Kinetic Meshless Method for Conservation Laws

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Summary

A new meshless approximation for hyperbolic conservation laws has been recently proposed by the authors [1] which uses a local Taylor series representation of the unknown. In the present work, we propose a variant of the above method which uses the shape function approach as in FEM together with a least squares approximation. This approach has the advantage of allowing us to use non-standard basis functions like those of the radial-type. The resulting scheme looks like a finite volume update equation. Numerical results for the Euler equations in 2-D will be presented.

Introduction

We present a meshless approximation called the Kinetic Meshless Method (KMM), for the numerical solution of hyperbolic conservation laws that can be obtained by taking moments of a Boltzmann-type transport equation. The meshless formulation requires the domain discretization to have very little topological information; a distribution of points in the domain together with local *connectivity* information is sufficient. For each node i, the connectivity consists of a set of nearby nodes $\mathcal{C}_i$ which are used to evaluate the spatial derivatives appearing in the conservation law. The derivatives are obtained using a local approximation in terms of a set of basis functions and a modified form of the least squares approximation known as the *dual least squares approximation*.

Meshless methods for compressible flows where the governing equations are hyperbolic have been developed by Ghosh and Deshpande [2], Batina [3], Morinishi [4], Balakrishnan [5] and Lohner [6], all of which make use of least squares in one form or the other. They differ from one another in the way they introduce upwinding. Even though we use shape functions as in FEM, we do not make use of an integral formulation. A similar approach is taken by Morinishi [4] and Lohner et. al. [6] but their way of upwinding is ad hoc without any relation to physics and is more like an artificial dissipation approach. The LSKUM [2] of Ghosh and Deshpande is obtained by taking moments of a meshless upwind discretization of the Boltzmann equation (moment-method strategy) and the same approach is taken in the present work. Note that since the Boltzmann equation is a scalar transport equation, its upwind directions are clearly known. The present method differs from LSKUM in that it uses a different upwind discretization of the Boltzmann equation which avoids stencil splitting and hence requires fewer points in each connectivity.

Kinetic Representation

We are interested in the numerical solution of hyperbolic conservation laws which can be obtained by taking suitable moments of a Boltzmann-type transport equation. The system of conservation laws is written in differential form as

$$\frac{\partial U}{\partial t} + \sum_{\alpha=1}^{d} \frac{\partial F^\alpha}{\partial x^\alpha} = 0, \;\; U : (\vec{x},t) \to \mathcal{U}_{ad} \subset \mathbb{R}^p, \;\; F^\alpha : \mathcal{U}_{ad} \to \mathbb{R}^p \tag{1}$$

where U is the vector of conserved variables, $\mathcal{U}_{ad}$ is the set of physically admissible values of U, and F^α are the flux vectors. We assume that the system of equations (1) has a kinetic representation, that is, it can be obtained by taking moments of

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{d} v_\alpha \frac{\partial f}{\partial x^\alpha} = J(f) \tag{2}$$

where the moments are defined as

$$\langle \cdot \rangle = \int_{\mathbb{R}^d \times \mathbb{R}_+} \Psi(\vec{v}, I)(\cdot) \mathrm{d}\vec{v}\mathrm{d}I \tag{3}$$

Equation (1) is obtained by taking moments of (2) provided

$$\langle f \rangle = U, \;\; \langle v_\alpha f \rangle = F^\alpha, \;\; \langle J(f) \rangle = 0 \tag{4}$$

The variable I is sometimes necessary to get the correct moment closure and represents non-translational degrees of freedom. The kinetic representation gives a particle picture of the system under consideration, where the particles are moving with velocity $\vec{v}$, with the function $f(\vec{v}, I; \vec{x}, t)$ giving the statistical distribution of molecular velocity. The macroscopic behaviour is obtained by taking average/moments of the microscopic behaviour. If (1) has the property that $F^\alpha(U) = A^\alpha(U)U$, with all the A^α having only real eigenvalues and a complete set of eigenvectors, then it has a kinetic representation. It is also enough if the system of equations (1) has a convex entropy, as shown by Harten, Lax and Van Leer [7]. Assuming that such a situation exists, we first obtain an *upwind* discretization of (2) and take moments to obtain a discretization for the conservation law (1). Note that we do not actually solve equation (2) but merely use it as an intermediary to obtain a discretization of (1).

Shape Functions and Dual Least Squares Approximation

Let $p_m : \mathbb{R}^d \to \mathbb{R}$, $1 \le m \le M$ be a set of basis functions and we approximate the unknown function f in a small neighbourhood of $\vec{x} = \vec{x}_i$ in the form

$$f(\vec{x}) \approx P(\vec{x}; \vec{x}_i) = \sum_{m=1}^{M} a_{im} p_m(\vec{x} - \vec{x}_i) \tag{5}$$

where we have shifted the origin to $\vec{x}_i$. Let $\vec{x}_{ij} = (\vec{x}_i + \vec{x}_j)/2$ be the midpoint of line segment $\vec{ij}$ and f_{ij} denote the value of f at $\vec{x}_{ij}$. The unknown coefficients a_{im} are determined by solving the following weighted minimization problem:

$$\text{minimize} \sum_{j \in C_i} w_{ij} \left[f_{ij} - P(\vec{x}_{ij}; \vec{x}_i) \right]^2, \quad \text{w.r.t. } a_{im}, \quad 1 \le m \le M \tag{6}$$

We can then obtain the derivative of f by differentiating equation (5)

$$\nabla f|_i \approx \nabla P(\vec{x}_i; \vec{x}_i) = \sum_{m=1}^{M} a_{im} \nabla p_m(\vec{0}) \tag{7}$$

