A New Upwind Least Squares Finite Difference Scheme (LSFD-U) for Euler Equations of Gas Dynamics§

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1. Introduction

One of the remarkable progresses made in the area of CFD, in recent years is the development of Grid Free Method [1-4] for numerically solving the conservation laws encountered in fluid dynamics. The fundamental principle underlying this method, is the representation of the derivatives of the fluxes appearing in the conservation laws, using a generalised finite difference strategy based on the method of least squares. This method which can operate upon any kind of grid (structured, unstructured or cartesian) requires only local connectivity information at any given node. The utility of this method in computing flow past complex configurations is extremely promising. The present work draws its inspiration from the fact that this method has been applied only in the framework of flux vector splitting schemes and not in the framework of flux difference splitting schemes. Here we have attempted to extend the applicability of this Grid Free Method to the framework of flux difference splitting schemes, and in the process arrived at an entirely new methodology equally applicable to flux vector splitting schemes.

In section 2, we present briefly the details regarding the Least Squares Kinetic Upwind Method (LSKUM) [1-4]. In section 3, the new least squares scheme is presented with a brief review of the flux difference splitting schemes. In section 4, we present the difficulties in extending the methodology to 2D and 3D flows, and also present two variations of the scheme, which would circumvent this problem. In section 5, we present the results and discussions. Concluding remarks are made in section 6.

2. Least Squares Kinetic Upwind Method (LSKUM)

The kinetic schemes for solving Euler equations of gas dynamics are obtained by exploiting the fact that these equations are moments of the Boltzmann equation of kinetic theory for gases. Consider the 1D Boltzmann equation :

$$f_t + v\, f_x = 0 \qquad (1)$$

where f is the velocity distribution function which is a Maxwellian and v is the molecular velocity. The fundamental principle underlying LSKUM is that the discrete approximation to f_x appearing in the Boltzmann equation is obtained using a least squares approximation given by,

$$f_x = \frac{(\Delta f, \Delta x)}{\|\Delta x\|^2} \qquad (2)$$

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The moment of the discretized Boltzmann equation will lead to an upwind scheme for the Euler equation, if the stencil of the grid points to be used in equation (2) is chosen, taking into account the direction of signal propagation. In other words, discrete approximation to f_x at any given point is obtained by using the grid points on its left if v>0 and vice versa. An interested reader is referred to the papers cited above for a number of interesting developments in LSKUM, including higher order schemes and the use of weighted least squares in equation (2).

This idea when extended to the flux vector split framework of Euler equations, given by,

$$U_t + F_x^+ + F_x^- = 0 \tag{3}$$

where U is the vector of conserved variables and F is the flux vector, the discrete approximation to F_x^+ at any given point will involve grid points to its left, and F_x^- will involve grid points to its right.

3. Upwind Least Squares Finite Difference Method (LSFD-U)

Inspired by the fact that the discrete least squares approximation to the derivative F_x involves the flux difference term ΔF, it was thought that it would be appropriate to make use of flux difference splitting in the least squares framework. Before we discuss the details of the present least squares algorithm, we briefly discuss the flux difference splitting scheme as applied to finite volume framework.

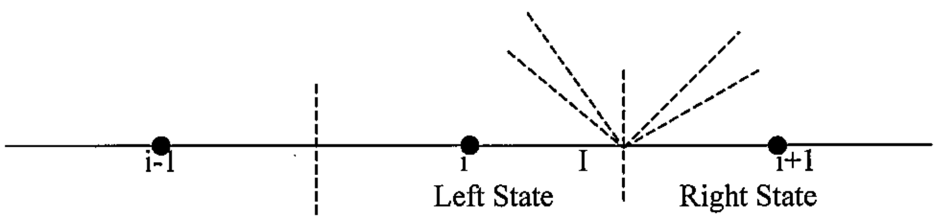

Fig 1. Typical 1D Finite Volume Computational Domain

Fig 1.depicts a typical 1D finite volume computational domain. In the flux difference splitting scheme[8], the total flux difference $\Delta F = F_R - F_L$ is split into a positive part $(\Delta F)^+$ corresponding to the right running waves and a negative part $(\Delta F)^-$ corresponding to the left running waves, based on a suitable linearization procedure, in such a way that the interfacial flux F_I is given by

$$\left.\begin{aligned} F_I &= F_L + (\Delta F)^- \\ F_I &= F_R - (\Delta F)^+ \end{aligned}\right\} \tag{4}$$

In a finite volume framework, an interfacial flux obtained as an average of the above two expressions is made use of in the state update formula.

