Kinetic energy preserving and entropy stable finite volume schemes for compressible Euler and Navier-Stokes equations

Deep Ray and Praveen Chandrashekar
TIFR Center for Applicable Mathematics, Bangalore, India
Email: (praveen,deep)@math.tifrbng.res.in

1 Introduction

Due to their non-linear hyperbolic nature, solutions of Euler equations can be discontinuous with the presence of shocks or contact discontinuities. It is well known that discontinuous solutions can be non-unique and an additional entropy condition has to be imposed in order to select the unique weak solution. In the case of Euler equations, there is a natural entropy condition which comes from the entropy condition in thermodynamics which must also be satisfied by the numerical scheme. Additionally other global balance equations like that for the total kinetic energy must also be consistently approximated by the numerical solutions. The finite volume method requires the computation of the inviscid and viscous fluxes across the boundaries of the finite volumes. The design of these fluxes must incorporate the properties of the Euler/NS equations like entropy condition and kinetic energy preservation. There exists a vast library of numerical flux functions for the Euler equations and some of these like the Godunov scheme and kinetic scheme can be shown to satisfy the entropy condition. The popular Roe scheme [2] does not satisfy the entropy condition and can give rise to entropy violating shocks near sonic points. Various entropy fixes for Roe scheme have been proposed which involve preventing the numerical dissipation from vanishing at sonic points. Tadmor [4] proposed the idea of entropy conservative numerical fluxes which can then be combined with some dissipation terms using entropy variables to obtain a scheme that respects the entropy condition. This is a more mathematically rigorous approach to construct entropy stable schemes for conservation laws. For the Euler equations, Roe [3] proposed entropy conservative numerical fluxes which are augmented by Roe-type dissipation terms using entropy variables. These schemes do not suffer from entropy violating solutions that are observed in the original Roe scheme. However for strong shocks, even the first order schemes can produce oscillations indicating that the amount of numerical dissipation is not sufficient. Roe [3] proposed modifying the eigenvalues of the dissipation matrix which lead to non-oscillatory solutions. The modification of the eigenvalues is such that the amount of entropy production is of the correct order of magnitude for weak shocks.

Faithful representation of kinetic energy evolution is another desirable property of a numerical scheme [1]. This is important for direct numerical simulation of turbulent flows where the kinetic energy balance plays an important role in the evolution of turbulence. The essential feature for a numerical flux to correctly capture the kinetic energy balance is that the momentum flux should be of the form $f^m_{j+\frac{1}{2}} = \tilde{p}_{j+\frac{1}{2}} + \overline{u}_{j+\frac{1}{2}} f^\rho_{j+\frac{1}{2}}$ where $\overline{u}_{j+\frac{1}{2}} = (u_j + u_{j+1})/2$ and $\tilde{p}_{j+\frac{1}{2}}, f^\rho_{j+\frac{1}{2}}$ are any consistent approximations to the pressure and the mass flux. This scheme thus leaves most terms in the numerical flux unspecified and various authors have used simple averaging. These fluxes are however not entropy conservative, while on the other hand, the entropy conservative flux of Roe does not have the kinetic energy preservation property.

2 Novel kinetic energy and entropy stable fluxes

In the present work, we construct centered numerical fluxes for the Euler equations which are entropy conservative and also preserve kinetic energy. Two versions of the numerical flux are constructed, one of which is only approximately entropy consistent but has simpler expressions, while the second one is exactly entropy conservative but involves certain logarithmic averages that require more computations. Due to lack of upwinding, the schemes are not stable for discontinuous solutions and for NS equations on coarse meshes for which shocks may not be well resolved. They yield stable solutions for Navier-Stokes equations when used on very fine meshes for which the cell Peclet number is of the order of one and the physical viscosity is enough to stabilize the scheme. However for Euler equations and for NS equations on coarse meshes, the centered fluxes are unstable and must be augmented with dissipation terms. Firstly, we construct scalar dissipation terms using second and fourth order differences as in the JST scheme. The second order dissipation terms which are active near shocks are kinetic energy and entropy stable while the fourth order dissipation is active only in smooth regions of the flow. Secondly, we also use entropy variable based matrix dissipation flux similar to the Roe scheme which can be shown to lead to entropy generation. The eigenvalue modification of Roe [3] is used to compute strong shocks without oscillations. All the schemes are shown to give entropy consistent solutions in cases where the Roe scheme would give entropy violating shocks. The performance of the schemes is demonstrated on inviscid and viscous test cases involving shocks and contact discontinuities.

3 Numerical results

We present sample 1-D results for the Euler and Navier-Stokes equations using the new flux functions. The final presentation will include results for 2-D problems.

