

i.e., the completion of the set of smooth functions having compact support with respect to the norm induced by the inner product $\int_{\Omega} \nabla u \nabla v$. In dimension $N \geq 3$ this Hilbert space embeds in the Lebesgue space $L^2(\Omega)$, whereas on unbounded domains of $\mathbb{R}^1$ or $\mathbb{R}^2$ the space $D^{1,2}_0(\Omega)$ cannot be identified in general with a space of functions (see [10] and also [26]). So henceforth we shall always assume $N \geq 3$, but we stress that up to minor modifications our results do hold on bounded domains when $N = 1, 2$. Now consider in $D^{1,2}_0(\Omega)$ the Rayleigh quotient

$$R(u) := \frac{\int_{\Omega} \{|
abla u|^2 + V u^2\}}{\left(\int_{\Omega} w|u|^p\right)^{2/p}} \quad (p \geq 2)$$

with domain of definition

$$\mathcal{M}_p := \left\{ u \in D^{1,2}_0(\Omega) : V u^2, w|u|^p \in L^1(\Omega) \text{ and } \int_{\Omega} w|u|^p > 0 \right\}.$$

Using these notations, we are more specifically interested in the existence of minimizers of the functional $R$ on $\mathcal{M}_p$ (the “ground states”). Due to the homogeneity of the ratio (1.3) the uniqueness of minimizers up to a scalar multiplication will also be discussed. One checks easily that such minimizer $\Phi$ satisfies (1.1) in the weak sense

$$\int_{\Omega} \{\nabla \Phi \nabla \xi + V \Phi \xi\} = \lambda(\Phi) \int_{\Omega} w|\Phi|^{p-2} \Phi \xi, \quad \forall \xi \in D^{1,2}_0(\Omega).$$

with $\lambda(\Phi) := \frac{\int_{\Omega} \{|
abla \Phi|^2 + V \Phi^2\}}{\int_{\Omega} w|\Phi|^p}$.

When $p = 2$, the minimizers of (1.4) are the “principal eigenfunction” of the problem

$$-\Delta u + V(x)u = \lambda w(x)u, \quad u \in D^{1,2}_0(\Omega) \setminus \{0\}, \quad u \geq 0,$$

with an associated eigenvalue $\lambda$ called “principal eigenvalue”. It arises when one linearizes semilinear equations $-\Delta u = \lambda f(x, u)$, which appear in the mathematical formulation of
phomena in physics, biology etc. In the fields of applications the nonlinearity may be discontinuous with indefinite sign in order to model strong attraction or repulsion reactions in the physical system. Linearization is an important tool to study bifurcation, uniqueness, stability and many other qualitative properties of semilinear elliptic problems, and in this setting the principal eigenvalue plays a particular role. Further, when such nonlinear problems admit blow up solutions (see for example [5]), the linearization around these solutions lead to (1.6) with singular weights $w$. Thus it is of relevance to identify a large class of weight functions, for which (1.6) admits a principal eigenfunction and to study its properties.

The existence of a principal eigenfunction for (1.6) and the fact that the associated eigen-space has dimension 1 is well known when $V, w \in C^0(\Omega)$ are positive on a bounded domain (see [8]). If $V \geq 0$ and $w$ has an indefinite sign with $w^+ \in L^r(\Omega) \setminus \{0\}$ with $r > \frac{N}{2}$ all these results can be extended thanks to the work of Manes and Micheletti [22]. In $\mathbb{R}^N$, many authors have given sufficient conditions for the existence of positive principal eigenvalue. One such sufficient condition introduced by Brown et al. in [6] is that the weight function $w$ be smooth, negative and bounded away from 0 at infinity. Another sufficient condition is that the weight function $w$ decays faster than $|x|^{-2}$ at infinity. The relaxation of the first condition, was introduced by Brown and Tertikas in [7], namely the positive part $w^+$ to be of compact support. Further improvement was given by Allegretto in [1], for $w^+ \in L^N_0(\Omega)$. Szulkin and Willem considered the problem (1.6) on a general domain (with $V \equiv 0$) and proved the existence of a positive principal eigenfunction in the space (1.2) whenever $w^+ \in L^N_0(\Omega)$ or having a faster decay than $|x|^{-2}$ at infinity and at any point in the domain.

In order to enlarge the class of weight functions beyond the Lebesgue spaces, Visciglia [31] considered problem (1.1) with positive weights $w$ lying in Lorentz spaces, a two-parameter family of spaces introduced by [18] that generalizes the Lebesgue spaces. For $p = 2$ it was already noted in [30, Lemma 3.3] that for some $w$ outside the Lorentz space $L^N_2(\Omega)$ the infimum of (1.3) may be 0. On the other hand, the space $L^N_2(\Omega)$ is still too large to ensure the existence of a minimizer, since typically for $V \equiv 0$, $w = |x|^{-2}$ and $\Omega = \mathbb{R}^N$ the infimum of (1.3) is strictly positive (by the Hardy inequality), but does not admit any extremal functions in $\mathcal{M}_2$. Hence when $p = 2$ we are led to look for a suitable class of weights $w$ belonging in $L^N_2(\Omega)$ that ensures existence of a minimizer to (1.3). In [21] we followed this direction by considering the linear problem (1.6) with

$$V \equiv 0, \quad w \in \bigcup_{1 \leq q < \infty} L^N_2(q), \quad w^+ \neq 0.$$  \hspace{1cm} (1.7)

Under these assumptions we proved the existence and uniqueness of the principal eigenfunction, but it was also shown that this class of weights and that of Szulkin and Willem in [28] are completely independent. For a general $p \in [2, 2^*)$, Visciglia [31] showed that the infimum of (1.3) is positive whenever $w \in L(\bar{p}, \infty)$ with $\frac{1}{p} + \frac{p}{2^*} = 1$ where $2^*$ denotes as usual the critical exponent (for $p = 2$ note that $\bar{p} = \frac{N}{2}$). Further under the assumption

$$w \geq 0, \quad w \in \bigcup_{1 \leq q < \infty} L(\bar{p}, q),$$  \hspace{1cm} (1.8)

it is also shown in [31] that $R$ attains its infimum at some $\Phi \in \mathcal{M}_p$ (see [31, Theorem 1.3]) which gives a non-negative solution to (1.1).

Our goal is to generalize all these previous works by exhibiting a general class of admissible weights $w$ that ensures existence of a minimizer to (1.3) in $\mathcal{M}_p$. With this aim the main
 novelty in our paper is to consider for each $s \geq 1$ the space:

\[ F_s := \text{closure of } C_0^\infty(\Omega) \text{ in the Lorentz space } L(s, \infty). \]

We will see that these spaces allow to relax the conditions (1.7), (1.8), or the one used in [28], and turn out also to be equivalent to a condition used earlier by Tertikas in [30]. Our main result reads as follows:

**Theorem 1.1** Let $p \in [2, 2^*)$ and $V, w \in L^1_{\text{loc}}(\Omega)$ satisfying

\[ V \geq 0, \quad w^+ \in F_{\frac{1}{p}} \setminus \{0\} \text{ with } \frac{1}{p} + \frac{p}{2^*} = 1. \quad (1.9) \]

(a) Then the infimum of $R$ is attained at some $\Phi \in \mathcal{M}_p$.

(b) Either $\Phi$ or $-\Phi$ is positive a.e. in $\Omega$. Further the nodal set of the precise representative of $\Phi$ has $H^1$-capacity zero.

Note that our result gives new sufficient conditions on $w$ ensuring the validity and the existence of extremals for the generalized Hardy–Sobolev inequality:

\[ \left( \int_{\Omega} w(x)|u|^p \right)^{\frac{1}{p}} \leq C \int_{\Omega} (|\nabla u|^2 + Vu^2), \quad \forall u \in \mathcal{M}_p. \quad (1.10) \]

Let us stress that in this theorem the negative part of $w$ is only assumed to lie in $L^1_{\text{loc}}(\Omega)$. Thus with such a weak assumption we cannot conclude the existence of a continuous representative minimizer positive everywhere. However we can always consider the precise representative $\tilde{\Phi}$ of a minimizer $\Phi$, for which a strong maximum principle due to Ancona [3] and Brezis, Ponce [4] allows to conclude that the nodal set $\tilde{\Phi}^{-1}(0)$ has $H^1$-capacity zero. Hence beside allowing more general weights than the one considered by Visciglia [31, Theorem 1.3], we also conclude that the minimizer $\tilde{\Phi}$ does satisfy (1.1) with $\pm \tilde{\Phi} > 0$ quasi-everywhere in $\Omega$, “q.e.” for short, i.e. everywhere except on a set of $H^1$-capacity zero. This rules out in particular the possibility for $\Phi$ of having a nodal set of positive measure.

In the case $p = 2$, the space $F^N_\infty$ (closure of $C_0^\infty(\Omega)$ in $L^N(\infty, \infty)$) defines a natural class of weights for which the existence and property of the principal eigenvalue can be studied.

In this linear setting, Theorem 1.1 can be restated more precisely as follows:

**Theorem 1.2** Let $V, w \in L^1_{\text{loc}}(\Omega)$ with $V \geq 0$ and $w^+ \in F^N_\infty \setminus \{0\}$. Then

\[
\lambda_1^+(V, w) := \inf \left\{ \frac{\int_{\Omega} (|\nabla u|^2 + Vu^2)}{\int_{\Omega} w u^2} : u \in \mathcal{M}_2 \right\}, \quad (1.11)
\]

is a positive principal eigenvalue for problem (1.6). Furthermore, the precise representative of the associated eigenfunction is positive q.e. in $\Omega$, is unique up to a constant multiple, and it is the unique positive eigenvalue admitting a non-negative eigenfunction.