Now we describe the present methodology. Equation 4 given above clearly suggests that the flux difference between any fictitious interface perpendicularly intersecting the line connecting the two grid points and the points themselves, can be obtained using a suitable linearization procedure. At the heart of the present methodology is the use of such flux differences based on upwinding principle in the discrete least squares approximation to F_x appearing in the Euler equations. This leads to an upwind scheme based on the least squares principle. It is also not necessary that the present methodology should be used only in conjunction with Flux Difference Splitting Schemes. The very fact that an upwind estimate of the flux at the fictitious interface is all that is required for the determination of flux differences involved in the present method, suggests the use of this method in conjunction with all upwind flux formula. As can be easily seen, the present methodology makes use of a global stencil of grid points in contrast to the methodology described in section 2, which requires an upwind stencil of grid points. Also, we make use of an upwind estimate of the interfacial flux for determining ΔF's appearing in the least squares formula, in contrast to the explicit use of the nodal values of the fluxes in the earlier framework.

4. LSFD-U in 2D

Here we present some of the interesting problems we face when extending the algorithm just described for solving 2D flows. Fig.2 gives a typical 2D stencil of grid points. Let I_j represent the fictitious interface drawn across the line connecting the point under consideration 'o', and its j^{th} neighbour, and $\hat{n}_j$ represent the unit vector along 'oj'. Representing the flux along oj by

$$F_{\uparrow j} = F \cdot n_{xj} + G \cdot n_{yj}$$

we have

$$\Delta F_{\uparrow j} = F_{\uparrow Ij} - F_{\uparrow o} \tag{5}$$

$$\Delta F_{\uparrow j} = \left(\vec{\nabla} F_o \cdot \vec{\Delta} r_{Ij}\right) n_{xj} + \left(\vec{\nabla} G_o \cdot \vec{\Delta} r_{Ij}\right) n_{yj}$$

(6)

where F and G are fluxes in x and y directions respectively. Now our job reduces to recovering the information regarding the gradients of the 2D fluxes namely $\vec{\nabla} F_0$ and $\vec{\nabla} G_0$ from the many 1D flux difference terms given in equation 6. The derivatives F_x and G_y thus recovered would eventually be used in the state update formula. Equation 6 represents an over determined system of equations and it appears that the straight forward way to obtain the gradients of the fluxes is to minimise $\sum E_j^2$ with respect to the derivatives of the fluxes, where,

$$E_j = \Delta F_{\uparrow j} - \left(\vec{\nabla} F_o \cdot \Delta r_{Ij}\right) n_{xj} - \left(\vec{\nabla} G_o \cdot \Delta r_{Ij}\right) n_{yj} \tag{7}$$

Unfortunately, simple algebra would demonstrate that such a procedure leads to a singular system. To circumvent this problem we suggest the following two methodologies.

4.1 Method 1

In this method we locally rotate the co-ordinate system from $(x,y) \rightarrow (\xi,\eta)$, in such a way that $\tilde{F}_\eta + \tilde{G}_\xi = 0$, where $\tilde{F}$ and $\tilde{G}$ represent the fluxes along the new co-ordinate directions ξ and η respectively. Note that the second and third components of the flux vectors $\tilde{F}$ and $\tilde{G}$ still represent the x and y momentum conservation. It can easily be demonstrated that a co-ordinate system rotated at an angle α, given by,

$$\alpha = \frac{1}{2} \tan^{-1} \left[\frac{F_y + G_x}{F_x - G_y} \right] \tag{8}$$

would satisfy the condition that $\tilde{F}_\eta + \tilde{G}_\xi = 0$. This leaves us with a non singular system with two unknowns, $\tilde{F}_\xi$ and $\tilde{G}_\eta$. The derivatives thus determined are used in the state update formula. The gradients of the fluxes used in the estimation of α are obtained using the least squares procedure [3] making use of a global stencil of points.