3.1 1-D Navier-Stokes shock structure

We compute the shock structure using the Navier-Stokes equations with the help of the entropy consistent flux and scalar artificial dissipation. The parameters defining the problem are: Mach number ahead of the shock is $M_1 = 1.5$, $\gamma = 5/3$, $\mathrm{Pr} = 2/3$ while the viscosity law is given by $\mu = \mu_1(T/T_1)^{0.8}$ where the subscript "1" denotes pre-shock conditions and $\mu_1 = 0.0005$. Figure (1) shows the solutions on $N = 50, 100, 200$ cells. On the coarse mesh, the stress and heat flux cannot be computed accurately since there are too few points inside the shock region, but the solution is still non-oscillatory. On the finer meshes, the scheme is able to compute the shock structure with good accuracy.

3.2 Modified Sod test case

This is a shock tube problem with the left state being $(\rho, u, p) = (1.0, 0.75, 1.0)$ and the right state being $(\rho, u, p) = (0.125, 0.0, 0.1)$. The original Roe scheme gives entropy violating jump in the expansion region as shown in the top of figure (2). The new entropy consistent scheme together with entropy variable based dissipation gives solutions which do not suffer from this problem as shown in the bottom of figure (2).

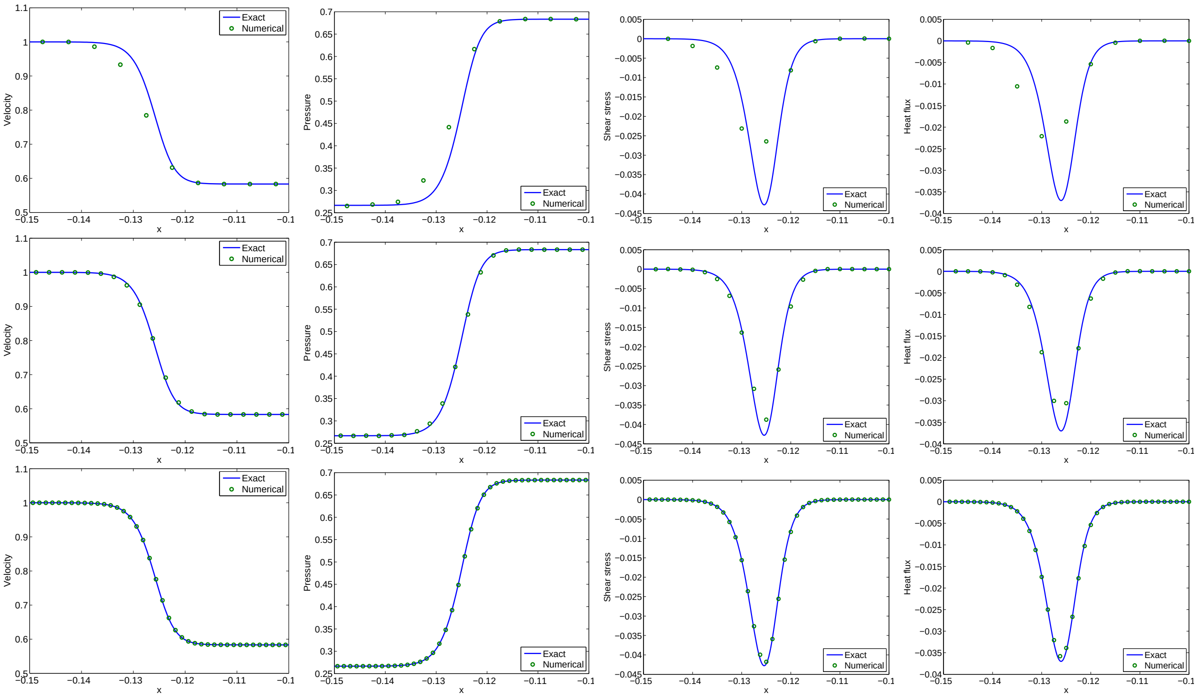


Figure 1: NS shock structure: (top) $N = 50$ cells, (middle) $N = 100$ cells, (bottom) $N = 200$ cells, KEPS-SD flux