If we make a further assumption $w^- \in F^N_\infty \setminus \{0\}$, Theorem 1.2 applied to the weight $-w$ shows that $-\lambda_1^+(V, -w)$ is a negative principal eigenvalue for (1.6). Theorem 1.2 generalizes considerably the results obtained until now and covers in particular the results of [21,28] (see Remark 5.4). Note that Theorem 1.2 brings two more information with respect to Theorem 1.1: the uniqueness up to a scalar multiple of the associated minimizer (“simplicity” of the principal eigenvalue) and the uniqueness of a positive value $\lambda$ for which the
associated eigenfunctions have constant sign. The simplicity of the principal eigenvalue is tightly related to the fact that minimizers do not change sign, whereas the second property relies on a “Piccone identity”.

Let us stress that Theorem 1.2 can be applied to study the bifurcation of solutions for the following problem,

\[- \Delta u = \lambda (a(x)u + b(x)r(u)), \quad u \in \mathcal{D}^{1,2}_0(\Omega)\]  

(1.12)  

for a real parameter \( \lambda \) when \( a, b \) have very less regularity. More specifically assume

\[ (H1) \quad r \in C^0(\mathbb{R}), \quad \lim_{s \to 0} \frac{r(s)}{s} = 0, \quad \limsup_{|s| \to \infty} \frac{|r(s)|}{|s|^{p-1}} < \infty \quad \text{for some } p \in [1, 2^*); \]

\[ (H2) \quad a \in \mathcal{F}_N, \quad b \in \mathcal{F}_N \cap \mathcal{F}_p \quad \text{where } \frac{1}{p} + \frac{2}{\tilde{p}} = 1 \text{ if } p \geq 2, \text{ whereas } \tilde{p} = \frac{N}{2} \text{ for } p < 2. \]

Since \( r(0) = 0 \), solutions may bifurcate from the trivial solution \( u \equiv 0 \). By assuming only \((H1), (H2)\) we can generalize some of the results obtained in [13, 21] and prove the following global bifurcation result:

**Theorem 1.3** Assume \((H1), (H2)\) and \( a^+ \neq 0 \). Define

\[ S := \left\{ (\lambda, u) \in \mathbb{R} \times \mathcal{D}^{1,2}_0(\Omega) : (\lambda, u) \text{ solves } (1.12), \ u \neq 0 \right\}, \]

\[ \sigma(a) := \left\{ \lambda \in \mathbb{R} : \text{ Null } (-\Delta - \lambda a(x)) \neq \{0\} \text{ in } \mathcal{D}^{1,2}_0(\Omega) \right\}. \]

Then, there exists a set \( C^+ \) connected in \( S \) bifurcating from \( \lambda_1^+(0, a) \). Moreover,

(i) \( C^+ \) is either unbounded,  
(ii) or \( C^+ \ni (\lambda, 0) \text{ with } \lambda \in \sigma(a) \setminus \{\lambda_1^+(0, a)\}. \]

Obviously if \( a^- \neq 0 \) a similar result holds. In particular if \( a^+, a^- \neq 0 \) we get existence of two branches \( C^+, C^- \) of solutions bifurcating from \( \lambda_1^+(0, a) \) and \( -\lambda_1^+(0, -a) \).

This paper is organized as follows. Section 2 recalls the definition and basic properties of Lorentz spaces. Section 3 discusses in more details the function space \( \mathcal{F}_r \). Several examples of functions belonging to \( \mathcal{F}_N \) are provided in Sect. 4. The existence part of Theorem 1.1 and Theorem 1.2 is proved in Section 5. The fact that these minimizers have constant sign and are unique up to a constant factor (for \( p = 2 \)) is the object of Sect. 6. The existence of a principal eigenvalue for linear problems is applied in Sect. 7 to establish the existence of branches of bifurcations for nonlinear equations of the type (1.12).

## 2 Prerequisites on Lorentz spaces

We recall here the definition and main properties of the Lorentz spaces that we will need. These spaces have been first discussed by Lorentz in [18], and for more details we refer to [15, 27, 29].

Given a measurable function \( f : \Omega \to \mathbb{R} \), we define the distribution function \( \alpha_f \) and nonincreasing rearrangement \( f^* \) of \( f \) as follows

\[ \alpha_f(s) := |\{ x \in \Omega : |f(x)| > s \}|, \quad f^*(t) := \inf \{ s > 0 : \alpha_f(s) \leq t \}. \]  

(2.1)  

Then \( \alpha_f : \mathbb{R} \to \mathbb{R} \cup \{\infty\} \) is nonnegative, nonincreasing. If \( \alpha_f \) is finite, it is also continuous from the right and we easily verify that
(i) \( f^* (\alpha_f(s)) \leq s, \ \alpha_f (f^*(t)) \leq t; \) under the further assumption that \( \alpha_f \) is continuous and strictly decreasing then \( f^* = \alpha_f^{-1}; \)

(ii) If \(|f| \leq |g|\) then \( \alpha_f \leq \alpha_g \) and \( f^* \leq g^*; \)

(iii) \((f + g)^*(t_1 + t_2) \leq f^*(t_1) + g^*(t_2);\)

(iv) \( \int_\Omega |f|^p \, dx = \int_0^\infty |f^*(t)|^p \, dt \) (Cavalieri principle).

Given a function \( f \in L_{1_{\text{loc}}}^1(\Omega) \) and \( p, q \in [1, \infty] \) we set

\[
\|f\|_{(p,q)}^* := \begin{cases} 
\left( \int_0^\infty \left[ \frac{1}{t^p} f^*(t) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}}, & \text{if } 1 \leq p, q < \infty, \\
\sup_{t>0} \left\{ \frac{1}{t^p} f^*(t) \right\}, & \text{if } 1 \leq p \leq \infty, q = \infty, 
\end{cases}
\]

and the Lorentz spaces are defined by

\[
L(p,q) := \{ f \in L_{1_{\text{loc}}}^1(\Omega) : \|f\|_{(p,q)}^* < \infty \}.
\]

In general the functional \( \| \cdot \|_{(p,q)}^* \) defines only a quasi-norm. In order to get a norm, we set \( f^{**}(t) := \frac{1}{t} \int_0^t f^*(r) \, dr \), and define

\[
\|f\|_{(p,q)} := \begin{cases} 
\left( \int_0^\infty \left[ \frac{1}{t^p} f^{**}(t) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}}, & \text{if } 1 \leq p, q < \infty, \\
\sup_{t>0} \left\{ \frac{1}{t^p} f^{**}(t) \right\}, & \text{if } 1 \leq p \leq \infty, q = \infty. 
\end{cases}
\]  \( \text{(2.2)} \)

For \( p > 1 \), the functional \( \| \cdot \|_{(p,q)} \) defines a norm which satisfies for some positive constant \( C := C(p,q) \) the inequalities \( C^{-1} \|f\|_{(p,q)}^* \leq \|f\|_{(p,q)} \leq C \|f\|_{(p,q)}^* \) (see Chapter V, Theorem 3.2, [27]). Endowed with this norm \( L(p,q) \) is a Banach space, whose main properties are summarized in the following proposition:

**Proposition 2.1** (a) Consider the Lebesgue space \( L^p(\Omega) \) with the norm \( \|u\|_p := (\int_\Omega |u|^p)^{1/p} \). Then \( L(p,p) = L^p(\Omega) \) and \( \| \cdot \|_p \leq \| \cdot \|_{(p,p)} \leq C_p \| \cdot \|_p \).

(b) If \( p \geq 1 \) and \( q_2 \geq q_1 \geq 1 \), then \( \| \cdot \|_{(p,q_2)} \leq \left( \frac{q_1}{p} \right)^{\frac{1}{q_1} - \frac{1}{q_2}} \| \cdot \|_{(p,q_1)} \).

(c) Hölder inequality (244): Given \( (f,g) \in L(p_1,q_1) \times L(p_2,q_2) \) and \( p,q \in [1,\infty] \) such that \( 1/p = 1/p_1 + 1/p_2 \), \( 1/q \leq 1/q_1 + 1/q_2 \), then

\[
\|fg\|_{(p,q)} \leq C_H \|f\|_{(p_1,q_1)} \|g\|_{(p_2,q_2)}, \quad \text{with } C_H = \begin{cases} 
1 & \text{if } p = 1 \\
p' & \text{if } p > 1.
\end{cases}
\]  \( \text{(2.3)} \)

(d) Let \( (p,q) \in (1,\infty) \times (1,\infty) \). Then the dual space of \( L(p,q) \) is isomorphic to \( L(p',q') \) where \( 1/p + 1/p' = 1 \) and \( 1/q + 1/q' = 1 \).

We also note the following property that will be useful: given \( \gamma \geq 1 \) then

\[
\|f^\gamma\|_{(p,q)} \leq \|f\|_{(p,q)}^{\gamma}. \quad \text{(2.4)}
\]

Indeed we have \( \alpha_{f^\gamma}(s) = \alpha_f(s^{1/\gamma}) \) and \( (|f|^{\gamma})^* (t) = (|f|^*(t))^{\gamma} \). Hence by Jensen inequality we get \( (|f|^{\gamma})^{**} (t) \leq (|f|^{*^*}(t))^{\gamma} \) and so (2.4) follows.

Let \( D_0^{1,2} (\Omega) \) be the Hilbert space defined by (1.2). The main interest of considering Lorentz spaces is that the usual Sobolev embedding \( D_0^{1,2} (\Omega) \hookrightarrow L^{2^*} (\Omega) \) can be improved as follows (see for example, appendix in [2]):

\[\text{Springer}\]
**Proposition 2.2** (Sobolev-Lorentz embedding). We have $\mathcal{D}^{1,2}_0(\Omega) \hookrightarrow L^2, 2)$, i.e. there exists $C_S > 0$ such that $\|u\|_{2, 2} \leq C_S \|\nabla u\|_2$ for any $u \in \mathcal{D}^{1,2}_0(\Omega)$.

