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A New Least Squares Based Finite Difference Scheme for Compressible Flows

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ABSTRACT *Inspired by the Flux Difference Splitting schemes, a new upwind Least Squares Finite Difference Scheme (LSFD-U), operating on a global stencil of grid points has been developed. This solution procedure is equally applicable to Flux Vector Splitting schemes also. The class of generalised finite difference schemes based on the least squares procedure require only local connectivity information pertaining to a cloud of grid points around any given node. The results obtained are very encouraging*

1 Introduction

A class of generalised finite difference schemes, wherein the derivatives of the fluxes appearing in the conservation laws are approximated using the least squares procedure, are becoming more and more important in the computation of flows past realistic configurations. The Least Squares Kinetic Upwind Method (LSKUM) is one such strategy based on the Boltzmann equation of Kinetic Theory of Gases[1,2]. The other option is to exploit the Flux Difference Splitting framework in constructing solution procedures based on the method of least squares[3]. The procedure thus developed is equally applicable to Flux Vector Splitting schemes as well. In the present work we discuss the latter strategy.

2 Upwind Least Squares Finite Difference Scheme (LSFD-U)

In this section, we present the new method as applied to 2-D Euler equations given by,

$$U_t + F_x + G_y = 0 \tag{1}$$

Figure 1 gives a typical 2-D stencil of grid points. Here I_j represents the fictitious interface placed at the center of the line connecting the point o (where the solution is sought) and its j^{th} neighbour. (n_{x_j}, n_{y_j}) represents the unit vector along oj. The flux along oj is given by,

$$F_{\uparrow_j} = F n_{x_j} + G n_{y_j} \tag{2}$$

The flux difference between points o and the fictitious interface along oj given by,

$$\Delta F_{\uparrow_j} = F_{\uparrow_{I_j}} - F_{\uparrow_o} \tag{3}$$

can be either directly computed making use of a flux difference splitting scheme or indirectly computed making use of any flux vector splitting scheme which would correspond to some linearisation procedure. Making use of Taylor series we have,

$$\Delta F_{\uparrow_j} = \left(\vec{\nabla} F_o \cdot \Delta \vec{r}_{I_j}\right) n_{x_j} + \left(\vec{\nabla} G_o \cdot \Delta \vec{r}_{I_j}\right) n_{y_j} \tag{4}$$

Now our job reduces to recovering the information regarding the gradients of the 2-D fluxes namely $\vec{\nabla} F_o$ and $\vec{\nabla} G_0$ from the many 1-D flux difference terms given in equation 4. The derivatives F_x and G_y thus recovered would eventually be used in the state update formula. Equation

4 represents an over determined system of equations and it appears that the straight forward way to obtain the gradients of the fluxes is to minimise ΣE_j^2 with respect to the derivatives of the fluxes, where,

$$E_j = \Delta F_{\dagger_j} - \left(\vec{\nabla} F_o \cdot \Delta \vec{r}_{I_j}\right) n_{x_j} - \left(\vec{\nabla} G_o \cdot \Delta \vec{r}_{I_j}\right) n_{y_j} \tag{5}$$

Unfortunately, simple algebra would demonstrate that such a procedure leads to a singular system. To circumvent this problem we suggest the following two methodologies.

2.1 Method I

In this method, following Ghosh and Deshpande [1], the local co-ordinate axes is rotated from (x,y) to (ξ,η), so that one of them, ξ, coincides with the the tangent to the local streamline. If $\widetilde{F}$ and $\widetilde{G}$ represent the fluxes along the new coordinate directions ξ and η respectively, the gradient information pertaining to $\widetilde{G}$ can be evaluated using a global stencil based on the conventional least squares procedure[1]. This leaves us with a non-singlar system involving the gradient of $\widetilde{F}$, which in turn can be determined making use of the upwind flux difference, as described in section 2.

2.2 Method II

In this method, we locally rotate the co-ordinate axes from (x,y) to (ξ, η), so that $\widetilde{F}_\eta + \widetilde{G}_\xi = 0$. It can be easily shown that a co-ordinate system rotated by an angle β, given by,

$$\beta = \frac{1}{2} \arctan \left[\frac{F_y + G_x}{F_x - G_y}\right] \tag{6}$$

would satisfy this condition. This leaves us with a non-singular system with two unknowns, $\widetilde{F}_\xi$ and $\widetilde{G}_\eta$, which are determined using the method of section 2. The gradients of the fluxes used in the estimation of β are obtained using the least squares procedure [3] making use of a global stencil.