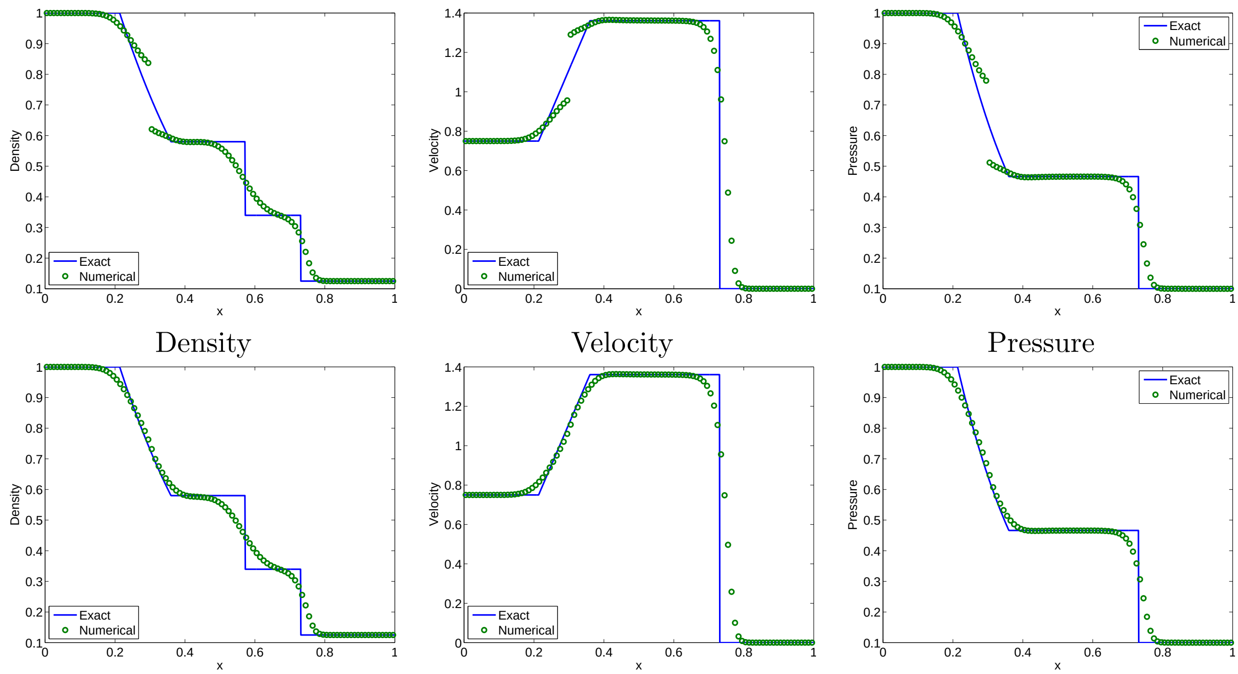
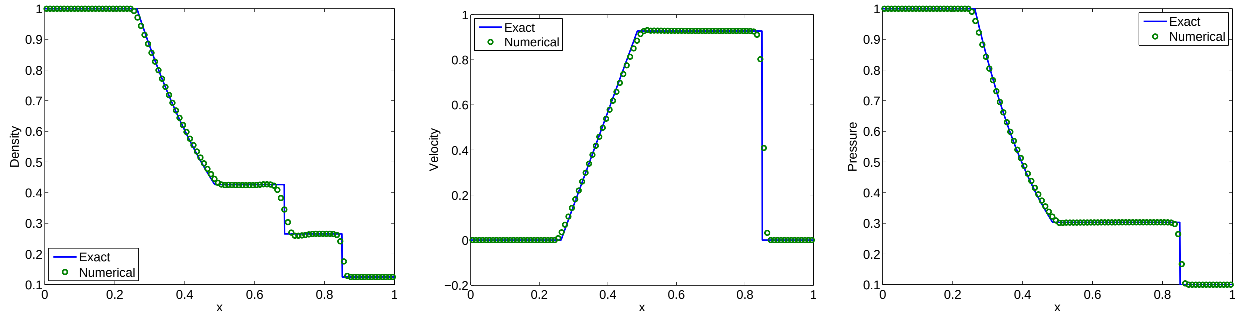


Figure 2: Modified Sod test case: $N = 100$ cells, (top) Roe, (bottom) KEPS-ED, $\beta = 1$



Density Velocity Pressure

Figure 3: Sod test case: $N = 100$ cells, KEPS-ED, minmod, $\beta = 1$

3.3 Sod test case

This is also a shock tube problem with the left state being $(\rho, u, p) = (1.0, 0.0, 1.0)$ and the right state being $(\rho, u, p) = (0.125, 0.0, 0.1)$. We compute the solution using entropy consistent scheme with entropy dissipation and using MUSCL scheme and minmod limiter. The solution shown in figure (3) indicates that the sharp resolution of contact and shocks can be achieved with the new schemes.

4 Summary and conclusions

We have proposed novel flux functions that conserve both kinetic energy and entropy. These fluxes are used together with scalar or matrix dissipation to obtain entropy stable schemes for Euler and Navier-Stokes equations. Preliminary results on model problems show that they behave as expected yielding accurate, non-oscillatory solutions that satisfy entropy condition. Future work will explore their further properties and applications to multi-dimensional flows.

References

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