Note that by this result any $u \in \mathcal{D}^{1,2}_0(\Omega)$ belongs to the Sobolev space $H^1_{loc}(\Omega)$, but on unbounded domain $u$ does not belong in general to $L^2(\Omega)$.

### 3 The function space $\mathcal{F}_p$

For $1 \leq p < \infty$ it is well known that $C_0^\infty(\Omega)$ is dense in the Lebesgue space $L^p(\Omega)$, whereas in $L^\infty(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the sup-norm is the space of functions vanishing on $\partial \Omega$ and at $\infty$ (if the domain is unbounded). A similar situation occurs in the Lorentz spaces. Indeed for $1 < p, q < \infty$ the set $C_0^\infty(\Omega)$ is dense in the Banach space $L(p, q)$ with respect to the norm $\| \cdot \|_{p,q}$ defined by (2.2). However the closure of $C_0^\infty(\Omega)$ in $L(p, \infty)$ defines a proper subspace that will henceforth be denoted by $\mathcal{F}_p$:

$$\mathcal{F}_p := \overline{C_0^\infty(\Omega) | \cdot |_{(p, \infty)}} \subset L(p, \infty) \quad (p > 1).$$

This space $\mathcal{F}_p$ can also be defined by considering the sets of linear combination of characteristic functions. More precisely we say that a measurable function $f : \Omega \to \mathbb{R}$ is a “simple function” if

$$f = \sum_{j=1}^m c_j \chi_{E_j}, \quad m \in \mathbb{N}, \ c_j \in \mathbb{R}, \quad (3.1)$$

where $E_1, \ldots, E_m$ are bounded measurable sets and $\chi_E$ stands for the characteristic function of the set $E$. Denote by

- $S$ the set of simple functions;
- and $S_1$ the class of functions of the type (3.1) with $E_j$ measurable set of finite measure (a class of functions considered in [15]).

Then both $S$ and $S_1$ are dense in $L(p, q)$ for $1 < p, q < \infty$ (see [15], Sect. 2) whereas

$$\mathcal{S}^p = \mathcal{S}_1^p = \mathcal{F}_p \subset L(p, \infty).$$

Our first result shows that $\mathcal{F}_p$ contains $L(p, q)$, for $1 < p, q < \infty$, and furthermore that $\mathcal{F}_p$ is a proper subspace of $L(p, \infty)$.

**Proposition 3.1**

(a) We have $L(p, q) \subset \mathcal{F}_p$ for each $1 < p, q < \infty$.

(b) For each $a \in \Omega$, the Hardy potential $x \mapsto |x - a|^{-N/p}$ does not belong to $\mathcal{F}_p$.

**Proof** (a) Let $f \in L(p, q)$. Since $C_0^\infty(\Omega)$ is dense in $L(p, q)$ with respect to the norm $\| \cdot \|_{p,q}$, there exists a sequence $f_n \in C_0^\infty(\Omega)$ such that $\lim_{n \to \infty} \|f - f_n\|_{p,q} = 0$. Since $\|f - f_n\|_{p,\infty} \leq C\|f - f_n\|_{p,q}$ (by Proposition 2.1) we immediately deduce $f \in \mathcal{F}_p$.

(b) Let $E \subset \Omega$ be any bounded measurable set. Setting $h_E(x) := |x - a|^{-N/p} - \chi_E(x)$, we claim $\|h_E\|_{p, \infty} \geq c > 0$ for some constant $c$ not depending on $E$. Fix a ball $B(a, r) \subset \Omega$ and without loss of generality we may assume $a = 0 \in \Omega$, $r = 1$ (by making a translation
and dilation. We then easily see that

\[ \alpha_{h_E}(s) \geq \left| \left\{ x \in B(0, 1) \cap E : |x|^{-N} - 1 > s \right\} \right| + \left| \left\{ x \in B(0, 1) \setminus E : |x|^{-N} > s \right\} \right| \]

\[ \geq \left| \left\{ x \in B(0, 1) : |x|^{-N} < (1 + s)^{-1} \right\} \right| \]

\[ = |B(0, 1)|(1 + s)^{-p}. \]

Hence

\[ h_E^*(t) = \inf \{ s > 0 : \alpha_{h_E}(s) \leq t \} \geq \inf \{ s > 0 : |B(0, 1)|(1 + s)^{-p} \leq t \} \]

\[ = |B(0, 1)|^{\frac{1}{p}} t^{-\frac{1}{p}} - 1, \]

which implies

\[ \sup_{t \in (0, |\Omega|)} \left\{ t^{\frac{1}{p}} h_E^*(t) \right\} \geq \sup_{t \in (0, |B(0, 1)|)} t^{\frac{1}{p}} \left( |B(0, 1)|^{\frac{1}{p}} t^{-\frac{1}{p}} - 1 \right) = |B(0, 1)|^{\frac{1}{p}}. \]

This shows that the Hardy potential never lies in \( \mathcal{F}_p \), but lies in \( L(p, \infty) \). □

The following characterization of \( \mathcal{F}_p \) will be useful.

**Proposition 3.2** Let \( f \in \mathcal{F}_p \) if and only if for every \( \varepsilon > 0 \) we can write \( f = f_\varepsilon + g_\varepsilon \) with

\[ f_\varepsilon \in L^\infty(\Omega), \quad |\text{supp}(f_\varepsilon)| < \infty, \quad ||g_\varepsilon||_{(p, \infty)} < \varepsilon. \] (3.2)

**Proof** Let \( f \in \mathcal{F}_p \). Given \( \varepsilon > 0 \), choose \( f_\varepsilon \in C_0^\infty(\Omega) \) such that \( ||f - f_\varepsilon||_{(p, \infty)} < \varepsilon \) and define \( g_\varepsilon := f - f_\varepsilon \). Conversely, for every \( \varepsilon > 0 \) there exists \( f_\varepsilon \in L^\infty(\Omega) \) whose support has finite measure, such that \( ||f - f_\varepsilon||_{(p, \infty)} < \frac{\varepsilon}{2} \). Since \( f_\varepsilon \in L^p(\Omega) \) we can find \( \varphi_\varepsilon \in C_0^\infty(\Omega) \) such that \( ||f_\varepsilon - \varphi_\varepsilon||_{(p, \infty)} < \frac{\varepsilon}{2} \). Hence \( ||f - \varphi_\varepsilon||_{(p, \infty)} < \varepsilon \), and therefore \( f \in \mathcal{F}_p \). □

We now show that the space \( \mathcal{F}_p \) can be characterized with another condition using the behavior of \( f^* \) introduced by Tertikas (see Remark 3.1 in [30]).

**Theorem 3.3** Let \( f \in L^1_{loc}(\Omega) \). Then \( f \in \mathcal{F}_p \) if and only if

\[ \lim_{t \to 0+} t^{\frac{1}{p}} f^*(t) = \lim_{t \to \infty} t^{\frac{1}{p}} f^*(t) = 0. \] (3.3)

**Proof** Let \( f \in \mathcal{F}_p \). For a given \( \varepsilon > 0 \) write \( f = f_\varepsilon + (f - f_\varepsilon) \) where \( f_\varepsilon \in C_0^\infty(\Omega) \) is such that \( ||f - f_\varepsilon||_{(p, \infty)} < \varepsilon \). Since \( (f_1 + f_2)^*(t_1 + t_2) \leq f_1^*(t_1) + f_2^*(t_2) \) for any \( t_1, t_2 > 0 \) (see [15]), we deduce \( f^*(2t) \leq (f - f_\varepsilon)^*(t) + f_\varepsilon^*(t) \). Hence

\[ (2t)^{\frac{1}{p}} f^*(2t) \leq 2^{\frac{1}{p}} \left\{ t^{\frac{1}{p}} (f - f_\varepsilon)^*(t) + t^{\frac{1}{p}} f_\varepsilon^*(t) \right\} \leq 2^{\frac{1}{p}} \left\{ \varepsilon + t^{\frac{1}{p}} f_\varepsilon^*(t) \right\}. \]

Since \( f_\varepsilon \in C_0^\infty(\Omega) \), the function \( f_\varepsilon^* \) has also compact support and satisfies therefore (3.3). Thus \( (2t)^{\frac{1}{p}} f^*(2t) \) can be made arbitrarily small for large and small values of \( t \), showing that (3.3) holds for any \( f \in \mathcal{F}_p \).