4.2 Method 2

Method 2 draws its inspiration from the work of Ghosh and Deshpande [2]. Here the local co-ordinates are rotated in such a way that one of the axes coincides with the streamwise direction. It is a well known fact that the fluxes normal to the streamwise co-ordinate direction involve only pressure terms and a global stencil can be used for approximating the derivatives of such fluxes without loss in stability. The streamwise rotation of the co-ordinate system leaves us with a non singular system involving $\tilde{F}_\xi$ and $\tilde{F}_\eta$. Similar to the previous method $\tilde{F}_\xi$ and $\tilde{G}_\eta$ are substituted in the state update formula. It should be remembered that in method 2, the second and third components of the fluxes $\tilde{F}$ and $\tilde{G}$ represent ξ and η momentum conservation, unlike method 1 in which they represent x and y momentum conservation.

5. Results and Discussions

The new least squares upwind finite difference method (LSFD-U) is validated using standard 1D and 2D test problems. In the computations higher order accuracy is achieved using the method of reconstruction[5]. Non physical oscillations in the solution are suppressed using Venkatakrishnan limiter [6]. In all the computations presented in this work the fictitious interface is placed at the mid point of the line segment under consideration.

Figure 3 gives the results obtained for the 1D shock tube problem of Sod [7]. The results are obtained on a non-uniform grid generated using cosine spacing for grid points. One hundred grid points have been used in the computation. Roe [8] flux has been used in these calculations.

The results obtained for subsonic and transonic flows past NACA 0012 airfoil are presented in Figures 4,5,6,7. The results are obtained on an unstructured mesh with 2779 mesh points (80 points are distributed on the wall). The 2D calculations have been made using KFVS [9] flux formula. The results obtained using LSFD-U are compared with those obtained using a cell vertex finite volume code. From the results it is evident that the new LSFD-U framework is capable of capturing all features expected out of inviscid compressible flows.

6. Conclusions

A new upwind least squares finite difference method has been developed. The new scheme by the virtue of using a least squares framework can be considered as a Grid Free Method. It has an added advantage of making use of a global stencil. The way the interfacial fluxes are calculated in the new scheme resembles that of finite volume method and therefore all the developments that have taken place in finite volume method, like the method of reconstruction for achieving higher order accuracy [5] can be directly adopted. The use of LSFD - U in the computation of flows past complex configurations is extremely promising.

7. References

1. S.M. Deshpande, A.K. Ghosh and J.C Mandal, "Least Squares Weak Upwind Method for Euler Equations", 89 FM 4, Fluid Mechanics Report, Department of Aerospace Engineering, Indian Institute of Science, Bangalore.
2. A.K. Ghosh, "Robust Least Squares Kinetic Upwind Method for Inviscid Compressible Flows", June 1996, PhD thesis.
3. S.M. Deshpande, P.S. Kulkarni, and A.K. Ghosh, "New Developments in Kinetic Schemes", Computers Math. Applic., Vol 35, No 1/2, pp. 75-93, 1998
4. K. Anandhanarayanan, D.B. Dhokrikar, V. Ramesh and S.M. Deshpande, "A Grid Free Method for 2D Euler Computations using Least Squares Kinetic Upwind Method", Proceedings of the third Asian Computational Fluid Dynamics Conference, pp. 390, vol 2, December 7th - 11th 1998, Bangalore, India.
5. T.J. Barth " Higher Order Solution of the Euler Equations on Unstructured Grids using Quadratic Reconstruction", AIAA-90-0013, 1990
6. V. Venkatakrishnan, "Convergence of Steady State Solutions of Euler Equations on Unstructured Grids with Limiters", JCP, vol 118, pp. 120, 1995.
7. G.A. Sod, "A Survey of Several Finite Difference Methods for system of Nonlinear Hyperbolic Conservation Laws", JCP, vol 27, pp. 1, 1978.
8. Roe, Phlip. L. "Approximate Reimann Solvers Parameter Vectors and Difference Schemes", Journal of Computational Physics, Vol. 43, pp. 357-372, 1981
9. J.C. Mandal and S.M. Deshpande, " Kinetic Flux Vector Splitting for Euler Equations", Computers and Fluids, vol 23, No 2, pp 447-478.