3 Stability Analysis

It is interesting to analyse the stability of a numerical scheme applied to a non-uniform grid, and find the effect of grid distortion on stability. The present LSFD-U has been applied to the 1-D linear scalar convection equation

$$u_t + a u_x = 0, \quad a > 0 \tag{7}$$

on a non-uniform grid. We define a grid distortion parameter, $\epsilon_j = \frac{\Delta_+ x_j}{\Delta_- x_j}$, where $\Delta_+ x_j = x_{j+1} - x_j$ and $\Delta_- x_j = x_j - x_{j-1}$. The first order LSFD-U can be written as

$$u_j^{n+1} = u_j^n - \left(\frac{2\lambda_j}{1+\epsilon_j^2}\right)\left(u_j^n - u_{j-1}^n\right), \quad \lambda_j = \frac{a\Delta t}{\Delta_- x_j} \tag{8}$$

von Neumann analysis of the above scheme leads to the following expression for the amplification factor,

$$g(\lambda_j, \beta_j, \epsilon_j) = 1 - \left(\frac{2\lambda_j}{1+\epsilon_j^2}\right)\left(1 - e^{-i\beta_j}\right), \quad \beta_j = 2\pi\xi\Delta_- x_j \tag{9}$$

where β_j is the dimensionless wave-number. Imposing the condition $|g| < 1$ leads to a CFL-type condition on λ_j given by

$$\lambda_j \leq \frac{1}{2}(1 + \epsilon_j^2) \tag{10}$$

It can be shown that a modified partial differential equation analysis also leads to the same condition for stability. Figure 2 shows the plots of the amplification factor for a given λ and for various values of ϵ. From the plots and from equation 10, it is amply clear that the constraint on the permissible timestep is enormously relaxed for larger and larger values of ϵ. Also, from equation 10, it is clear that the present framework is stable for $\lambda \leq 0.5$ for all values of ϵ. The observations emphatically bring out the fact that the mesh distortion can have considerable effect on the stability of any given scheme.

4 Results and Discussions

The results for NACA-0012 for $M_\infty = 0.8$ and $\alpha = 1.25$, and for a ramp in a channel for an inflow Mach number of 2.0 are presented here. The computations were done on an unstructured grid of 4733 points for NACA-0012 and 3231 points for the ramp. The flux calculations have been done using KFVS [4]. High resolution is obtained using a linear reconstruction procedure [5]. Non-physical oscillations in the solution are suppressed using Venkatakrishnan limiter [6]. Fig.[3] shows the mach contours and the pressure distribution for the transonic case with a weak shock on the lower surface, which is also captured in the solution. The pressure distribution compares very well with that obtained from a cell-vertex finite-volume computation. Fig.[4] shows the pressure contours for supersonic flow over a ramp. Both the methods capture the shock and its reflection very accurately.

5 Conclusions

A new upwind least squares finite difference method (LSFD-U) has been developed. This method, which can be considered as a grid-free method, operates on a global stencil of grid points. The linear stability of such a scheme on distorted meshes is established. It is also demonstrated that the accuracy of such a scheme can be easily improved using a reconstruction procedure. The results obtained using the two variations of LSFD-U are very encouraging.

References

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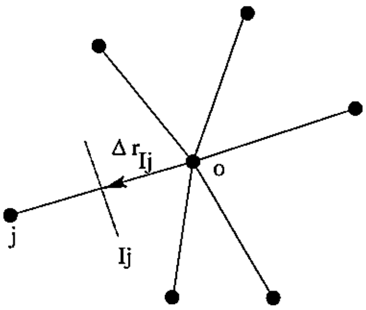