Conversely assume (3.3) holds. Let \( \varepsilon > 0 \) be given. By (3.3) there exist \( t_0, t_1 \) such that

\[ t^{\frac{1}{p}} f^*(t) < \varepsilon, \quad \forall t \in (0, t_0] \cup [t_1, \infty). \] (3.4)

For each \( n \in \mathbb{N} \), define

\[ A_n := \{ x \in \Omega : |f(x)| \leq n, \ |x| \leq n \}, \quad f_n := \chi_{A_n} f, \quad g_n := f - f_n = \chi_{\Omega \setminus A_n} f. \]
Clearly $A_n$ is bounded and $f_n \in L^\infty(\Omega)$. Thus by Proposition 3.2 it is enough to show that $\lim_{n \to \infty} \|g_n\|_{(p, \infty)} = 0$. For $n \in \mathbb{N}$, $s > 0$, let $B_{n,s} := \{x \in \Omega : |f(x)| > s, |x| > n\}$. Observe that for a fixed $s > 0$ and for every $n > s$

$$\{x \in \Omega : |g_n(x)| > s\} = \{x \in \Omega : |f(x)| > n\} \cup B_{n,s}.$$  

Therefore

$$\alpha_{g_n}(s) \leq \alpha_f(n) + |B_{n,s}|.$$  

Clearly for each $s > 0$, $\bigcap_n B_{n,s} = \emptyset$ and $|B_{1,s}| < \infty$. Thus for each $s > 0$, from the right continuity of Lebesgue measure, $\lim_{n \to \infty} |B_{n,s}| = 0$. Further we have $\lim_{n \to \infty} \alpha_f(n) = 0$ (if not there exists $k > 0$ such that $\alpha_f(n) > k$, $\forall n$ and $f^*(t) = \infty$ for every $t \leq k$, contradicting (3.3)). Thus for every $s > 0$, $\lim_{n \to \infty} \alpha_{g_n}(s) = 0$. Since $|g_n| < |f|$, $g_n^s(t) \leq f^*(t)$. Thus from (3.4) we have the following,

$$t^\frac{1}{p} g_n^s(t) < \varepsilon, \quad \forall t \in (0, t_0] \cup [t_1, \infty), \quad \forall n \in \mathbb{N}. \quad (3.5)$$

Since for every $s > 0$, $\lim_{n \to \infty} \alpha_{g_n}(s) = 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that, $\alpha_{g_n}(\varepsilon t_1^{-\frac{1}{p}}) < t_0$, $\forall n \geq n_\varepsilon$. Hence $g_n^s(t_0) \leq \varepsilon t_1^{-\frac{1}{p}}$, $\forall n \geq n_\varepsilon$. Thus for $t \in (t_0, t_1)$,

$$t^\frac{1}{p} g_n^s(t) \leq t_1^\frac{1}{p} g_n^s(t_0) \leq \varepsilon, \quad \forall n \geq n_\varepsilon. \quad (3.6)$$

From (3.5) and (3.6), we get $\frac{1}{p} g_n^s(t) < \varepsilon$, $\forall t > 0$ and hence $\|g_n\|_{(p, \infty)} < \varepsilon$, $\forall n \geq n_\varepsilon$. \(\square\)

4 Some examples

This section provides several explicit examples of functions belonging to $\mathcal{F}_N^\frac{1}{q}$. Proposition 3.1 already shows that $\mathcal{F}_N^\frac{1}{q}$ contains the weights satisfying

$$w \in \bigcup_{1 \leq q < \infty} L\left(\frac{N}{2}, q\right). \quad (4.1)$$

Another class of functions contained in $\mathcal{F}_N^\frac{1}{q}$ is provided by the work of Szulkin and Willem in [28]. More specifically consider the functions $w$ defined by the conditions:

$$\left\{ \begin{array}{l}
w \in L^1_{\text{loc}}(\Omega), \quad w^+ = w_1 + w_2 \neq 0, \quad w_1 \in L^\frac{N}{2}(\Omega), \\
\lim_{|x| \to \infty, x \in \Omega} |x|^2 w_2(x) = 0, \quad \lim_{x \to a, x \in \overline{\Omega}} |x - a|^2 w_2(x) = 0 \quad \forall a \in \overline{\Omega}. \end{array} \right. \quad (4.2)$$

We claim that the positive part of any function satisfying (4.2) belongs to the space $\mathcal{F}_N^\frac{1}{q}$. To get this result we will need the following lemma:

**Lemma 4.1** Let $f : \Omega \to \mathbb{R}$ be a measurable function satisfying

(i) $\lim_{|x| \to \infty} |x|^2 f(x) = 0$, \hspace{1cm} (ii) $\lim_{x \to a} |x - a|^2 f(x) = 0$ \forall $a \in \overline{\Omega}. \quad (4.3)
Then there exist finite number of points \( a_1, \ldots, a_m \in \Omega \) with the following property: for every \( \varepsilon > 0 \) there exists \( R := R(\varepsilon) > 0 \) such that

\[
|f(x)| < \frac{\varepsilon}{|x|^2} \quad \text{a.e } x \in \Omega \setminus B(0, R),
\]

\[
|f(x)| < \frac{\varepsilon}{|x - a_i|^2} \quad \text{a.e } x \in \Omega \cap B(a_i, R^{-1}), \quad i = 1, \ldots, m,
\]

and setting \( A_\varepsilon := \bigcup_{i=1}^m B(a_i, R^{-1}) \cap \Omega \)

\[
\Omega \setminus A_\varepsilon \neq \emptyset \quad \text{and } f \in L^\infty(\Omega \setminus A_\varepsilon).
\]

**Proof** Using condition (i) we can find \( r \) such that

\[
|f(x)| < \frac{1}{|x|^2} \quad \text{a.e } x \in \Omega \setminus B(0, r).
\]

Now by condition (ii) for each \( a \in \Omega \cap B(0, r) \) there exists \( r_a > 0 \) such that \( |f(x)| < \frac{1}{|x-a|^2} \) for a.e. \( x \in \Omega \cap B(a_r, r_a) \). Since \( \Omega \cap B(0, r) \) is compact, we can find finitely many points \( a_1, \ldots, a_m \in \Omega \cap B(0, r) \) such that \( \Omega \cap B(0, r) \subset \bigcup_{i=1}^m B(a_i, r_a) \).

Let now \( \varepsilon > 0 \). Using again condition (i), choose \( R = R(\varepsilon) \geq r, r_{a_1}, \ldots, r_{a_m} \) so that

\[
|f(x)| < \frac{\varepsilon}{|x - a_i|^2}, \quad \text{a.e } x \in \Omega \cap B(a_i, R^{-1}) \quad \text{and } f \in L^\infty(\Omega \setminus A_\varepsilon).
\]

Observe that any larger \( R \) suffices for the proof since both the above inequalities continue to hold and \( \Omega \setminus A_\varepsilon \) also can be made nonempty. \( \square \)

**Proposition 4.2** (a) Any measurable function \( f : \Omega \to \mathbb{R} \) satisfying (4.3) lies in \( \mathcal{F}_{\frac{N}{2}} \).

(b) In particular if \( w \in L^1_{\text{loc}}(\Omega) \) satisfies (4.2), then \( w^+ \in \mathcal{F}_{\frac{N}{2}} \).

**Proof** (a) By applying Lemma 4.1, we readily see that \( f \in L^1_{\text{loc}}(\Omega) \). We prove that \( f \in \mathcal{F}_{\frac{N}{2}} \) by showing that \( f \) satisfies condition (3.3). Let \( \varepsilon > 0 \) be given, and consider the balls \( B(0, R), B(a_1, R^{-1}), \ldots, B(a_m, R^{-1}) \) given by Lemma 4.1.

We first show that \( \lim_{t \to \infty} t^{\frac{N}{2}} f^*(t) = 0 \). Using (4.4), for each \( s \in (0, \varepsilon R^{-2}) \) we have,

\[
B(0, R) \subset B(0, \left(\frac{\varepsilon}{s}\right)^{1/2}) \quad \text{and } |f(x)| < s \quad \forall x \in \Omega \setminus B(0, \left(\frac{\varepsilon}{s}\right)^{1/2}).
\]

Therefore, letting \( s_0 := \varepsilon R^{-2} \) the distribution function \( \alpha_f(s) \) can be estimated for each \( s \in (0, s_0) \) as follows:

\[
\alpha_f(s) = \left| \{ x \in \Omega \cap B(0, \left(\frac{\varepsilon}{s}\right)^{1/2}) : |f(x)| > s \} \right| \leq \omega \left(\frac{\varepsilon}{s}\right)^\frac{N}{2},
\]

where \( \omega \) is the volume of unit ball in \( \mathbb{R}^N \). Hence letting \( t_0 := \omega \left(\frac{\varepsilon}{s_0}\right)^\frac{N}{2} = \omega R^N \), for any \( t > t_0 \) we get

\[
f^*(t) = \inf \left\{ s > 0 : \alpha_f(s) \leq t \right\} \leq \inf \left\{ s > 0 : \omega \left(\frac{\varepsilon}{s}\right)^\frac{N}{2} \leq t \right\} = \left(\frac{\omega}{t}\right)^\frac{N}{2} \varepsilon.
\]
Thus
\[ t^{2/N} f^*(t) < \omega^{2/N} \forall t > t_0 := \omega R^N, \]
which shows that \( \lim_{t \to \infty} t^{2/N} f^*(t) = 0. \)

To show \( \lim_{t \to 0^+} t^{2/N} f^*(t) = 0 \), consider the set \( A_\varepsilon = \bigcup_{i=1}^m B(a_i, R^{-1}) \cap \Omega \) and let \( s_1 := \| f \|_{L^\infty(\Omega \setminus A_\varepsilon)} \). For \( s > s_1 \), using (4.5) the distribution function can be estimated as follows:
\[
\alpha_f(s) = |\{ x \in \Omega : |f(x)| > s \}| = |\{ x \in A_\varepsilon : |f(x)| > s \}|
\leq \sum_{i=1}^m \left| \{ x \in B(a_i, R^{-1}) \cap \Omega : |f(x)| > s \} \right|
\leq \sum_{i=1}^m \left| \{ x \in B(a_i, R^{-1}) : \varepsilon |x - a_i|^{-2} > s \} \right|
= \sum_{i=1}^m \omega \left( \frac{\varepsilon}{s} \right)^{N/2}
\]
therefore
\[ \alpha_f(s) \leq m \omega \left( \frac{\varepsilon}{s} \right)^{N/2} \forall s > s_1. \]

Setting \( t_1 := m \omega \left( \frac{\varepsilon}{s_1} \right)^{N/2} \), for each \( t < t_1 \) the function \( f^* \) is estimated by
\[ f^*(t) \leq \inf \left\{ s : m \omega \left( \frac{\varepsilon}{s} \right)^{N/2} \leq t \right\} = \left( \frac{m \omega}{t} \right)^{2/N} \varepsilon. \]
This gives
\[ t^{2/N} f^*(t) \leq (m \omega)^{2/N} \varepsilon \forall t < t_1, \]
which shows \( \lim_{t \to 0^+} t^{2/N} f^*(t) = 0 \). This concludes the proof that \( f \in F_{N/2}. \)