In the present work the weight function is chosen as $w_{ij} = |\vec{x}_i - \vec{x}_j|^{-2}$. A more convenient form of the above expression is

$$\text{Dual Least Squares:} \;\; \left. \frac{\partial f}{\partial x^\alpha} \right|_i = \sum_{j \in C_i} \mu_{ij}^\alpha f_{ij} \tag{8}$$

Let us assume that for any $c \neq 0$, the shape functions have the following *scaling property*:

$$\vec{\xi} = c\vec{x} \implies \nabla_{\vec{\xi}} P(\vec{\xi}_i; \vec{\xi}_i) = \frac{1}{c} \nabla_{\vec{x}} P(\vec{x}_i; \vec{x}_i) \tag{9}$$

Then we can use the solution of the dual least squares problem to obtain another formula for the derivative which uses nodal values of f instead of the midpoint values. This is given by

$$\text{Least Squares: } \left.\frac{\partial f}{\partial x^\alpha}\right|_i = \frac{1}{2}\sum_{j\in C_i} \mu_{ij}^\alpha f_j \tag{10}$$

The idea of using mid-point values f_{ij} instead of the nodal values has been used previously by Morinishi [4] and Balakrishnan [5].

Kinetic Meshless Method

The KMM is built on the dual least squares approximation described in the last section and the moment-method strategy. The advantage of the dual least squares is that the value of f_{ij} is not known a priori, and we can introduce upwinding into the scheme by defining f_{ij} in a consistent and upwind manner. Substituting the dual least squares estimate given by equation (8) into the Boltzmann equation and assuming that the collision term is identically zero, we obtain

$$\frac{\mathrm{d}f_i}{\mathrm{d}t} + \sum_\alpha \sum_{j\in C_i} \mu_{ij}^\alpha v_\alpha f_{ij} = 0 \tag{11}$$

Taking moments we obtain the semi-discrete form of the update equation for the conservation law at node i

$$\frac{\mathrm{d}U_i}{\mathrm{d}t} + \sum_\alpha \sum_{j\in C_i} \mu_{ij}^\alpha F_{ij}^\alpha = 0 \tag{12}$$

where $F_{ij}^\alpha = \langle v_\alpha f_{ij}\rangle$. In order to upwind-bias the approximation given by equation (12), we construct f_{ij} from the particle distribution functions of nodes i and j, depending on whether the particle velocity has a component towards node i or towards node j. If V is a complete set of macroscopic variables which uniquely define f, then this approximation is given by

$$f_{ij}(\vec{v},I) = \begin{cases} f(\vec{v},I;V_{ij}^+), & \vec{v}\cdot\hat{e}_{ij} \geq 0 \\ f(\vec{v},I;V_{ij}^-), & \vec{v}\cdot\hat{e}_{ij} \leq 0 \end{cases} \tag{13}$$

where $\hat{e}_{ij}$ is the unit vector which is directed from node i to node j. The $V_{ij}^\pm$ must be atleast second-order accurate interpolants [4,5] of V from $\vec{x}_i$, $\vec{x}_j$ to the midpoint of $\vec{ij}$. This can be achieved by using linear interpolation and a MUSCL-type limiting [8]. Assuming that the scaling property holds, the derivatives of V are calculated using the least squares approximation given by equation (10), which avoids the solution of another least squares problem for ∇V.

Similarity with Finite Volume Method

We can write equation (12) in a form that resembles the finite volume method. For this we define

$$\mu_{ij} = \sqrt{\sum_\alpha \mu_{ij}^\alpha \mu_{ij}^\alpha}, \quad n_{ij}^\alpha = \frac{\mu_{ij}^\alpha}{\mu_{ij}} \tag{14}$$

We also define a flux function

$$\mathcal{F}_{ij} = \sum_\alpha n_{ij}^\alpha F_{ij}^\alpha \tag{15}$$

In terms of these we can write equation (12) as

$$\frac{\mathrm{d}U_i}{\mathrm{d}t} + \sum_{j \in C_i} \mu_{ij} \mathcal{F}_{ij} = 0 \tag{16}$$

whereas a finite volume discretization will be of the form

$$\frac{\mathrm{d}U_i}{\mathrm{d}t} + \sum_{j \in C_i} \frac{S_{ij}}{V_i} \mathcal{G}_{ij} = 0 \tag{17}$$

provided that the points in C_i define a finite volume around node i, and $\mathcal{G}_{ij}$ is a numerical flux function. The update equations (16) and (17) have a similar structure if we identify μ_{ij} as the ratio of a generalized edge-length to a generalized volume. The conservation property of finite volume methods is due to the following flux-balance condition: $\mathcal{G}_{ij} + \mathcal{G}_{ji} = 0$. We do not know at present whether an analogous property holds for equation (16).

Numerical Results

We have applied KMM to solve the 2-D Euler equations governing inviscid compressible flows. These are obtained by taking moments of the Boltzmann equation together with the Maxwellian distribution function. For the shape functions we take linear polynomials and for interpolation, the primitive variables $V = [\rho, \quad \vec{u}, \quad p]^T$ are used. The wall boundary conditions are based on the strong formulation [9], while the outer boundary condition for subsonic and transonic flows is based on the point vortex model of Thomas and Salas. Time integration is performed using a 3-stage scheme of Shu and Osher. The point distribution is obtained from standard triangular grid generation techniques and the average number of points in the connectivity is six. Results for 2-D compressible flows will be presented.

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