Fig.1 Typical 2-D grid distribution for LSFD-U

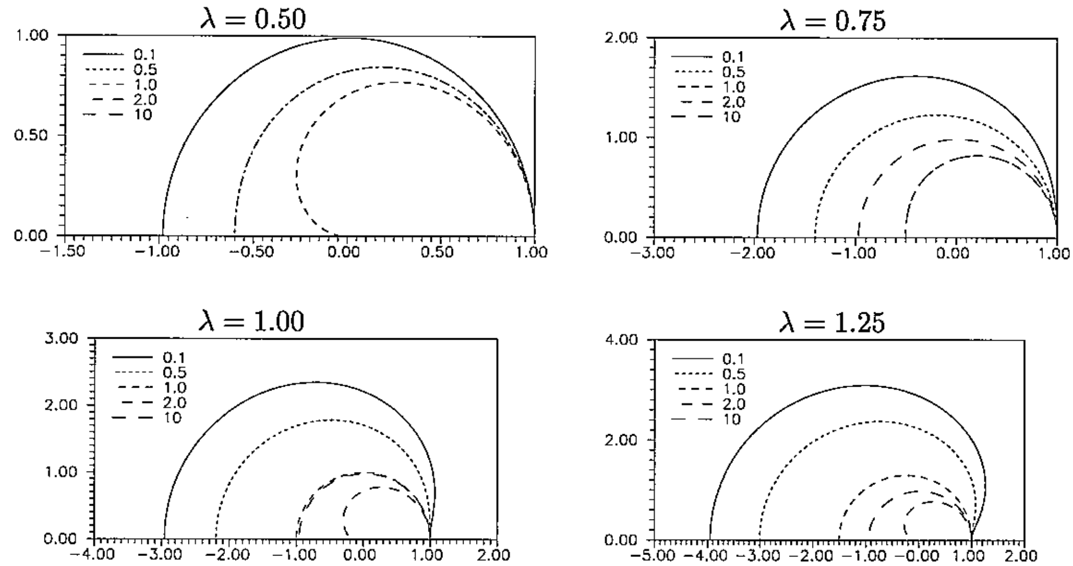

Fig. 2 Polar plots of amplication factor for first order LSFD-U. The numbers in the figure indicate the grid distortion parameter, ϵ

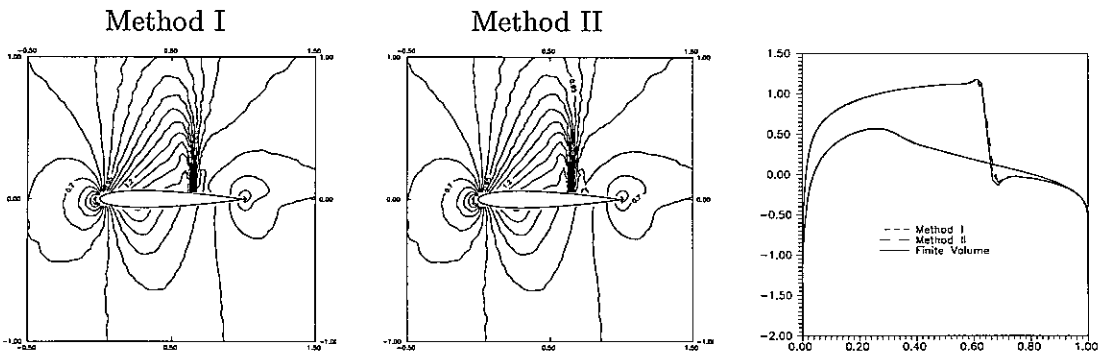

Fig.3 Mach contours for $M_\infty = 0.8$ and $\alpha = 1.25$ and corresponding pressure distribution.

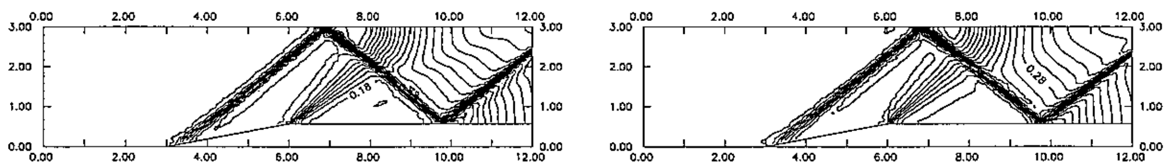

Fig. 4 Pressure contours for flow over a ramp using Method I and Method II, $M_\infty = 2.0$.