(b) Let \( w \) be a function satisfying (4.2). On one hand \( w_1 \in L^{N/2}(\Omega) \subset F_{N/2} \) (by Proposition 3.1). On the other hand, by the first part of this proposition we know that \( w_2 \in F_{N/2} \). Hence \( w = w_1 + w_2 \in F_{N/2}. \)

In order to understand better how the conditions (4.1), (4.2) and the property defining \( F_{N/2} \) are related to one another, consider first the following functions borrowed from [28]:
\[
W_1(x) = \frac{1}{1 + |x|^2}, \quad W_2(x) = \frac{1}{|x|^2(1 + |x|^2)};
\]
\[ \bar{W}_1(x) = \frac{W_1(x)}{\log(2 + |x|^2)^{N/2}}, \quad \bar{W}_2(x) = \frac{W_2(x)}{\left( \log(2 + \frac{1}{|x|^2}) \right)^{N/2}}. \]
(4.8)

For these functions we note the following:

- As shown in [28] none of the functions \( W_1, W_2 \) satisfy (4.2) and the associated eigenvalue problems (1.6) (with \( V \equiv 0 \)) do not possess any eigenvalue (see also [30]). One may also check easily that these two functions do not satisfy (4.1), and do not belong to \( F_{N/2} \).
The functions $\tilde{W}_1, \tilde{W}_2$ satisfy both condition (4.1) and (4.2). In particular they belong to $F_N^2$ (by Proposition 3.1).

As already explained in [21] with examples conditions (4.1) and (4.2) are independent. Indeed there are functions satisfying (4.1) but not (4.2):

**Example 4.3** In the cube $\Omega = \{(x_1, \ldots, x_N) \in \mathbb{R}^N : |x_i| < R\}$ with $0 < R < 1$ consider the function defined by

$$W_3(x) = |x_1 \log(|x_1|)|^{-\frac{2}{N}}, \quad x_1 \neq 0.$$  

(4.9)

Straight computations show that $W_3 \in L^{(N^2, q)}$ for some $q \in (1, \infty)$, and in particular $W_3 \in F_N^2$ (by Proposition 3.1). But this function does not satisfy (4.2). Indeed the limit of $|x|^2 W_3(x)$, as $x$ tends to 0 along the curve $x_2 = (x_1)_{1 = 2}^N$, tends to infinity, while along the $x_1$ axis it tends to 0. Thus the limit does not exist and therefore $W_3$ does not satisfy the condition (4.2).

Conversely there are functions satisfying (4.2) but not (4.1):

**Example 4.4** Consider the function

$$W_4(x) = \frac{W_1(x)}{\log \log (2 + |x|^2)}.$$  

This function does not satisfy condition (4.1). But this function does satisfy (4.2) as $|x|$ tends to infinity (see [21] for the calculations).

Finally let us emphasize that conditions (4.1) and (4.2) do not exhaust the space $F_N^2$. This is illustrated by the following example:

**Example 4.5** There are functions lying in $F_N^2$ which fail to satisfy both (4.1) and (4.2). Let $\Omega = \{(x_1, \ldots, x_n) : |x_i| < R\}$ with $R = 2^{\frac{1}{N} - 1}$, and consider

$$W_5(x) = \begin{cases} |x_1|^{-\frac{2}{N}} \log(|\log(|x_1|)|)^{-1} & \text{if } x \in \Omega \text{ and } x_1 \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$W_5$ does not satisfy (4.2), since it fails to satisfy the conclusions of Lemma 4.1. To show that $W_5$ does not satisfy (4.1) but satisfies (3.3), we first note that the distribution function of $W_5$ is given by:

$$\alpha_{W_5}(s) = 2^N R^{N-1} |x_1(s)|,$$

where $x_1(s)$ is the first coordinate of $x \in \Omega$, such that $s = W_5(x)$. Now for $t \in (0, (2R)^N)$,

$$W_5^*(t) = \inf \left\{ s : \alpha_{W_5}(s) \leq t \right\} = \inf \left\{ s : 2^N R^{N-1} |x_1(s)| \leq t \right\} = \inf \left\{ |x_1(s)|^{-\frac{2}{N}} \log(|\log(|x_1(s)|)|)^{-1} : 2^N R^{N-1} |x_1(s)| \leq t \right\} = \left( \frac{t}{2^N R^{N-1}} \right)^{-\frac{2}{N}} \left| \log \left( \left| \log \left( \frac{t}{2^N R^{N-1}} \right) \right| \right) \right|^{-1} = t^{-\frac{2}{N}} |\log(|\log(t)|)|^{-1}$$
while \( W^q_s(t) = 0 \) for \( t \in [(2R)^N, \infty) \). Thus \( W_5 \) satisfies Tertikas’s condition (3.3). But \( W_5 \notin L^{(N/2, q)} \) for any \( q \in [1, \infty) \) since a straight computation shows

\[
\int_0^\infty \left\{ \frac{1}{t^N} W^q_s(t) \right\}^q \frac{dt}{t} = \int_0^{(2R)^N} \frac{1}{|\log(|\log t|)|^q} \frac{dt}{|\log(2R)^N|} \int_0^\infty \log|y|^q dy
\]

and this last integral is divergent for any \( q \geq 1 \).

5 Extremals for Hardy–Sobolev inequalities

To study the existence part of Theorem 1.1, for given \( p \geq 2, V, w \in L^1_{loc}(\Omega) \) we define the sets:

\[
\mathcal{M}_p := \left\{ u \in D^{1,2}_0(\Omega) : Vu^2, w|u|^p \in L^1(\Omega) \text{ and } \int \omega w|u|^p > 0 \right\},
\]

\[
\mathcal{\tilde{M}}_p := \left\{ u \in D^{1,2}_0(\Omega) : Vu^2, w|u|^p \in L^1(\Omega) \text{ and } \int \omega w|u|^p = 1 \right\},
\]

and consider the following minimization problems

(P) Minimize \( R(u) = \int_\Omega \left\{ |\nabla u|^2 + Vu^2 \right\} \) on \( \mathcal{M}_p \),

(\tilde{P}) Minimize \( I(u) = \int_\Omega \left\{ |\nabla u|^2 + Vu^2 \right\} \) on \( \mathcal{\tilde{M}}_p \).

Observe that due to the homogeneity of the Rayleigh quotient \( R \), problems (P) and (\tilde{P}) are equivalent. Furthermore if problem (P) admits a solution \( \Phi \), by noting that the mapping \( R \) admits a directional derivative in each direction \( \xi \in C_0^\infty(\Omega) \), we necessarily have

\[
\frac{d}{dt} R(\Phi + t\xi)\big|_{t=0} = 0, \quad \forall \xi \in C_0^\infty(\Omega).
\]

We then easily deduce that \( \Phi \) must satisfy (1.5) for any \( \xi \in C_0^\infty(\Omega) \). An approximation argument and Vitali’s convergence Theorem allows to conclude that (1.5) holds for any \( \xi \in D^{1,2}_0(\Omega) \).

In [31] it is shown that the minimizing problem (P) admits a solution whenever \( w \) satisfies (1.8). This result can be extended to the larger class of weights \( \mathcal{F}_p \) without restriction on the sign. To this aim we first note the following compactness result

**Lemma 5.1** Let \( r \in C_0(\mathbb{R}) \) satisfying \( |r(s)| \leq C|s|^\alpha-1 \) for some \( \alpha \in [2, 2^*) \) and \( w \in \mathcal{F}_{\alpha} \) with \( \frac{1}{\alpha} + \frac{\alpha}{2^*} = 1 \). Then the operator

\[
N : D^{1,2}_0(\Omega) \to [L(2^*, 2)]' \quad u \mapsto w(x)r(u),
\]

is continuous and compact.

**Proof** We only give the arguments for the compactness. Given \( u_n \rightharpoonup \tilde{u} \) converging weakly in \( D^{1,2}_0(\Omega) \), we need to prove that \( N(u_n) \rightharpoonup N(\tilde{u}) \) in \( [L(2^*, 2)]' \). Let \( \epsilon > 0 \) be given. Since

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w ∈ \mathcal{F}_{\tilde{\alpha}}$, we can find
\[ w_\varepsilon \in C_0^\infty (\Omega), \quad ||w - w_\varepsilon||_{(\tilde{\alpha}, \infty)} < \frac{\varepsilon}{2C_1CC_1}, \quad (5.1) \]
where \( C_1 := \sup_{n \in \mathbb{N}} ||u_n||_{(\alpha-1, 2^*)} + ||\bar{u}||_{(\alpha-1, 2^*)} \) and \( C_2 > 0 \) is the constant in Proposition 2.2.