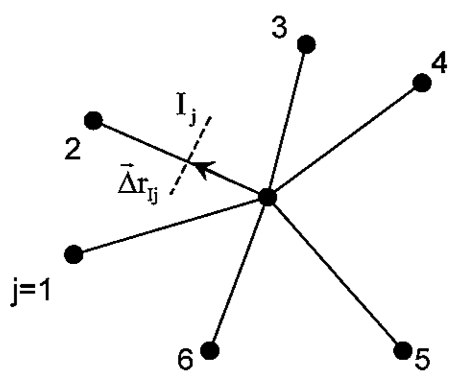

Fig 2. Typical 2D grid distribution for LSFD-U

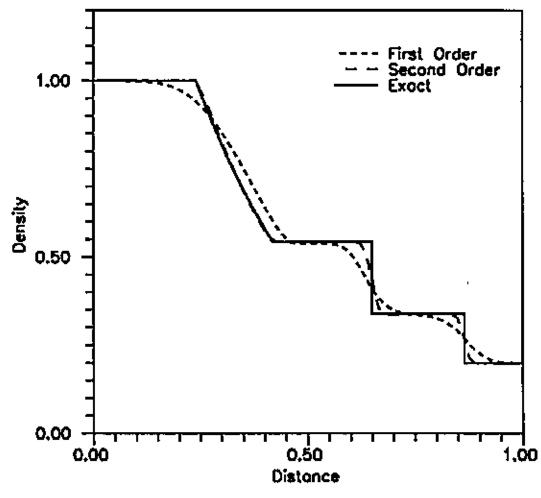

Fig 3. LSFD-U applied to Shock tube using Roe flux and 100 grid points

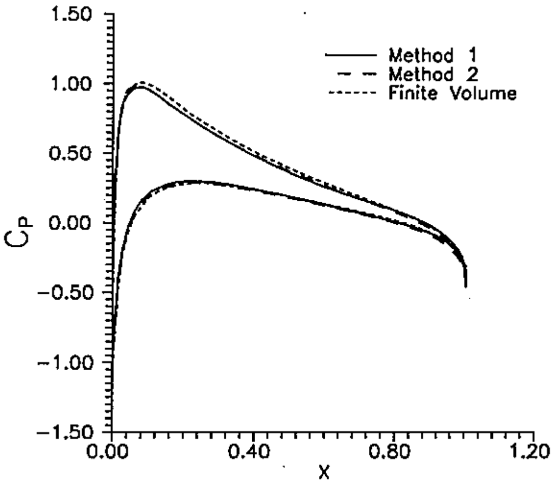

Fig 4. C_p distribution for $M_\infty = 0.63$ and $\alpha = 2^\circ$

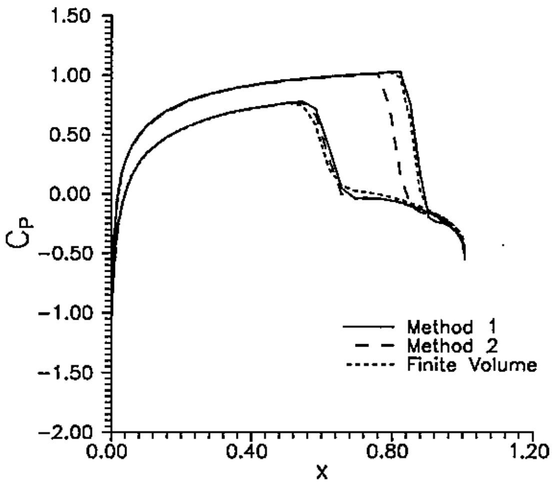