For each \( v \in L(2^*, 2) \) we then get
\[ |N(u_n)(v) - N(\bar{u})(v)| \leq \int_{\Omega} |w_\varepsilon (r(u_n) - r(\bar{u}))/v| + \int_{\Omega} |w - w_\varepsilon||r(u_n) - r(\bar{u})||v|. \quad (5.2) \]

To estimate the first integral in the right hand side of (5.2), we use the Hölder inequality (2.3) and the Sobolev-Lorentz embedding
\[ \int |w_\varepsilon (r(u_n) - r(\bar{u}))/v| \leq 2C_S \|w_\varepsilon \| (2^*, 2^*) \|\nabla v\|_2. \quad (5.3) \]

With our growth assumption we have
\[ |r(u_n) - r(\bar{u})| \leq C_2 \left( ||u_n||^{(\alpha-1)|2^*\prime|_2} + ||\bar{u}||^{(\alpha-1)|2^*\prime|_2} \right). \quad (5.4) \]

Since \( u_n \in H^1_{locc} (\Omega) \) and for each ball \( B \subset \subset \Omega \) the embedding \( H^1(B) \rightarrow L^q(B) \) is compact for \( q \in [1, 2^*) \), the right hand side of (5.4) converges in \( L^1_{locc}(\Omega) \) and a.e. in \( \Omega \). Hence by applying the generalized Lebesgue dominated convergence theorem, we have \( r(u_n) \rightarrow r(\bar{u}) \) in \( L^1_{locc}(\Omega) \). Therefore from (5.3) and the fact that \( w_\varepsilon \in C_0^\infty (\Omega) \) we deduce the existence of \( n_0 \in \mathbb{N} \) such that
\[ \int_{\Omega} |w_\varepsilon||r(u_n) - r(\bar{u})||v| < \varepsilon \|\nabla v\|_2, \quad \forall n \geq n_0. \quad (5.5) \]

The second integral in (5.2) can be estimated using the growth assumption on \( r \), Proposition 2.1, (2.4), (5.1) and the Sobolev-Lorentz embedding:
\[ \int_{\Omega} |w - w_\varepsilon||r(u_n) - r(\bar{u})||v| \leq C \int_{\Omega} |w - w_\varepsilon||u_n||^{(\alpha-1)} + ||\bar{u}||^{(\alpha-1)}||v| \]
\[ \leq 2C \|w - w_\varepsilon\|_{(\tilde{\alpha}, \infty)} \|u_n||^{(\alpha-1)} + ||\bar{u}||^{(\alpha-1)}\|v\|_2. \]
\[ \leq \frac{\varepsilon}{C_SCC_1} \left( ||u_n||^{(\alpha-1)|2^*\prime|_2} + ||\bar{u}||^{(\alpha-1)|2^*\prime|_2} \right) \|v\|_2 \]
\[ \leq \varepsilon \|\nabla v\|_2. \quad (5.6) \]

Putting together (5.2), (5.5) and (5.6) we conclude
\[ \int_{\Omega} |w||r(u_n) - r(\bar{u})||v| < 2\varepsilon \|\nabla v\|_2, \quad \forall n \geq n_0. \]

This shows that \( N(u_n) \rightarrow N(\bar{u}) \) in \([L(2^*, 2)]\). \( \square \)

We are now able to prove the main result of this section

**Proposition 5.2 (Existence)** Let \( p \in [2, 2^*) \) and \( V, w \in L^1_{locc}(\Omega) \) satisfying (1.9). Then \( I \) has a minimizer on \( \tilde{M}_p \).

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Proof Since \( w \in L^1_{\text{loc}}(\Omega) \) and \( w^+ \neq 0 \), there exists \( \varphi \in C^\infty_0(\Omega) \) such that \( \int_\Omega w|\varphi|^p > 0 \) (see for example [17, Prop. 4.2]). In particular \( V\varphi^2 \in L^1(\Omega) \) (because \( V \in L^1_{\text{loc}}(\Omega) \)) and therefore \( \widetilde{M}_p \neq \emptyset \). Let \( u_n \) be a minimizing sequence:

\[
 u_n \in \widetilde{M}_p, \quad I(u_n) \to \inf_{u \in \mathcal{M}_p} I(u) .
\]

Since \( V \geq 0 \) we have \( C \geq I(u_n) = I(|u_n|) \geq \|\nabla |u_n| \|_2^2 \) and so \( |u_n| \) converges weakly to some \( \Phi \geq 0 \) (up to a subsequence). Since \( I \) is also weakly lower semicontinuous, in order to show that \( \Phi \) is a minimizer, we only need to prove that \( \int_\Omega w|\Phi|^p = 1 \). We claim that

\[
 \lim_{n \to \infty} \int_\Omega w^+(|u_n|^p - |\Phi|^p) = 0 .
\]

Indeed we have

\[
 \left| \int_\Omega w^+ (|u_n|^p - |\Phi|^p) \right| \leq \int_\Omega w^+ (|u_n|^{p-1} - |\Phi|^{p-1}) |u_n| \]

\[
 + \int_\Omega w^+ |\Phi|^{p-1} (|u_n| - |\Phi|) \]

\[
 \leq \left\| w^+ (|u_n|^{p-1} - |\Phi|^{p-1}) \right\|_{(2^*,2)} \|u_n\|_{(2^*,2)} + \int_\Omega w^+ |\Phi|^{p-1} (|u_n| - |\Phi|) .
\]

By the Sobolev-Lorentz embedding \( |u_n| \) converges weakly to \( \Phi \) in \( L(2^*,2) \) and is also bounded in \( L(2^*,2) \). Thus the first term in the right hand side of (5.8) converges to zero by Lemma 5.1, and since \( w^+ |\Phi|^{p-1} \in [L(2^*,2)]' \) the integral in the right hand side of (5.8) also converges to zero. Therefore (5.7) holds and we conclude

\[
 \int_\Omega w^+|\Phi|^p = \lim_{n \to \infty} \int_\Omega w^+|u_n|^p = 1 .
\]

Since \( \int_\Omega w^-u_n^2 = \int_\Omega w^+u_n^2 - 1 \), Fatou’s Lemma and (5.9) yield

\[
 \int_\Omega w^-|\Phi|^p \leq \int_\Omega w^+|\Phi|^p - 1 ,
\]

which shows that \( \int_\Omega w|\Phi|^p \geq 1 \). Setting \( \tilde{u} := (\int_\Omega w|\Phi|^p)^{-2/p} \Phi \), the non-negativity of \( V \) together with the weak lower-semicontinuity of \( I \) yield

\[
 \inf_{u \in \mathcal{M}_p} I(u) \leq I(\tilde{u}) = \frac{\int_\Omega (|\nabla \Phi|^2 + V\Phi^2)}{(\int_\Omega w|\Phi|^p)^{2/p}} \leq \frac{\int (|\nabla \Phi|^2 + V\Phi^2)}{\int_\Omega w|\Phi|^p} \leq \inf_{u \in \mathcal{M}_p} I(u) .
\]

Hence equality must hold at each step and therefore \( \int_\Omega w|\Phi|^p = 1 \), i.e. \( \Phi \in \tilde{M}_p \).

In the particular case \( p = 2 \), Proposition 5.2 shows that the linear problem (1.6) admits a positive principal eigenvalue under the only condition \( w \in L^1_{\text{loc}}, w^+ \in \mathcal{F}_{N,2} \).
Corollary 5.3 Let $V, w \in L^1_{\text{loc}}(\Omega)$ with $V \geq 0$ and $w^+ \in F_{\frac{N}{2}}$. Then

$$\lambda_1^+(V, w) := \inf \left\{ \int_{\Omega} (|\nabla u|^2 + Vu^2) \int_{\Omega} w u^2 : u \in M_2 \right\}$$

is a positive principal eigenvalue for problem (1.6).

This latter proposition generalizes several results in the literature and it deserves some remarks.

Remark 5.4 (a) In [28] Corollary 5.3 is proved for $w \in L^1_{\text{loc}}(\Omega)$ with $w^+ \in F_{N}$. This latter proposition generalizes several results in the literature and it deserves some remarks.

(b) Since $L(N, q) \subset F_{N}$ for each $q \in [1, \infty)$ (by Proposition 3.1), Corollary 5.3 covers the result obtained in [21].

(c) Let us stress that the condition $V \geq 0$ is important, since for sign-changing potential $V$ the linear problem (1.6) may have no principal eigenvalues (see [12]).

Remark 5.5 Under the stronger assumption $w \in F_{\frac{N}{2}}$, we know by Lemma 5.1 that the mapping

$$\tilde{L} : D^{1,2}_0(\Omega) \to \left[ L(2^*, 2) \right]', \quad u \mapsto w u$$

is a well-defined compact operator. Hence by considering the inverse of the Laplacian $\Delta^{-1} : \left[ L(2^*, 2) \right]' \to D^{1,2}_0(\Omega)$ (well-defined and continuous by Lax-Milgram theorem), and setting $L := (-\Delta)^{-1} \circ L$, problem (1.6) is equivalent to:

$$L(u) = \lambda^{-1} u \quad u \in D^{1,2}_0(\Omega).$$

Since the operator $L$ is compact and self-adjoint, the Hilbert–Schmidt Theorem yields the existence of a sequence $(\lambda_n, \varphi_n) \in (\mathbb{R} \times D^{1,2}_0(\Omega)) \setminus \{(0, 0)\}$ solving $L(\varphi_n) = \lambda_n^{-1} \varphi_n$. Furthermore the sequence of eigenvectors is a Hilbert basis of $D^{1,2}_0(\Omega)$ and $|\lambda_n| \to \infty$.

Remark 5.6 A natural question is whether one can get the existence of a principal eigenfunction when $w \notin F_{\frac{N}{2}}$. Tertikas in [30] introduced the notion of subcritical potential and showed that for every subcritical potential $w$, the problem (1.6) with $V \equiv 0$ admits a principal eigenvalue (see [30, Corollary 3.6]). He also showed that such existence still holds if $w = \frac{1}{|x|^2} + g(x)$, when $g(x)$ is any subcritical potential (see [30, Theorem 1.7]). One can verify that all positive functions in $F_{\frac{N}{2}}$ are subcritical potentials. Clearly for any positive weight $g \in F_{\frac{N}{2}}$, $\frac{1}{|x|^2} + g(x)$ cannot be in $F_{\frac{N}{2}}$, but it admits a principal eigenfunction.

6 Sign and uniqueness of the minimizers

In this section we discuss the sign and uniqueness of the minimizers obtained in Proposition 5.2. For $p = 2$ and $V, w^- \in L^q_{\text{loc}}(\Omega)$ with $q > \frac{N}{2}$, this is derived in [22] by using the Harnack inequality. When these functions have less regularities, it was shown in [17,19,20] that one can still reach similar conclusions, and for the sake of completeness we give here the main ideas.

A crucial tool is a weaker version of the strong maximum principle. Before stating this result let us recall the notion of “variational capacity” (for an extended discussion we refer
to [11, 14]). Given a compact set $K$ contained in an open subset $U$ of $\mathbb{R}^N$, the $H^1$-capacity of the pair $(K, U)$ is defined as

$$\text{Cap}_2(K, U) := \inf \left\{ \int_U |\nabla \varphi|^2 : \varphi \in C_c^\infty(U), \varphi \geq 1 \text{ on } K \right\}.$$ 

If $U'$ is an open subset of $U$, the corresponding $H^1$-capacity is defined as

$$\text{Cap}_2(U', U) := \sup \left\{ \text{Cap}_2(K, U) : K \subset U', K \text{ compact} \right\},$$

and the definition is extended to a general set $E \subset U$ as follows:

$$\text{Cap}_2(E, U) := \inf \left\{ \text{Cap}_2(U', U) : U' \text{ open}, E \subset U' \subset U \right\}.$$ 

A set $E \subset \mathbb{R}^N$ is said to be of $H^1$-capacity zero if $\text{Cap}_2(E \cap U, U) = 0$ for any open set $U \subset \mathbb{R}^N$. We also recall that the precise representative of a function $f \in L^1_{loc}(\mathbb{R}^N)$ is defined as:

$$f^*(x) := \begin{cases} \lim_{r \to 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} f \text{ if this limit exists}, \\ 0 \text{ otherwise.} \end{cases}$$

A function $f : \Omega \to \mathbb{R}^N$ is $H^1$-quasi continuous if for each $\epsilon > 0$ there is an open set $U \subset \Omega$ such that $\text{Cap}_2(U, \Omega) < \epsilon$ and $f|_{\Omega \setminus U}$ is continuous. It is known that the precise representative of a Sobolev function $f \in H^1_{loc}(\Omega)$ is $H^1$-quasi continuous (see [11], p.160). We can now state the following version of the strong maximum principle due to Ancona [3] and Brezis, Ponce [4, Corollary 4]:

**Theorem 6.1** (Strong Maximum Principle) Let $O \subset \mathbb{R}^N$ be a non-empty open connected bounded set and $a \in L^1_{loc}(O)$ with $a \geq 0$. Assume $u \in L^1(O)$ is a quasi-continuous function such that $\Delta u$ is a Radon measure on $O$ and satisfies

$$-\Delta u + au \geq 0 \text{ in } D'(O), \quad au \in L^1_{loc}(O), \quad u \geq 0 \text{ a.e. in } O.$$ 

Then either $u \equiv 0$, or else the $H^1$-capacity of the nodal set $u^{-1}(0)$ is zero.

Thanks to this result we can obtain the following information on the nodal set of functions solving problem (P):

**Proposition 6.2** (Positivity of extremals) Let $p \geq 2$ and assume $V$, $w \in L^1_{loc}(\Omega)$ are such that the minimizing problem (P) admits a solution. If $\Phi$ is the precise representative of a minimizer, then its nodal set $\Phi^{-1}(0)$ has $H^1$-capacity zero and in particular $\pm \Phi > 0$ a.e. in $\Omega$.

**Proof** Since $\Phi$ minimizes the Rayleigh quotient, it satisfies (1.5). Using $\Phi^+, \Phi^-$ as test functions in (1.5), we easily see that $\int_{\Omega} w|\Phi^\pm|^p > 0$ and also

$$R(\Phi^\pm) = R(\Phi) \left( \frac{\int_{\Omega} w|\Phi|^p}{\int_{\Omega} w|\Phi^\pm|^p} \right)^{\frac{2}{p} - 1} \leq R(\Phi) \quad (\text{since } p \geq 2).$$

Therefore $\Phi^\pm$ are also minimizer of (P) and satisfy in the weak sense:

$$-\Delta \Phi^\pm + V \Phi^\pm = \lambda(\Phi^\pm) w |\Phi^\pm|^{p-2} \Phi^\pm, \quad \Phi^\pm \geq 0.$$  

(6.1)
Since $V|\Phi^\pm|^2 \in L^1(\Omega)$ and $V \in L^1_{loc}(\Omega)$ we get $V\Phi^\pm \in L^1_{loc}(\Omega)$ (by Cauchy-Schwarz inequality). Similarly we have $\lambda(\Phi^\pm)w|\Phi^\pm|^\frac{p-2}{2} \in L^1_{loc}(\Omega)$. Hence Eq. 6.1 shows that $\Delta \Phi^\pm \in L^1_{loc}(\Omega)$ and is in particular a Radon measure on $\Omega$. Also we have (in the sense of distribution)
\[-\Delta \Phi^\pm + (V(x) + \lambda(\Phi^\pm)w)\Phi^\pm \geq 0 \text{ in } D'(\Omega) \quad \text{and} \quad \Phi^\pm \geq 0.
\]
There are two cases: either $\Phi^+ \not= 0$, or $\Phi^+ \equiv 0$. If $\Phi^+ \not= 0$ by the Strong Maximum Principle (Theorem 6.1), we deduce $\Phi^+ > 0$ q.e. in $\Omega$ (i.e. outside a set of $H^1$-capacity zero). Therefore $\Phi > 0$ q.e. in $\Omega$, and in particular $\Phi > 0$ a.e. in $\Omega$. If $\Phi^+ \equiv 0$, then we necessarily have $\Phi^- \not= 0$ (since $\Phi \not= 0$). Then arguing in the same way we deduce $\Phi < 0$ q.e. in $\Omega$. \hfill $\Box$

In the linear case $p = 2$, problem (P) admits a unique solution and the minimizers of problem (P) are completely characterized by the sign. More precisely,

**Proposition 6.3** (Uniqueness) Assume $p = 2$, and $V, w \in L^1_{loc}(\Omega)$ are such that the minimizing problem (P) admits a solution.

(a) Up to a constant multiple, the minimizer is unique.

(b) If there exists $(\lambda, \varphi) \in (0, \infty) \times D^{1,2}_0(\Omega)$ satisfying
\[-\Delta \varphi + V\varphi = \lambda w\varphi, \quad \varphi \geq 0, \quad \varphi \not= 0, \quad \text{Eq. 6.2} \]
then $\lambda = \lambda_1^+(V, w)$, where $\lambda_1^+(V, w)$ is defined by (1.11).

**Proof** (a) We can prove the uniqueness of the minimizer by adapting the continuity argument of [22] as follows. Let $\Phi_1, \Phi_2$ be two non-negative minimizers. Since $\Phi_1 + \alpha \Phi_2$ is also a minimizer (for any $\alpha \in \mathbb{R}$), by proposition 6.2 the function $\Phi_1 + \alpha \Phi_2$ has constant sign and vanishes only on a set of zero measure. So consider the sets
\[A^+ := \{\alpha \in \mathbb{R} : \Phi_1 + \alpha \Phi_2 > 0 \text{ a.e.} \}, \quad A^- := \{\alpha \in \mathbb{R} : \Phi_1 + \alpha \Phi_2 < 0 \text{ a.e.} \}.
\]
These sets are clearly disjoint and the arguments in [19] shows that they are non-empty and open (an obvious fact if $\Phi_1, \Phi_2$ are continuous). Indeed since $\Phi_1 > 0$ a.e. we have $0 \in A^+$. To see that $A^- \not= \emptyset$ we note the existence of $m, n \in \mathbb{N}$ such that the set
\[E_{m,n} := \{x \in \Omega : \Phi_1 < m, \Phi_2 > n^{-1}\}
\]
has positive measure. Therefore by choosing $\alpha_0 < -mn$, we get
\[\Phi_1(x) + \alpha_0 \Phi_2(x) < 0 \text{ a.e. } x \in E_{m,n} \quad \text{and} \quad |E_{m,n}| > 0.
\]
Hence $\Phi_1 + \alpha_0 \Phi_2$ is negative on a set of positive measure. Proposition 6.2 implies that $\Phi_1 + \alpha_0 \Phi_2 < 0$ a.e. in $\Omega$ and so $\alpha_0 \in A^- \not= \emptyset$. A similar argument shows that $A^+$ and $A^-$ are open. Since $\mathbb{R}$ is connected we deduce that $\mathbb{R} \setminus (A^+ \cup A^-) \not= \emptyset$. Hence there exists $\alpha_0$ such that $\Phi_1 + \alpha_0 \Phi_2$ vanishes on a set of positive measure and therefore must vanish a.e. in $\Omega$ (by Proposition 6.2). This shows that $\Phi_1$ and $\Phi_2$ are linearly dependent.