Fig 5. C_p distribution for $M_\infty = 0.85$ and $\alpha = 1^\circ$

Method 2 **Method 1**

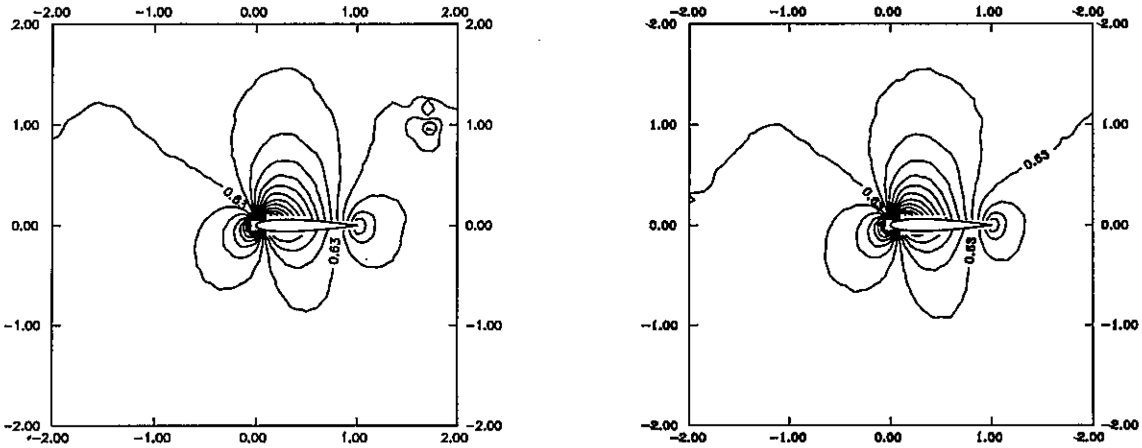

Fig 6. Mach Contours for $M_\infty = 0.63$, and $\alpha = 2^\circ$

Method 2 **Method 1**

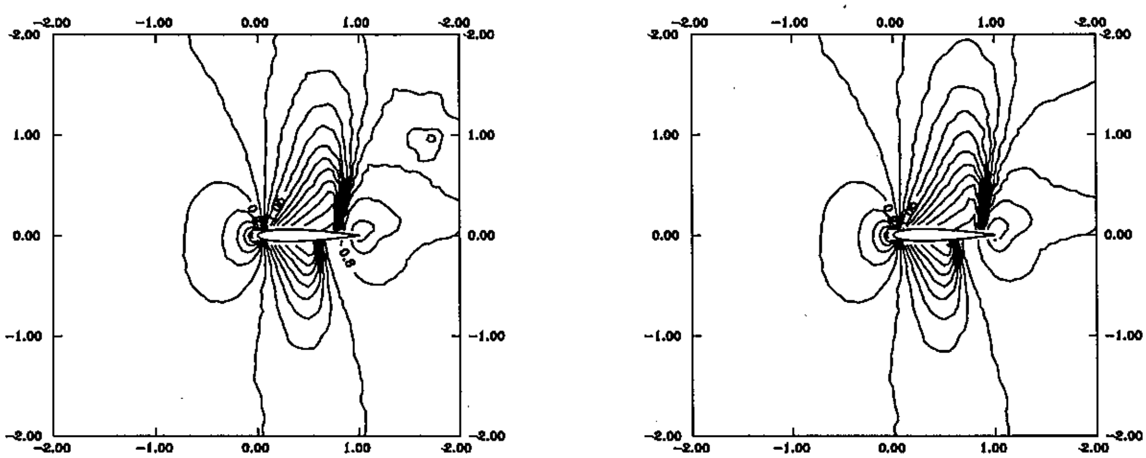

Fig 7. Mach Contours for $M_\infty = 0.85$ and $\alpha = 1^\circ$