(b) From the definition (1.11), we immediately see that $\lambda \geq \lambda_1^+(V, w)$. To show that equality holds we sketch the arguments of Cuesta [9, Prop. 4.1] and [19]. Let $\Phi$ be a non-negative minimizer of Problem (P) and define for each $k \geq 0$ the truncated function:
\[\Phi_k(x) := \begin{cases} k & \text{if } \Phi(x) \geq k, \\
\Phi(x) & \text{if } \Phi(x) \in [0, k).
\end{cases}
\]
Since $\Phi_k \in L^\infty(\Omega) \cap D_0^{1,2}(\Omega)$, both $\Phi_k$ and $\Phi_k^2/\varphi + \epsilon$ are legitimate trial functions in (1.5) (with $p = 2$). We therefore get

$$
\int_{\Omega} |\nabla \Phi_k|^2 - \int_{\Omega} \nabla \varphi \nabla \left( \frac{\Phi_k^2}{\varphi + \epsilon} \right) = \int_{\Omega} \lambda_1^+(V, w) w \Phi_k - \int_{\Omega} \lambda w \varphi \frac{\Phi_k^2}{\varphi + \epsilon}.
$$

(6.3)

But a straight calculation shows that the following “Picone’s identity” holds:

$$
|\nabla \Phi_k|^2 - \nabla \varphi \left( \frac{\Phi_k^2}{\varphi + \epsilon} \right) = \nabla \Phi_k - \left( \frac{\Phi_k}{\varphi + \epsilon} \right)^2 \nabla \varphi.
$$

(6.4)

By plugging (6.4) in (6.3) and using the fact that the set $\{\varphi = 0\}$ is of measure zero (by Theorem 6.1), we deduce

$$
0 \leq \int_{\{\varphi > 0\}} \left| \nabla \Phi_k - \left( \frac{\Phi_k}{\varphi + \epsilon} \right) \nabla \varphi \right|^2 = \int_{\{\varphi > 0\}} \left\{ \lambda_1^+(V, w) w \Phi_k - \lambda w \varphi \frac{\Phi_k^2}{\varphi + \epsilon} \right\}.
$$

(6.5)

Now, letting $\epsilon \to 0$ and $k \to \infty$ in (6.5) and applying Lebesgue dominated Theorem to the right hand side, we get $0 \leq (\lambda_1^+(V, w) - \lambda) \int_{\Omega} w \Phi_k^2$. Since $\int_{\Omega} w \Phi^2 > 0$, we conclude $\lambda \leq \lambda_1^+(V, w)$. Therefore $\lambda = \lambda_1^+(V, w)$. \qed

Remark 6.4  
(a) Above proposition is stated in [19] for $w \in L^\infty(\Omega)$ and $V \equiv 0$, but the arguments hold whenever the problem (P) admits a minimizer. For quasilinear problems of the type $-\Delta_p u = \lambda w(x)|u|^{p-2}u$ ($p > 2$), an analogue of above proposition holds but the proof is more involved (see [17,20]).

(b) The minimizers to problem (P) when $p > 2$ have constant sign (by proposition 6.2), but may not be unique (up to a constant multiple). For example in [16,23] it is observed that in a ball such minimizers with $p > 2$ are not necessarily radial and so cannot be multiple of each others.

(c) Part (b) of Proposition 6.3 does not hold when $p > 2$. Indeed if $(\lambda_0, \varphi_0)$ solves (6.2) with $\lambda_0 > 0$, then $(\lambda_0/a)^{1/p} \varphi_0$ yields a non-negative solution to (6.2) for any $\lambda > 0$.

7 Application to a bifurcation problem

In this section, we give an application of Theorem 1.2 to study bifurcation for Eq. 1.12. First let us formulate our problem in a suitable functional framework. Note that as a consequence of Lax-Milgram theorem and Sobolev-Lorentz embedding, the mapping $(-\Delta)^{-1} : [L(2^*, 2)]^* \to D_0^{1,2}(\Omega)$ is continuous. Hence by defining

$$
L : D_0^{1,2}(\Omega) \to D_0^{1,2}(\Omega) \quad u \mapsto (-\Delta)^{-1}(a(x)u) \\
H : \mathbb{R} \times D_0^{1,2}(\Omega) \to D_0^{1,2}(\Omega) \quad (\lambda, u) \mapsto (-\Delta)^{-1}(\lambda b(x)r(u))
$$

(7.1)

the problem (1.12) is equivalent to solving

$$
u = \lambda L(u) + H(\lambda, u), \quad u \in D_0^{1,2}(\Omega).
$$

(7.2)

Under hypotheses (H1–H2), we will show that $H$ and $L$ satisfy all the requirements for applying the global bifurcation Theorem of Rabinowitz [25].
**Proposition 7.1** Assume (H1–H2) hold. Then the mappings \( L \) and \( H \) defined by (7.1) are continuous, compact and furthermore

\[
\lim_{\|u\| \to 0} \frac{\|H(\cdot, u)\|}{\|u\|} = 0. \tag{7.3}
\]

**Proof** Using (H1), we can write \( r(s) = r_1(s) + r_2(s) \) with \( r_1, r_2 \in C^0(\mathbb{R}) \) satisfying

\[
\lim_{s \to 0} \frac{r_1(s)}{s} = 0, \quad |r_1(s)| \leq C_0 |s|, \quad |r_2(s)| \leq \begin{cases} 0 & \text{if } p \in [1, 2], \\ C_0 |s|^{p-1} & \text{if } p > 2. \end{cases} \tag{7.4}
\]

Let us consider the following maps defined from \( D_0^{1,2}(\Omega) \) to \([L(2^*, 2)]^\prime\)

\[
\tilde{L}(u) = a(x)u, \quad \tilde{H}(u) = b(x)r(u), \quad \tilde{H}_i(u) = b(x)r_i(u) \quad (i = 1, 2).
\]

By applying Lemma 5.1 we deduce that \( \tilde{L}, \tilde{H}_1, \tilde{H}_2 \) are continuous and compact. In particular \( \tilde{H} = \tilde{H}_1 + \tilde{H}_2 \) is also continuous and compact. Now it is easy to see that the continuity and compactness of \( H \) and \( L \) follow from those of \( \tilde{L} \) and \( \tilde{H} \).

In order to prove (7.3), it is enough to prove

\[
\lim_{\|u\| \to 0} \frac{\| \tilde{H}_1(u) \|_{((2^*, 2), 2)} }{\|u\|} = \lim_{\|u\| \to 0} \frac{\| \tilde{H}_2(u) \|_{((2^*, 2), 2)} }{\|u\|} = 0. \tag{7.5}
\]

Concerning \( \tilde{H}_2 \), we simply note for \( p > 2 \) that

\[
\|b r_2(u) \chi_F \|_{((2^*, 2), 2)} \leq C_0 \| b |u|^{p-1} \chi_F \|_{((2^*, 2), 2)} \leq 2^* C_0 \| b \|_{(\tilde{p}, \infty)} \|u|^{p-1} \|_{(2^*, 2)} \leq 2^* C_0 \| b \|_{(\tilde{p}, \infty)} \|u|^{p-1} \|_{(2^*, 2)} \leq 2^* C_0 C_{\tilde{S}}^{p-1} \| b \|_{(\tilde{p}, \infty)} \| \nabla u \|_2^{p-1}
\]

and therefore (7.5) holds for \( \tilde{H}_2 \).

To prove the property (7.5) for \( \tilde{H}_1 \), let us fix \( \varepsilon > 0 \). Using (7.4), we get \( s_0, C_1 > 0 \) depending only on \( \varepsilon \) such that

\[
|r_1(s)| \leq \frac{\varepsilon |s|}{2^* C_{\tilde{S}} |b|_{(\tilde{p}, \infty)}} \forall |s| < s_0, \quad \text{and} \quad |r_1(s)| \leq C_1 |s|^{2^*-1} \forall |s| \geq s_0. \tag{7.6}
\]

For each \( u \in D_0^{1,2}(\Omega) \), let us introduce the sets

\[
E := \{ x \in \Omega : |u(x)| < s_0 \} \quad \text{and} \quad F := \{ x \in \Omega : |u(x)| \geq s_0 \}.
\]

Using triangle inequality, we obtain

\[
\|br_1(u)\|_{((2^*, 2), 2)} \leq \|br_1(u)\chi_E\|_{((2^*, 2), 2)} + \|br_1(u)\chi_F\|_{((2^*, 2), 2)}. \tag{7.7}
\]

Let us estimate each term in the right hand side of (7.7) using Hölder’s inequality and Sobolev-Lorentz embedding. The first term is handled as follows:

\[
\|br_1(u)\chi_E\|_{((2^*, 2), 2)} \leq 2^* \|b\|_{(\frac{p}{2}, \infty)} \|r_1(u)\chi_E\|_{(2^*, 2)} \leq \frac{\varepsilon}{C_{\tilde{S}}} \|u\|_{(2^*, 2)} \leq \varepsilon \|\nabla u\|_2. \tag{7.8}
\]

To estimate \( \|br_1(u)\chi_F\|_{((2^*, 2), 2)} \) write \( b = b_\varepsilon + (b - b_\varepsilon) \) with

\[
b_\varepsilon \in C_0^\infty(\Omega), \quad \|b - b_\varepsilon\|_{(\frac{p}{2}, \infty)} < \frac{\varepsilon}{2^* C_{\tilde{S}} C_0}. \tag{7.9}
\]
Then by using (7.4), (7.6), (7.9), the second term in (7.7) is estimated as follows
\[
\|br_1(u)\chi_F\|((2^*)',2) \leq C_1 \|b_\varepsilon|u|^{2^*-1}\chi_F\|((2^*)',2) + C_0 \|(b-b_\varepsilon)|u|\chi_F\|((2^*)',2) \\
\leq C_1 \|b_\varepsilon\|_\infty \|u|^{2^*-1}\|((2^*)',2) + 2^*C_0 \|b-b_\varepsilon\|((2^*)',2) \\
\leq C \|b_\varepsilon\|_\infty \|\nabla u|^{2^{*}-1} + \varepsilon\|\nabla u\|_2, \tag{7.10}
\]
with \( C := C_1C_S^{2^*-1} \). From (7.7), (7.8) and (7.10), we deduce that (7.5) holds for \( \tilde{H}_1 \).

Now we are in a position to prove the existence of a global branch of solutions for problem (1.12):

**Proof of Theorem 1.3** Theorem 1.2 shows that \( \lambda_1^+ (0, a) \) is an eigenvalue of multiplicity one. This fact with Lemma 7.1 shows that all the conditions of the Rabinowitz global bifurcation theorem (see [25]) are satisfied, and the proof follows.

**References**